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Control de fluctuaciones en  
sistemas dinámicos discretos:  
aplicaciones a dinámica de  
poblaciones

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*A mis hijas. A Silvia. A mis padres.*



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## RESUMEN

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Los sistemas dinámicos discretos se utilizan para modelar fenómenos en casi todos los campos científicos. En muchas ocasiones estos modelos presentan dinámicas caóticas. El objetivo de esta tesis doctoral es el estudio analítico y numérico de algunos métodos de control de caos para sistemas dinámicos discretos que actúan directamente sobre la variable de estado.

El primer método de control de caos, método OGY<sup>1</sup>, fue presentado hace un cuarto de siglo (Ott et al., 1990). Desde entonces han aparecido varios métodos que se ajustan a las necesidades de los fenómenos a los que se aplican (Andrievskii and Fradkov, 2003). Para implementar algunos de estos métodos de control es necesario realizar continuas modificaciones en los parámetros que rigen el sistema. Sin embargo, en ciertos casos los parámetros intrínsecos que aparecen en el modelo son difíciles o imposibles de modificar. En consecuencia, los métodos de control basados en modificaciones de dichos parámetros no son aplicables. Y se considera más adecuado introducir nuevos parámetros en el sistema que sean más fácilmente modificables. En esta memoria nos centraremos en este último tipo de métodos.

Para ilustrar los resultados consideraremos modelos discretos de dinámica de poblaciones, que describen la evolución temporal del número de individuos de una especie o grupo de especies en una región determinada. Estos modelos son ricos desde el punto de visto matemático ya que, tras un periodo transitorio inicial, la evolución de los mismos puede tender asintóticamente hacia un punto de equilibrio positivo o un ciclo, pero también puede presentar una evolución

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<sup>1</sup> Las letras OGY hacen referencia a las iniciales de los creadores del método: Ott, Grebogy y Yorke.

que acabe con la extinción de la especie o una evolución muy irregular con grandes fluctuaciones e incluso dinámica caótica (May et al., 1974). Pero además del interés puramente teórico-matemático, estos modelos son importantes desde el punto de vista práctico puesto que se utilizan para tomar decisiones de tipo económico o de conservación medioambiental (Beverton and Holt, 1957). También debemos destacar que la dinámica de poblaciones ha sido una de las áreas en las que las estrategias de control de caos han despertado más interés en los últimos años, e.g. (Solé et al., 1999; Tung et al., 2014).

Los resultados de esta memoria están agrupados atendiendo a la dimensión del sistema dinámico. Así, recogemos en una parte los resultados relativos a sistemas unidimensionales y en otra los relativos a sistemas de dimensión dos. En ambos casos extendemos y complementamos resultados sobre métodos de control de caos propuestos por otros autores. Y en el caso unidimensional plantearemos y estudiaremos una nueva estrategia de control de caos.

En el primer capítulo sobre modelos unidimensionales analizamos el método *proportional feedback* (PF), que consiste en multiplicar la variable de estado por una constante positiva ( $\gamma > 0$ ) cada cierto número de iteraciones. Aunque el método, propuesto por Güémez and Matías (1993), ha sido utilizado y generalizado en varias situaciones, fue recientemente cuando se hizo una validación teórica sobre su aplicación en sistemas discretos. Concretamente, Liz (2010b) demostró que utilizando un rango de la constante positiva  $\gamma$  menor que 1 para una clase de ecuaciones definidas por funciones unimodales a la que pertenecen algunos modelos de población clásicos, como por ejemplo la ecuación de Ricker y la ecuación cuadrática, el método PF hace que la solución del sistema tienda asintóticamente hacia un equilibrio positivo. En este primer capítulo sobre modelos unidimensionales recogemos la publicación (Carmona and Franco, 2011), en la cual se mejoran y complementan resultados relativos al método PF en las siguientes tres direcciones principales:

- Definimos una clase más amplia de funciones para las que se puede garantizar *a priori* el éxito del método PF en la estabilización de un equilibrio positivo.

- Relajamos las condiciones para garantizar la estabilidad asintótica global del equilibrio positivo.
- Demostramos que el método PF se puede emplear en algunos casos para transformar el origen de atractor a repulsor. Lo que en el caso de dinámica de poblaciones se traduce en que el método PF permite evitar la extinción de especies provocada por el efecto Allee (Schreiber, 2003).

También en la parte de la memoria dedicada a los sistemas unidimensionales estudiaremos un nuevo método de control de caos que hemos denominado *método de control no lineal* (CN). Dado que CN fue motivado por otro método, llamado *target oriented control* (TOC), necesitamos describir brevemente TOC para explicar en qué consiste la aportación de CN.

El método target oriented control depende de dos parámetros: básicamente un parámetro sirve para establecer una población objetivo mientras que el otro parámetro tiene que ver con la estabilización producida Dattani et al. (2011); Franco and Liz (2013). TOC tiene algunas propiedades que lo hacen interesante. Por ejemplo, Dattani et al. (2011) calcularon numéricamente los costes de las intervenciones de TOC y obtuvieron mejores resultados en comparación con otros métodos de control de caos, lo cual es una ventaja desde el punto de vista económico. Por otro lado, desde un punto de vista del estudio de las propiedades del método, TOC presenta la ventaja de que realiza una transformación lineal en la variable de estado, que puede usarse para demostrar la estabilidad global si la función a controlar es unimodal y tiene derivada Schwarziana negativa en todo el dominio Singer (1978). Como principal inconveniente del método podría considerarse que se requiere modificar sensiblemente el sistema para asegurar la estabilización: el origen deja de ser un punto fijo para el sistema controlado aunque lo fuera para el sistema de partida, y si la función que define el sistema no controlado es unimodal, la función controlada puede convertirse en estrictamente decreciente.

CN tiene similitudes con TOC pero permite que la función que define el sistema tras el control no cambie tan significativamente. El

nuevo método también utiliza dos parámetros que juegan el mismo papel que para TOC, es decir, uno sirve para fijar la población objetivo y el otro se utiliza para estabilizar el sistema. Entre otras cosas, en el capítulo dedicado al método CN mostramos que al igual que TOC este método:

- Es capaz de estabilizar cualquier función unimodal hacia su equilibrio positivo.
- El aumento del parámetro de control después de conseguir la estabilización hace que el equilibrio sea aún más atractivo, es decir, que un sobrecontrol no produce la extinción de la población. Algo que puede ocurrir, por ejemplo, con un sobreuso del control en el método PF.
- Permite eliminar el efecto Allee, es decir, es un método adecuado para especies para las que tamaños pequeños de la población representan un riesgo para su supervivencia.

Pero al contrario que TOC, aquí mostramos que CN tiene las siguientes propiedades:

- Si el origen es un punto fijo la función original, entonces será un punto fijo de la función tras el control.
- La función controlada es unimodal, si lo era la función de partida.

El método CN realiza una transformación no lineal en la variable de estado, algo que complica mucho el estudio de la estabilidad global. Por este motivo, el análisis de las propiedades de estabilización global del método CN lo realizaremos únicamente para el caso particular, aunque muy relevante, de la función de Ricker. Comprobaremos que el signo de la derivada Schwarziana de la función controlada puede variar aunque ese signo sea constante para la función a controlar de partida. Por lo tanto, no es posible utilizar la misma estrategia que en el caso de TOC para tratar de garantizar la estabilidad global del equilibrio positivo (Franco and Liz, 2013). Lo que nos llevará a utilizar otra herramienta para analizar la estabilidad global, concretamente la de envolvimiento por una transformación fraccionaria lineal

que desarrolló Paul Cull (Cull et al., 2005; Cull, 2007). Los resultados que aparecen en este segundo capítulo sobre sistemas unidimensionales fueron presentados por el autor de esta memoria en el *Workshop on Dynamical Systems and Applications* celebrado en la UNED en 2012.

En la parte dedicada a los sistemas dinámicos de dimensión mayor que uno estudiamos un tipo particular de sistemas que se forma acomplando dos sistemas unidimensionales. Este acomplamiento, para el caso concreto de modelos de dinámica de poblaciones, permite simular el papel que juega la migración si la dinámica de la población se produce en varias subregiones (también denominadas celdas o parcelas) conectadas a través de procesos de migración. En varios artículos se proponen y estudian modelos de evolución de una población distribuida de forma discreta en parcelas o celdas interconectadas, frecuentemente llamadas metapoblaciones. Estos modelos pueden ser relativamente sencillos, como por ejemplo dos celdas conectadas de forma unidireccional o bidireccional (Lloyd, 1995), pero muy prácticos para poder entender el papel que juega la conexión de las celdas para la estabilidad del sistema (Hastings, 1993). Otros modelos algo más complicados permiten modelizar, por ejemplo, el hábitat de la orilla de un lago suponiendo celdas conectadas en serie con condiciones de frontera periódicas (Doebeli, 1995; Doebeli and Ruxton, 1997). En cualquier caso, estos sistemas tienen en común la siguiente idea: modelan la evolución de la población mediante una fase de reproducción gracias a un sistema de ecuaciones en diferencias y una fase de dispersión definida por una ley de reparto entre las celdas del sistema. Estos sistemas pueden presentar grandes fluctuaciones que afectan a dos propiedades críticas para la población como son su constancia y su persistencia; además en estos sistemas puede presentarse multiestabilidad, es decir, el tamaño de la población tras un transitorio inicial puede tender a atractores diferentes dependiendo de las condiciones iniciales.

Esta parte de la memoria cuenta con un capítulo en el que consideraremos sistemas bidimensionales. En él, estudiamos el papel de la propia conexión como método de control para el sistema. Desde el punto de vista de la dinámica de poblaciones este estudio resulta

muy interesante porque facilitar u obstaculizar el movimiento de individuos a través de diferentes regiones ha sido utilizado como una herramienta de control del medio ambiente (corredores ecológicos, barreras de dispersión, etc.). Por lo tanto, para diseñar y gestionar estas estrategias, es necesario entender qué consecuencias pueden aparecer por la modificación de la migración entre regiones. Con el objetivo de comprender la influencia de la migración como método de control en la publicación (Carmona and Franco, 2015), que da lugar a este capítulo, realizamos estudios numéricos sobre el comportamiento de:

- El tamaño total medio de la población ante distintos escenarios de dinámica local y del tipo de conexión entre regiones (unidireccional o bidireccional).
- La estabilidad en el sentido de la constancia y de la persistencia.

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**ABSTRACT**

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Discrete dynamical systems are used to model phenomena in almost all scientific fields. These models often exhibit chaotic dynamics. The aim of this thesis is the analytical and numerical study of methods for chaos control in discrete dynamical systems which act directly on the state variable.

The first method for chaos control, the OGY method<sup>1</sup>, was introduced a quarter of century ago (Ott et al., 1990). Since then, several methods have been introduced and analyzed to meet the needs of different phenomena and situations (Andrievskii and Fradkov, 2003). To implement some of these methods it is required continuous modifications in the parameters that govern the system. However, in certain cases the intrinsic parameters that are present in the model are difficult or even impossible to modify. Consequently, the methods for chaos control based on modifications of these parameters are not applicable and it is considered more appropriate to introduce new parameters in the system that are easily modifiable. In this thesis, we focus on the latter type of methods.

In order to illustrate the results, we will consider discrete models of population dynamics, which describe the temporal evolution of the number of individuals of a species or group of species in a particular region. These models are rich from a Mathematical point of view, because after an initial transition period, the evolution of the population may tend asymptotically towards a positive equilibrium point or a cycle, but it is also possible an evolution towards the extinction of the species or the presence of irregular behavior with large fluctuations

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<sup>1</sup> The letters OGY stand for the initials of the creators of the method: Ott, Yorke and Grebogy.

or even chaotic dynamics (May et al., 1974). But besides the purely theoretical mathematical interest, these models are important from the practical point of view since they are used to make economic or environmental conservation decisions (Beverton and Holt, 1957). We should also note that the dynamics of populations has been one of the areas where strategies for chaos control have received more attention in the recent years, e.g. (Solé et al., 1999; Tung et al., 2014).

The results of this thesis are arranged attending to the dimension of the dynamic system. So, we consider in the first part the results for one-dimensional systems and in the last part we consider systems of dimension two. In both cases, we complement and extend recent results on methods proposed by other authors in the literature. And in the unidimensional case, we propose and study a new strategy for chaos control.

In the first chapter regarding one-dimensional models, we analyze the method *proportional feedback* (PF), which consists in multiplying the state variable by a positive constant ( $\gamma > 0$ ) every certain number of iterations. This method was proposed by Güemez and Matías (1993), and although it has been used and widespread in many applications, it was not until recently when a theoretical validation on its application in discrete systems was made. Specifically, Liz (2010b) showed that using a range of the positive constant  $\gamma$  less than 1 PF method is able to force the system to asymptotically reach a positive equilibrium for a class of maps defined by unimodal functions—to which belong some classic population models such as the Ricker and the quadratic map. In this first chapter on one-dimensional models, we collect the paper (Carmona and Franco, 2011), which improves and complements results relative to the PF method in the following three main directions:

- We define a broader class of functions for which it can be ensured *a priori* that the PF method will be successful in stabilizing a positive equilibrium.
- We relax the conditions to ensure the global asymptotic stability of the positive equilibrium.



- We show that PF method can be used in some cases to change the nature of the origin, from an attractor to a repeller. What, in the case of population dynamics, means that PF method is able to prevent the extinction of species caused by the Allee effect (Schreiber, 2003).

Also in the part of the thesis regarding one-dimensional systems, we study a new method for chaos control that we have called *non-linear control method* (CN). Since CN was inspired by another method, called *target oriented control* (TOC), we need to describe briefly TOC in order to understand what is the contribution of CN.

TOC introduces two parameters to control the system: basically, one parameter is used to establish a population target while the other parameter deals with the stabilization itself Dattani et al. (2011); Franco and Liz (2013). TOC has some interesting features. For example, Dattani et al. (2011) numerically calculated the costs of interventions with TOC and they obtain that it performs better than other methods for chaos control, which is an advantage from the economic point of view. On the other hand, TOC has the advantage of performing a linear transformation in the variable of state, which can be used to prove the global stability of the positive equilibrium if the uncontrolled function is unimodal and has a negative Schwarzian derivative in the whole domain Singer (1978). The main drawback of TOC could be that in some cases it is required a significant modification of the system to ensure the stabilization: the origin is not a fixed point of the controlled map anymore though it was for the uncontrolled system, and if the function of the uncontrolled map is unimodal, the controlled function may become strictly decreasing.

CN has some similarities with TOC but it allows that the function that defines the system does not change significantly with the control. The new method also consists in the use of two parameters and they play the same role as for TOC, that is, one parameter sets the population target and the other parameter is modified to stabilize the system. Among other things, in the chapter dedicated to CN method we show that (as TOC does):

- It is able to stabilize any unimodal function to its positive equilibrium.
- The increase in the control parameter after reaching the stabilization of the equilibrium makes it even more attractive, i.e., an excessive use of the control does not cause the extinction of the population. Something that might occur, for example, with an overuse of the control in PF method.
- It eliminates the Allee effect, i.e., it is a method suitable for species in which small population sizes are dangerous for their survival.

But unlike TOC, here we show that CN has also the following properties:

- If the origin is a fixed point of the original function, then it will be a fixed point of the function after control.
- The controlled function is unimodal, if it was the original function.

The CN method performs a nonlinear transformation in the variable of state, something that complicates a lot the study of the global stability. For that reason, the analysis of the global stability for the method CN will be done for the particular case of the Ricker map. We will find that the sign of the Schwarzian derivative of the controlled function may change, even when the sign is constant for the initial uncontrolled function. Thus, it is not possible to use the same strategy that for TOC to achieve a sufficient condition for the global stability Franco and Liz (2013). This led us to employ another tool to analyze the global stability, the so-called technique of enveloping using a fractional linear function that developed Paul Cull (Cull et al., 2005; Cull, 2007). The results shown in this second chapter on one-dimensional systems were presented by the author of this thesis in the *Workshop on Dynamical Systems and Applications* held at UNED in the year 2012.

In the part of the thesis about dynamical systems of dimension higher than one, we study a particular type of systems generated by

coupling two one-dimensional systems. That is, in the case of the models of population dynamics, we are considering the role of migration in the evolution of the population occurring in several subregions (also called cells or patches) connected through migration processes. In several papers are proposed and studied models of the dynamics of a population discretely distributed in patches or interconnected cells, often called metapopulation. These models may be relatively simple, such as two patches connected in a unidirectional or bidirectional way (Lloyd, 1995), but very handy in order to understand the fundamental role that the connection of the cells plays in the stability of the system (Hastings, 1993). Other more sophisticated schemes model, for example, the habitat of the shore of a lake assuming that the cells are connected in series with periodic boundary conditions Doebeli (1995); Doebeli and Ruxton (1997). In all cases, these systems share this basic idea: they model the evolution of the population with a phase of reproduction defined by a system of difference equations and a dispersion phase defined by a law of distribution among the cells of the system. These systems may exhibit large fluctuations that affect two critical properties of the population such as constancy and persistence; moreover this systems may show multistability, that is, the size of the population after an initial transient may tend to alternative attractors depending on the initial conditions.

This part of the thesis has one chapter in which we consider two-dimensional systems. There, we study the role of the connection itself as a method of control of the system. From the point of view of the population dynamics, this study is very interesting because the strategy of facilitating/hindering the movement of individuals across different regions has been used as a tool of environmental control (ecological corridors, dispersal barriers, etc.). Therefore, to design and manage these strategies, it is necessary to understand what consequences can arise due to the modification of migration among regions. In order to understand the impact of migration as a potential method of control, in the paper (Carmona and Franco, 2015), which corresponds to this chapter, we perform numerical simulations to study the behavior of:

- The average of the total population size in different scenarios of local dynamics and the type of connection among regions (unidirectional or bidirectional);
- The stability in the sense of constancy and persistence.

Parte I

SISTEMAS UNIDIMENSIONALES



# 3

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## CONTROL PROPORCIONAL

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### 3.1 INTRODUCTION

Many population models experience an irreversible period doubling route to chaos as intrinsic parameters change (see May et al., 1976). For example, it is known (Getz, 1996) that the generalized Beverton-Holt model (Beverton and Holt, 1957; De Vries et al., 2006)

$$x_{n+1} = \frac{rx_n}{1 + x_n^B}, \quad B > 1, \quad (3.1)$$

admits chaotic solutions when the density dependent *abruptness* parameter  $B$  is sufficiently large in relation to the per capita growth rate parameter  $r$ .

Although in wild ecological systems is difficult to observe chaotic dynamics, it has been observed in laboratory (for more details see Costantino et al., 1997; Hastings, 2009) and researchers have built strategies of control to avoid the not desirable situations predicted by the models. From an ecological point of view, changing some of the intrinsic biological parameters that appear in the model as the growth rate is difficult or even impossible. In consequence, in many of these strategies of control the authors introduce a new parameter in the system which is easy to modify. For example, a constant immigration or depletion in each generation transforms an uncontrolled system

$$x_{n+1} = f(x_n)$$

into the controlled system  $x_{n+1} = f(x_n) + C$  with  $C$  a positive fixed constant in case of immigration or a negative constant in case of emigration. This method is known as *constant feedback control* and has

been deeply studied in the literature; see (Liz, 2010a) and references therein for a nice state of the art of the method.

In this chapter, we are interested in the stabilization of one-dimension discrete systems into a positive equilibrium. We focus our attention on the *proportional feedback control* method introduced in (Güémez and Matías, 1993). This method, to which we will refer in the sequel as (PF) for the sake of brevity, consists in multiplying the state variable by a constant positive factor  $\gamma$  each  $p$  iterations, where  $p$  is an integer greater than zero. We restrict our study to the case  $p = 1$ .

When the state variable  $x_n$  represents a population, the interpretation of the (PF) method is clear. Taking  $0 < \gamma < 1$ , means that a fixed percentage (exactly  $1 - \gamma$ ) of the population is removed by *migration or harvesting* in each generation; whereas if  $\gamma > 1$ , there is a population supplement proportional to the size of the previous generation by a factor of  $\gamma - 1$ . In a wild ecological model we can understand this proportional supplement as a *calling effect* depending on the success rate. We note that in both cases the control occurs prior to reproduction. This last fact has a great influence in the effects of a PF control; see (Seno, 2008).

Güémez and Matías in (Güémez and Matías, 1993; Matías and Güémez, 1994) showed numerically that the (PF) method stabilizes some chaotic systems. They noted that the method successes in the stabilization both for a  $\gamma_1 > 1$  and a  $\gamma_2 < 1$  in some cases. When this occurs they observed that the equilibrium state was not the same for the different values of the parameter of control.

We note that not every smooth map from an interval  $I = [0, b)$  into  $I$ , where  $b$  can be  $\infty$ , can be controlled by the (PF) method. For example, take the map  $f: [0, \infty) \rightarrow [0, \infty)$  given by  $f(x) = x^2$ , which has as unique unstable positive equilibrium  $K = 1$ . If the (PF) method stabilizes an equilibrium  $x_0 > 0$  for  $f$ , then  $x_0$  would have to satisfy together  $x_0 = (\gamma x_0)^2$  and  $2\gamma^2 x_0 \leq 1$ . What is clearly impossible.

If we consider a linear homogeneous map for which zero is the only equilibrium, it is clear that the (PF) method can change the stability of the origin, but not create an asymptotically stable positive equilibrium. Just, for one critical value of the parameter  $\gamma$  in which



the origin changes its stability, every point is an stable but not asymptotically stable positive equilibrium. Similar difficulties could arise if we consider a piecewise-linear dynamical system (Bernardo et al., 2008). Therefore, sufficient theoretical conditions for guaranteeing either that the (PF) asymptotically stabilizes a positive equilibrium or creates a global attractor for the controlled system are of high interest.

Although the (PF) method proposed by Güémez and Matías has been used and generalized in several situations and directions (Läu et al., 2005; Solé et al., 1999; Saito and Mitsubori, 1995), there was not until very recently a theoretical validation for discrete systems. Concretely, assuming  $0 < \gamma < 1$ , Liz (2010b) proved that for a class of unimodal maps to which some classic population models belong, as for example Ricker and quadratic, the (PF) method globally stabilizes the system towards a positive equilibrium. More precisely, in (Liz, 2010b) it was assumed that the map that defined the system had to satisfy the following conditions:

(A1)  $f$  has only two fixed points:  $x = 0$ , and  $x = K > 0$ ,  
with  $f'(0) > 1$ .

(A2)  $f$  has a unique critical point  $c < K$  in such a way that  
 $f'(x) > 0$  for all  $x \in (0, c)$ ,  $f'(x) < 0$  for all  $x \in (c, b)$ .

(A3)  $f''(x) < 0$  for all  $x \in (0, c)$ .

(A4)  $(Sf)(x) < 0$  for all  $x \neq c$ , where

$$(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2$$

is the Schwarzian derivative of  $f$ .

Hypotheses (A1)-(A3) were necessary to guarantee the existence of a unique asymptotically stable equilibrium whereas adding hypotheses (A4) had the objective of guarantee the global attraction of that equilibrium. These assumptions were motivated by the fact that many maps usually employed in discrete models fulfill them. Similar conditions were assumed for constant feedback control in (Gueron, 1998; Schreiber, 2001), and for *predictive control* in (Liz and Franco, 2010).

Singer (1978) showed that if  $f$  satisfies (A1)-(A2) and (A4), and  $f'(K) \geq -1$ , then  $K$  is a global attractor of system  $x_{n+1} = f(x_n)$ . Therefore, condition (A3) is in some sense additional but, as it was remarked in (Liz, 2010b), not removable since that would lead to a possible Allee effect when applying the (PF) method.

We recall that an Allee effect, see (Schreiber, 2003) for a detailed description and discussion of this effect, is sometimes related to the extinction of small populations because of the difficulty of finding a mate. Most of the maps having an Allee effect have exactly three equilibrium points  $0 < K_1 < K_2$ : the origin is the *attracting extinction state*,  $K_1$  is the repelling fixed point, or *Allee threshold*, and  $K_2$  is the attracting fixed point or *carrying capacity*.

Our intention in this chapter is to improve and complement the theoretical knowledge of the (PF) method in the following three main directions:

1. Define a bigger class of maps for which it can be *a priori* guaranteed the success of the (PF) method in stabilizing a positive equilibrium without the restriction of  $0 < \gamma < 1$ .
2. Relax the conditions to guarantee the global asymptotic stability of the positive equilibrium. In particular, avoiding the constant sign condition in  $f''$  and the Schwarzian derivative, and the necessity of the uncontrolled map to have a unique positive equilibrium.
3. Show that the (PF) method with  $\gamma > 1$  can be employed in some cases to eliminate the Allee effect, and therefore when dealing with population models, be used as a tool to prevent the extinction of the species.

We finish this introduction clarifying that throw this chapter by unimodal map we mean a  $C^1$  map  $f: I \rightarrow I$  with a unique local maximum in the interior of  $I$  and not having any local minimum in the interior of  $I$ .

## 3.2 RESULTS AND DISCUSSION

For a given map  $f: I \rightarrow I$  we denote by  $g_\gamma$  the controlled map that we obtain after applying the (PF) method with control parameter  $\gamma$ . Therefore,  $g_\gamma$  satisfies

$$g_\gamma(x) = f(\gamma x), \quad \forall x \in I.$$

From a practical point of view, it would be interesting to have conditions for a general class of maps which a priori establish if the (PF) method will be able to stabilize a positive equilibrium. The following result gives a complete characterization of the maps, from a wide class, for which the method works.

*Theorem 3.1:* Denote  $I = [0, b]$ , a real interval ( $b = +\infty$  is allowed). Let  $f: I \rightarrow I$  be a  $C^1((0, b))$  map. Then, the following characterization holds:

There exist  $\gamma_1, \gamma_2 \in [0, \infty]$ ,  $\gamma_1 < \gamma_2$  such that for any  $\gamma \in (\gamma_1, \gamma_2)$  the map  $g_\gamma$  has at least one hyperbolic asymptotically stable positive equilibrium if and only if there exists  $z \in (0, b)$  such that

$$|zf'(z)| < f(z). \quad (3.2)$$

*Proof.* On the one hand, suppose that there exist  $\gamma_1, \gamma_2 \in [0, \infty]$ ,  $\gamma_1 < \gamma_2$  such that for any  $\gamma \in (\gamma_1, \gamma_2)$  the map  $g_\gamma$  has at least one hyperbolic asymptotically stable positive equilibrium. Let us fix  $\gamma^* \in (\gamma_1, \gamma_2)$  and denote by  $K_{\gamma^*}$  a hyperbolic asymptotically stable positive fixed point of  $g_{\gamma^*}$ . We know that  $K_{\gamma^*}$  satisfies  $|g'_{\gamma^*}(K_{\gamma^*})| < 1$  and  $K_{\gamma^*} = g_{\gamma^*}(K_{\gamma^*}) = f(\gamma^* K_{\gamma^*})$ .

Take  $z = \gamma^* K_{\gamma^*}$  and recall that  $g'_{\gamma^*}(x) = \gamma^* f'(\gamma^* x)$ . The equality  $K_{\gamma^*} = f(\gamma^* K_{\gamma^*})$  implies  $f(\gamma^* K_{\gamma^*})/K_{\gamma^*} = 1$  and therefore

$$|g'_{\gamma^*}(K_{\gamma^*})| < 1 \quad \Leftrightarrow \quad |\gamma^* f'(\gamma^* K_{\gamma^*})| < 1 = \frac{f(\gamma^* K_{\gamma^*})}{K_{\gamma^*}},$$

and  $z = \gamma^* K_{\gamma^*}$  satisfies the inequality (3.2).

On the other hand, suppose that there exists  $z \in (0, b)$  such that the inequality (3.2) holds. Firstly, note that in such  $z$  the map  $f$  has to be positive.

The continuity of  $f$  and  $f'$  implies that there exist  $z_1, z_2 \in [0, b]$  such that the inequality (3.2) holds for any  $z \in (z_1, z_2)$  and  $zf'(z) - f(z)$  is negative in  $(z_1, z_2)$ . Therefore, we define the interval  $(\gamma_1, \gamma_2)$  as the image of  $(z_1, z_2)$  by the well defined and strictly monotone increasing continuous function  $h: (z_1, z_2) \rightarrow \mathbb{R}$  given by  $h(x) = x/f(x)$ .

Now, take  $\gamma^* \in (\gamma_1, \gamma_2)$ . By construction there exists one  $z^* \in (z_1, z_2)$  such that  $h(z^*) = z^*/f(z^*) = \gamma^*$ . We claim that for such  $\gamma^*$  the map  $g_{\gamma^*}$  has at least a positive asymptotically stable fixed point given by  $f(z^*)$ .

Indeed,

$$f(z^*) = f\left(\frac{z^*}{f(z^*)}f(z^*)\right) = f(\gamma^* f(z^*)) = g_{\gamma^*}(f(z^*))$$

so  $f(z^*)$  is a fixed point of  $g_{\gamma^*}$ .

And, using that  $z^*$  satisfies (3.2) we have

$$|g'_{\gamma^*}(f(z^*))| = |\gamma^* f'(\gamma^* f(z^*))| = \left| \frac{z^*}{f(z^*)} f'(z^*) \right| < 1.$$

□

Theorem 3.1 does not impose the control  $\gamma$  to be strictly smaller than 1 which allows strategies of control with a constant proportional supplement. We also note that Theorem 3.1 is a complete characterization for the wide class of functions with continuous derivative. Therefore, we can use it to decide if the (PF) method will work or not. For example, for the map  $f(x) = x^2$  considered in the introduction we get

$$|xf'(x)| = 2x^2 > x^2 = f(x), \quad \forall x \in (0, \infty),$$

and Theorem 3.1 implies that the (PF) method is not able to stabilize any positive equilibrium.

We study how the (PF) method can be applied to two maps belonging to the following families recently proposed by Braverman and Haroutunian (2010):

$$\begin{aligned} h(x) &= \frac{rx}{1+x^B} + \frac{Cx}{1+x}, \\ j(x) &= \frac{rx}{1+x^B} + C(1-10^{-rx}), \end{aligned} \tag{3.3}$$

for  $r, B, C \in (0, +\infty)$ .

Both families of maps are perturbations of the generalized Beverton-Holt map (3.1) and have the common characteristic that tend asymptotically to a positive value at infinite instead that going to zero. Although, very similar, the two maps are sensibly different since one of them is unimodal and the other is not, see (Braverman and Haroutunian, 2010) for more details. We fix in both cases  $r = 4$ ,  $B = 6$  and  $C = 0.25$  as in the examples of (Braverman and Haroutunian, 2010). It is easy to verify that for such values of the parameters in both cases there is a unique positive equilibrium ( $K \approx 1.23245$  for  $h$  and  $K \approx 1.2594$  for  $j$ ) which is unstable.

In order to apply Theorem 3.1 we need to study the sign of the following expressions

$$|xh'(x)| - h(x) = \frac{x|x^{12} - 80x^8 - 160x^7 - 78x^6 + 16x^2 + 32x + 17|}{4x^{14} + 8x^{13} + 4x^{12} + 8x^8 + 16x^7 + 8x^6 + 4x^2 + 8x + 4} + \frac{-x^7 - 16x^2 - 17x}{4x^7 + 4x^6 + 4x + 4},$$

and

$$|xj'(x)| - j(x) = x \left| \frac{1 \log(10)}{10^{4x}} + \frac{4}{x^6 + 1} - \frac{24x^6}{(x^6 + 1)^2} \right| - \frac{1}{4} \left( 1 - \frac{1}{10^{4x}} \right) - \frac{4x}{x^6 + 1}.$$

In Figure 3.1 we plot the graphs of  $|xh'(x)| - h(x)$  and  $|xj'(x)| - j(x)$ , which suggest the existence of two different intervals, one bounded and other unbounded, where inequality (3.2) holds. Indeed, that is what happen but we will not show it in detail. We will just show how to prove that the method works both for  $\gamma > 1$  and  $\gamma < 1$ , that is, we can choose between increasing or decreasing proportionally the population. We note that  $|0.5 h'(0.5)| - h(0.5)$ ,  $|3 h'(3)| - h(3)$ ,  $|0.5 j'(0.5)| - j(0.5)$  and  $|3 j'(3)| - j(3)$  are negative,  $0.5/h(0.5)$  and  $0.5/j(0.5)$  are smaller than 1, and  $3/h(3)$  and  $3/j(3)$  are greater than 1. Now, in view of the proof of Theorem 3.1 we can affirm for both maps that there exist two intervals  $I_1 \subset (0, 1)$  and  $I_2 \subset (1, \infty)$  such that (PF) method works for  $\gamma \in I_1 \cup I_2$ . Moreover, since

$$\lim_{x \rightarrow \infty} |xh'(x)| - h(x) = \lim_{x \rightarrow \infty} |xj'(x)| - j(x) = -\frac{1}{4}$$

the interval  $I_2$  can be chosen unbounded.

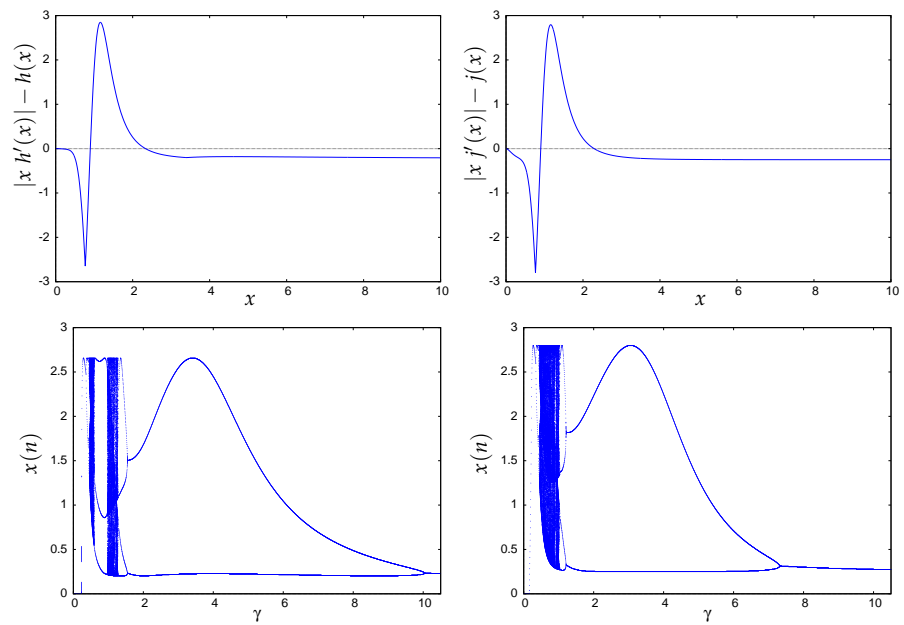


Figure 3.1: Superior panels show that maps  $h$  and  $j$  satisfy the condition (3.2) in two disconnected regions. Inferior panels show the bifurcation diagrams as  $\gamma$  varies. On the left side we consider map  $h$  of (3.3) and on the right side map  $j$  of (3.3), in both cases  $r = 4$ ,  $B = 6$  and  $C = 0.25$ .

In Figure 3.1 we include the bifurcation diagrams as well. We note that if we choose to control the system using  $\gamma < 1$  we have the advantage that we can obtain any population size from zero to the maximum of the carrying capacity, but we have the disadvantage that the control parameter should be selected with precision to avoid a chaotic situation, an undesirable population size or even worse, to induce the extinction of the species. However, if we opt for the other strategy,  $\gamma > 1$ , the range of parameters is infinite. The bifurcation diagrams have been constructed using *Maxima* by calculating 300 iterations of the map and representing the last 100, starting with the same initial conditions  $x_0 = 2$  and varying  $\gamma$  from 0 to 10.5 with a step of 0.005.

We give an immediate corollary of Theorem 3.1, which is of special interest for maps related to population models since those maps are usually  $C^1(I)$  unimodal maps with a local maximum.

*Corollary 3.1:* Denote  $I = [0, b]$ , a real interval ( $b = +\infty$  is allowed). Let  $f: I \rightarrow I$  be a  $C^1((0, b))$  map with a critical point in  $(0, b)$  where the map  $f$  takes a strictly positive value. Then, there exist  $\gamma_1, \gamma_2 \in (0, \infty]$ ,  $\gamma_1 < \gamma_2$  such that for any  $\gamma \in (\gamma_1, \gamma_2)$  the map  $g_\gamma$  has at least one asymptotically stable positive equilibrium.

Using the above result, a glance to the graph of the map, looking for critical points, is enough to obtain a sufficient condition for the (PF) method to work.

Corollary 3.1 complements the result in (Liz, 2010b) since conditions (A1)-(A3) are not assumed. However, we do not get the uniqueness of the positive equilibrium as in (Liz, 2010b). In the following subsections, adding some extra conditions we are able to guarantee the uniqueness of the positive equilibrium for the controlled map and if, in addition, the Schwarzian derivative is negative in a strict subinterval of  $I = [0, b]$ , we have that its basin of attraction is the whole open subinterval of positive initial conditions  $(0, b)$ , i.e. it is a global attractor.

To illustrate our results in the following subsections we consider the family of unimodal maps

$$f(x) = \frac{A(x^2 + Bx)}{x^3 + C}, \quad A, B, C \in (0, \infty). \quad (3.4)$$

Unimodal maps defined by rational functions are not strange in population models, think for example in the Beverton-Holt model and its generalization presented in (3.1) or some classes with Allee effect considered by Elaydi and Sacker (2010).

We state the following result from (El-Morshedy and López, 2008) that will be useful in the sequel. We point out that such result was employed in (Liz and Franco, 2010) as the main tool to prove the global stabilization of fixed points using predictive control.

*Lemma 3.1:* (El-Morshedy and López, 2008, Corollary 2.9)

Let  $g: (0, b) \rightarrow [0, b]$  be a continuous map with a unique fixed point  $K$  such that  $(g(x) - x)(x - K) < 0$  for all  $x \neq K$ . Assume that there are points  $0 \leq a < K < d \leq b$  such that the restriction of  $g$  to  $(a, d)$  has at most one turning point and (whenever it makes sense)  $g(x) \leq g(a)$  for every  $x \leq a$  and  $g(x) \geq g(d)$  for every  $x \geq d$ . If  $g$  is decreasing at  $K$ , assume additionally that  $(Sg)(x) < 0$  for all  $x \in (a, d)$  except at most one critical point of  $g$ , and  $-1 \leq g'(K) < 0$ . Then  $K$  is a global attractor of  $g$ .

### 3.2.1 Global stability for unimodal maps

In the following result we want to cover the situation of a system given by a unimodal map with a local maximum without Allee effect and having an unstable positive equilibrium.

*Theorem 3.2:* Denote  $I = [0, b]$ , a real interval ( $b = +\infty$  is allowed). Let  $f: I \rightarrow I$  be a  $C^3(I)$  unimodal map with  $f'(0) > 1$  or  $f(0) > 0$ ,  $f$  having a unstable positive equilibrium and an absolute maximum at  $c$ , and such that

$$cf(x) - xf(c) > 0, \quad \forall x \in (0, c). \quad (3.5)$$



Then, there exist  $\gamma_1, \gamma_2 \in (0, 1)$ ,  $\gamma_1 < \gamma_2$  such that for any  $\gamma \in [\gamma_1, \gamma_2)$  the map  $g_\gamma$  has exactly one asymptotically stable positive equilibrium.

Moreover, either if  $\gamma = \gamma_1$  or if  $\gamma \in (\gamma_1, \gamma_2)$  and  $Sf(x) < 0$  for all  $x \in (c, b)$ , then the above equilibrium is globally asymptotically stable for the controlled system in the sense that all orbits starting in a positive initial condition converge to that equilibrium.

*Proof.* Since  $f'(c) = 0$  and  $f(c) > 0$  we have that  $c$  satisfies inequality (3.2). By Theorem 3.1 we know that there exists an interval  $(\hat{\gamma}_1, \gamma_2) \subset [0, +\infty]$  such that the controlled map has at least one asymptotically stable positive equilibrium for any  $\gamma \in (\hat{\gamma}_1, \gamma_2)$ . Following the proof of Theorem 3.1, it is not a restriction to assume  $c/f(c) \in (\hat{\gamma}_1, \gamma_2)$ . Since the map  $f$  has a unique unstable positive equilibrium and is unimodal we conclude that  $c/f(c) < 1$  and  $(\hat{\gamma}_1, \gamma_2) \subset (0, 1)$ .

So, for proving the first affirmation of the result, it is enough to show that the controlled map has exactly one positive equilibrium for  $\gamma$  belonging to a subinterval of  $(\hat{\gamma}_1, \gamma_2)$ . In the hypotheses of our result such subinterval is  $[\gamma_1, \gamma_2)$  with  $\gamma_1 = c/f(c)$ . In order to verify it, let us fix  $\gamma \in [\gamma_1, \gamma_2)$ .

Now, we note that the equation  $z = g_\gamma(z)$  has a solution in the interval  $(0, c/\gamma)$  if and only if the equation  $x/\gamma = f(x)$  has a solution in the interval  $(0, c)$ . But, if there exists a point  $x \in (0, c)$  such that  $x = \gamma f(x)$ , then we get the following contradiction with hypothesis (3.5)

$$x = \gamma f(x) \geq \frac{c}{f(c)} f(x) \quad \Rightarrow \quad xf(c) \geq cf(x).$$

Therefore,  $g_\gamma$  has not positive equilibrium points in  $(0, c/\gamma)$ . For  $z = \frac{c}{\gamma}$  we have that  $g'_\gamma(z) = \gamma f'(\gamma z) = 0$  and for  $z > \frac{c}{\gamma}$  we have that  $g'_\gamma(z) = \gamma f'(\gamma z) < 0$ . In consequence,  $g_\gamma$  has at most one positive fixed point in  $(0, b)$  and we have finished the proof of the first affirmation.

Let's move to the second affirmation. Note that, since  $f$  is unimodal, then  $g_\gamma$  is unimodal for any  $\gamma \in (0, +\infty)$  and  $f$  and  $g_\gamma$  attain the same absolute maximum value at a single point of  $(0, b)$ . We shall divide the proof in two cases:

- Case  $\gamma = \frac{c}{f(c)}$ . For such  $\gamma$  the asymptotic stability of the positive equilibrium is obtained without any extra condition and besides the equilibrium is superstable. Effectively, it is easy to verify that the unique positive equilibrium coincides with the point where the maximum of  $g_\gamma$  is attained. In consequence the equilibrium is a global superstable attractor.
- Case  $\gamma \in (\frac{c}{f(c)}, \gamma_2)$ . In this situation we need to assume the extra condition that the Schwarzian derivative of the map  $f$  has constant negative sign in the interval  $(c, b)$  to apply Lemma 3.1. The desired result follows by taking  $F = g_\gamma$ ,  $a = \frac{c}{\gamma}$  and  $d = b$  in Lemma 3.1.

□

*Remark 3.1:* Condition (3.5) has the geometric interpretation that the graph of the map  $f$  is above the segment which joins the origin and its global maximum at  $c$ .

To illustrate a situation where Theorem 3.2 is applicable, we consider the map  $f$  given by (3.4) with  $A = 2.5$ ,  $B = 3$  and  $C = 1.5$ . In Figure 3.2 we can observe that for these values the graph of the map is above the segment which joins the origin and its global maximum at  $c$  and condition (3.5) holds, this can be checked analytically in the following way. Firstly, we calculate  $c$  having that  $c = 34409619/33554432 \approx 1.0254$ , secondly we calculate the rational expression  $f(x) - (\frac{f(c)}{c})x$  and we observe that its denominator has constant sign in  $(0, \infty)$  and its numerator, a fourth order polynomial, has exactly two real roots  $x = 0$  and  $x = c$  and it is positive in  $(0, c)$ . In blue, we plot the graph of the first derivative of  $f$  from which it is immediate that  $f''$  has not constant sign in  $(0, c)$ .

The bifurcation diagram in Figure 3.3 shows how the behaviour of the controlled map  $g_\gamma$  changes as we modify the value of  $\gamma$ . For  $\gamma = 1$  we have the uncontrolled map (3.4) with  $A = 2.5$ ,  $B = 3$  and  $C = 1.5$  which has an unstable positive equilibrium and a period two stable orbit. As  $\gamma$  decreases the amplitude of this period two orbit decreases until it collapses with the positive equilibrium in a period halving bifurcation at  $\gamma \approx 0.6$ . For  $\gamma$  smaller than that critical value  $\gamma \approx$

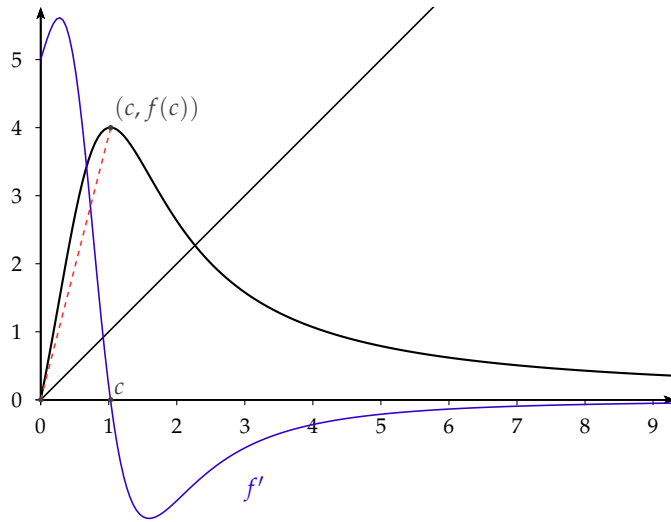


Figure 3.2: In black we plot the graph of the map (3.4) for  $A = 2.5$ ,  $B = 3$  and  $C = 1.5$ . Clearly, it is above the segment which joins the origin and its global maximum at  $c$  as condition (3.5) imposes. In blue, we plot its derivative to illustrate that  $f''$  changes sign.

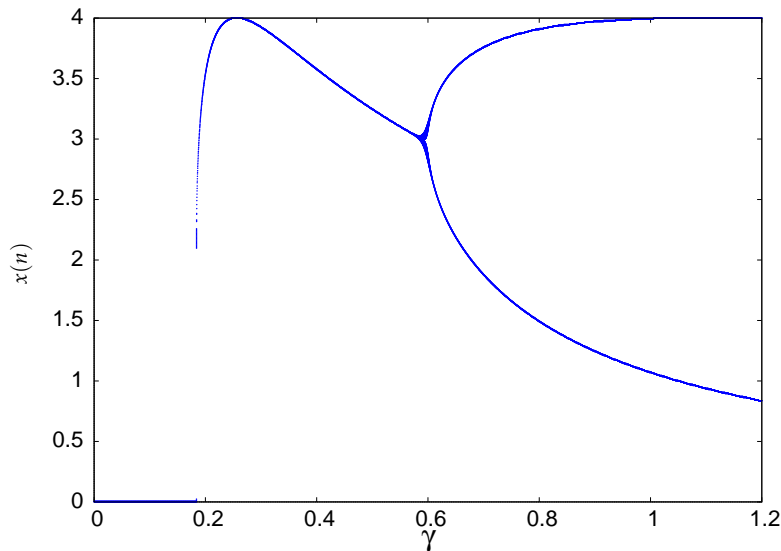


Figure 3.3: Bifurcation diagram of the (PF) method applied to (3.4) with  $A = 2.5$ ,  $B = 3$  and  $C = 1.5$ . We observe that the uncontrolled map has an unstable equilibrium and a period two stable orbit. After applying the control with  $\gamma < 1$  we obtain an asymptotically stable positive equilibrium.

0.6 and until  $\gamma \approx 0.25621$  the positive equilibrium is asymptotically stable and numerically it seems a global attractor. The Schwarzian derivative of  $f$  is negative in  $(0, \infty)$  since

$$Sf(x) = -18 \left( \frac{12x^4 + 8x^3 + 36x^2 + 72x + 3}{(2x^4 + 12x^3 - 6x - 9)^2} \right),$$

and the positive equilibrium is a global attractor at least for  $\gamma$  in the interval  $(0.25621, 0.6)$ . Theorem 3.2 could also be applicable to the map  $j(x)$  of (3.3) with  $r = 4$ ,  $B = 6$  and  $C = 0.25$ , for this map the Schwarzian derivative changes sign in  $(0, \infty)$  but, numerically, it seems to be negative in  $(c, \infty)$ , so we could guarantee the global attraction for an interval of control parameters contained in  $(0, 1)$ .

### 3.2.2 Preventing extinction for unimodal maps

The characterization of the maps for which the (PF) method controls an unstable positive equilibrium given in Theorem 3.1 does not impose a parameter of control  $\gamma$  smaller than one. In this subsection we consider a class of maps having an Allee effect for which the method is not guaranteed to work for  $\gamma < 1$ , but it does for  $\gamma > 1$ . Moreover, we are able to prove a global stability result even in this case. In Figure 3.4 we represent an example of the situation we want to cover. The map  $f$  is given by (3.4) with  $A = 3$ ,  $B = 3$  and  $C = 10$ . We observe that the uncontrolled map has an Allee effect with two positive equilibria and an attracting extinction state and the graph of  $f$  is above the segment which joins the origin and its global maximum at  $c$ . Both facts are easy to check since one can reduce them to study the sign of at most fourth order polynomials as in the previous case considered in Subsection 2.1.

*Theorem 3.3:* Denote  $I = [0, b]$ , a real interval ( $b = +\infty$  is allowed). Let  $f: I \rightarrow I$  be a  $C^3(I)$  unimodal map satisfying that attains an absolute maximum at  $c$ ,  $f(0) = 0$ ,  $f'(0) < 1$ , has two positive equilibria  $K_1$  and  $K_2$  such that  $K_1 < K_2 < c$ , and (3.5) holds.

Then, there exist  $\gamma_1, \gamma_2 \in (1, \infty]$ ,  $\gamma_1 < \gamma_2$  such that for any  $\gamma \in [\gamma_1, \gamma_2)$  the map  $g_\gamma$  has exactly one asymptotically stable positive equilibrium.

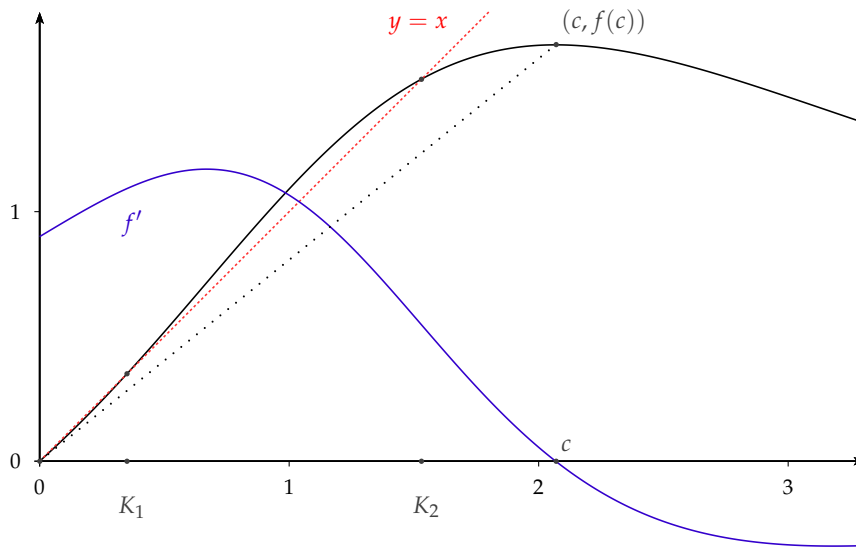


Figure 3.4: The map  $f$  without control has an Allee effect with Allee threshold  $K_1$  and carrying capacity  $K_2$ . In blue appears the derivative of  $f$  from which it is immediate that  $f''$  changes sign in  $(0, c)$ .

Moreover, either if  $\gamma = \gamma_1$  or if  $\gamma \in (\gamma_1, \gamma_2)$  and  $Sf(x) < 0$  for all  $x \in (c, b)$ , then the above equilibrium is globally asymptotically stable for the controlled system in the sense that all orbits starting in a positive initial condition converge to that equilibrium.

*Proof.* It follows the ideas in the proof of Theorem 3.2 so we omit it.  $\square$

Let us consider the map  $f$  given by (3.4) with  $A = 3$ ,  $B = 3$  and  $C = 10$ . As we have previously said, this uncontrolled map has an Allee effect with two positive equilibria. Increasing the parameter of control of the (PF) method, we are able to make disappear the *Allee threshold* and obtain a unique asymptotically stable positive equilibrium for  $\gamma \in (1.111, 3.333)$ , moreover if  $\gamma \approx 1.23945$  we can guarantee that the positive equilibrium is a superstable global attractor. In order to observe how the Allee effect disappears and the control prevents the extinction, in one of the plots of Figure 3.5 we have represented the bifurcation diagram with initial condition  $x_0 = 0.05$ , whereas in the other we have taken  $x_0 = 1$ . We see that in the case  $x_0 = 0.05$ , which appears at the left side, the population goes to extinction for

the controlled system until  $\gamma$  is big enough. However, in the case  $x_0 = 1$ , which appears at the right side, the population is attracted by the carrying capacity. In  $\gamma \approx 3.333$  a period doubling bifurcation occurs and a stable period two orbit appears, losing the positive equilibrium its stability. Since

$$Sf(x) = -6 \left( \frac{9x^4 + 40x^3 + 180x^2 + 360x + 100}{(x^4 + 6x^3 - 20x - 30)^2} \right),$$

the positive equilibrium is a global attractor for  $\gamma \in (1.23945, 3.333)$ .

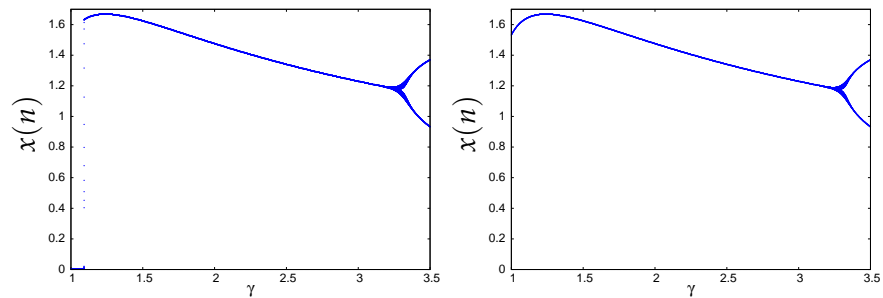


Figure 3.5: Bifurcation diagrams of the (PF) method applied to  $f$  given by (3.4) with  $A = 3$ ,  $B = 3$  and  $C = 10$ . The uncontrolled map has an Allee effect with two positive equilibria. Applying the control with  $\gamma$  greater than one a unique asymptotic equilibrium appears. In the first image we have represented the bifurcation diagram with initial condition  $x_0 = 0.05$ , whereas in the second we take  $x_0 = 1$ . Observe the different behaviours for  $\gamma$  near 1.

### 3.3 CONCLUSIONS

We have rigorously proved some properties of the (PF) method which generalize and complement the existing results in the literature. We have focused our attention in the stabilization of positive equilibria and our results cover both local and global behaviours.

Our first main goal has been to give a complete characterization of the maps (right hand sides of difference equations) for which the (PF) method succeeds in the local stabilization of a positive equilibrium without imposing the parameter  $\gamma$  to be smaller than 1. We remark the fact that we give a characterization and not just a sufficient condition for a class of maps. Moreover, the class of maps for which this

characterization is applicable is big enough to include all the typical models employed in ecology and many others.

Our second main goal has been to present sufficient conditions to ensure the global stability of the positive equilibrium in the sense that all orbits starting in a non zero initial condition converge to the equilibrium. We have proved that for a class of unimodal maps it is possible to show that the (PF) method stabilizes a unique positive equilibrium and that this equilibrium is a superstable global attractor for a specific value of the parameter without assuming constant sign conditions in the second derivative. We also show that it is possible to extend the global attraction of the equilibrium by adding a condition on the sign of the Schwarzian derivative of the map. Finally, we have showed how the (PF) method with  $\gamma > 1$  can be employed to stabilize towards a global attractor a map with Allee effect, preventing the extinction of the species.





# 4

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## CONTROL NO LINEAL

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### 4.1 INTRODUCCIÓN

Muchos de los controles que se aplican a un sistema discreto unidimensional mediante perturbaciones de la variable de estado vienen dados por una función de control de la forma

$$CL(x) = \gamma x + \beta, \quad \gamma, \beta \in \mathbb{R}. \quad (4.6)$$

Llevando a un sistema no controlado, dado por una función  $f$ , a un sistema controlado, dado por  $f \circ CL$  o bien  $CL \circ f$  (dependiendo de si el control se efectúa inmediatamente antes o después del momento en el que la función de producción actúa).

En este capítulo veremos que es posible obtener resultados de estabilización similares con una función de control no lineal. Además, exploraremos las ventajas e inconvenientes que tiene utilizar dicho control no lineal.

El nuevo control que proponemos en este capítulo comparte varias propiedades con un control lineal conocido como Target Oriented Control (TOC). TOC fue introducido por Dattani et al. (2011) y, posteriormente, ha sido considerado por varios autores (Franco and Liz, 2013; Braverman and Chan, 2014; Tung et al., 2014; Braverman and Franco, 2015). TOC, que se expresa mediante (4.6) haciendo  $\gamma = (1 - c)$  y  $\beta = T$ , utiliza dos parámetros de control: el parámetro  $T$  se emplea para establecer el tamaño objetivo para la variable de estado<sup>1</sup>, mientras que el otro parámetro se utiliza para la estabilización de la dinámica del sistema. Nuestro control también tendrá dos

<sup>1</sup> Utilizar la letra  $T$  hace referencia al término inglés *Target* para objetivo.

parámetros y, además, los roles de esos parámetros serán los mismos que acabamos de describir.

#### 4.2 PRESENTACIÓN DEL CONTROL

El control que vamos a presentar busca estabilizar la dinámica de sistemas discretos dados por funciones que verifiquen la siguiente condición:

(B1)  $f: [0, b] \rightarrow [0, b]$  (donde  $[0, b] = [0, \infty)$  en caso de  $b = \infty$ ) tiene derivada continua,  $f(0) = 0$ , y  $f(x) > 0$  para todo  $x \in (0, b)$ .

Es importante notar que la condición (A1) es muy general y la gran mayoría de funciones empleadas en dinámica de poblaciones la verifica. En particular permite considerar poblaciones con cualquier tipo de competencia intraespecífica (e.g., compensatoria o sobrecompensatoria).

La función de control no lineal que utilizaremos para reemplazar a la función de control lineal  $CL$  que aparece en (4.6) es la siguiente:

$$CN(x) := T \left( \frac{x}{T} \right)^{1-c}. \quad (4.7)$$

Como vemos, la función  $CN$  depende de dos parámetros. Al igual que para TOC, uno de los parámetros, que denotaremos por  $T$  y que deberá tomar valores en el intervalo  $(0, b)$  de condiciones iniciales<sup>2</sup>, se emplea para seleccionar el tamaño objetivo de la variable de estado; mientras que el otro, que denotaremos por  $c$  y que tomará valores en el intervalo  $[0, 1)$ , controla la intensidad del control.

Es conveniente que nos detengamos a comentar algunas de las propiedades de la función  $CN$  para comprender cómo afecta a la dinámica de las funciones controladas  $CN \circ f$  y  $f \circ CN$ .

- Claramente  $CN: [0, b] \rightarrow [0, b]$  es una función continuamente diferenciable, con  $CN(0) = 0$  y  $CN((0, b]) \subset (0, b]$ . Por lo tanto, si la condición (A1) se verifica para la función a controlar  $f$ ,

<sup>2</sup> Desde un punto de vista teórico podemos tomar como  $T$  cualquier punto del intervalo  $(0, b)$ , pero en aplicaciones reales parece natural seleccionarlo en el rango de  $f$ , esto es, en el intervalo  $(0, \sup_{x \in [0, b]} f(x))$ .

entonces también se verifica para la función controlada. Por lo tanto, el control respeta las condiciones esenciales de la función a controlar.

- Para  $x \neq 0$ , podemos reescribir  $T \left(\frac{x}{T}\right)^{1-c}$  como  $x \left(\frac{T}{x}\right)^c$ . Recordando que  $c \in [0, 1)$ , vemos claramente que al aplicar el control estamos aumentando el tamaño de la variable de estado cuando ésta está por debajo del tamaño que hemos fijado como objetivo,  $T$ . Mientras que estamos reduciendo esa variable de estado cuando su tamaño está por encima del valor dado por  $T$ . Esto también ocurría con TOC, por lo tanto en este aspecto TOC y CN solamente se diferencian en la forma en la que se aumenta y reduce la población.

- Puesto que

$$f(CN(T)) = f\left(T \left(\frac{T}{T}\right)^{1-c}\right) = f(T),$$

siempre que seleccionemos el tamaño objetivo como un punto fijo positivo de la función a controlar, tendremos que la función controlada hereda ese mismo punto fijo con independencia de la intensidad del control  $c$  que apliquemos. Esta propiedad, al igual que las siguientes, también las tiene TOC.

- Establecer  $c = 0$ , con independencia del valor del tamaño de estado objetivo  $T$ , deja al sistema inalterado. Esto es, apaga el control.
- Aplicar el control antes o después de la reproducción nos lleva a sistemas topológicamente equivalentes (dados por las funciones  $f \circ CN$  y  $CN \circ f$ ). Por lo tanto, este cambio en el momento de aplicación del control no tiene consecuencias relativas a la estabilidad o las cuencas de atracción. Sin embargo, sí puede tener implicaciones en otros aspectos como, por ejemplo, la visualización del efecto Hydra (Hilker and Liz, 2013).

Realicemos un ejemplo para ilustrar la acción del control sobre un sistema. Aplicaremos el control  $CN$  inmediatamente antes de la reproducción a una población que sigue la función de Ricker  $f(x) =$

$xe^{r(1-x)}$ . Como tamaño de población objetivo tomaremos  $T = 1$ , que es el valor del único punto fijo positivo que la función de Ricker tiene para todo valor de  $r$ . Así, el sistema controlado viene dado por

$$x_{n+1} = x_n^{1-c} e^{r(1-x_n^{1-c})}, \quad c \in [0, 1]. \quad (4.8)$$

Ahora busquemos las intensidades de control que debemos aplicar para que el equilibrio positivo  $x^* = 1$  sea asintóticamente estable. Linealizando, obtenemos que para que el equilibrio sea asintóticamente estable es suficiente que

$$|(1-r)(1-c)| < 1,$$

mientras que

$$|(1-r)(1-c)| > 1,$$

garantiza que el equilibrio es inestable.

En consecuencia, dependiendo del valor de  $r$  tenemos dos escenarios distintos. Primero, si  $r < 2$ , entonces podemos tomar cualquier intensidad de control  $c \in [0, 1)$  sin que CN altere la estabilidad del equilibrio del sistema de partida. Segundo, si  $r \geq 2$ , entonces debemos tomar

$$c > c_r := \frac{r-2}{r-1} \quad (4.9)$$

para garantizar que el sistema controlado mediante CN estabiliza al equilibrio inestable del sistema no controlado.

La Figura 4.6 muestra el diagrama de bifurcación para este ejemplo en el caso particular de seleccionar la función de Ricker con  $r = 3$ . Observamos que para  $c > 0,5$  el equilibrio positivo se estabiliza después de una cascada de bifurcaciones de *plegamiento* de período.

De este ejemplo deducimos dos cosas que trataremos de generalizar en la siguiente sección. La primera, que el control no desestabiliza un equilibrio que era estable para el sistema de partida. La segunda, que siempre es posible encontrar una intensidad de control a partir de la cual el sistema controlado estabiliza a un equilibrio.

Para terminar con el ejemplo, incluimos un diagrama de bifurcación que considera valores de  $c$  superiores a 1 en la Figura 4.7. Con esto tan solo buscamos mostrar que el comportamiento de nuestro

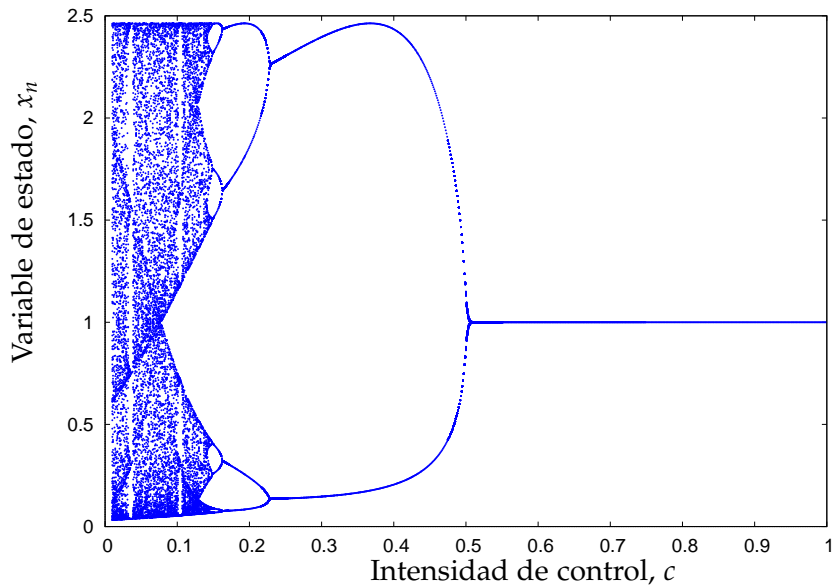


Figura 4.6: Diagrama de bifurcación para la función de Ricker  $r = 3$  tomando como tamaño objetivo para el control CN el valor  $T = 1$ . Para  $c > 0,5$  el equilibrio positivo se estabiliza después de varias bifurcaciones de plegamiento de período.

control es similar, aunque no idéntico, al descrito por Dattani et al. (2011) para TOC. Por ejemplo, para TOC se produce una bifurcación silla nodo que no se observa en CN; o para TOC puede darse la coexistencia de dos puntos fijos estables, algo que tampoco se observa para CN. Pero como hemos dicho también hay similitudes. Por ejemplo, en ambos casos para  $c$  suficientemente grande el control tiene un efecto desestabilizante sobre un equilibrio estable que parece conducir a caos siguiendo una ruta de bifurcaciones de doblamiento de período; o aparecen puntos fijos adicionales a  $x^* = 1$  y el punto fijo  $x^* = 1$  pierde su estabilidad en una bifurcación transcítica. Desde luego, analizar la dinámica para  $c \notin (0, 1)$  parece un problema interesante desde el punto de vista teórico. Pero puesto que ni siquiera se ha realizado para TOC y la utilidad práctica no es clara, preferimos no acometer ese estudio en esta memoria.

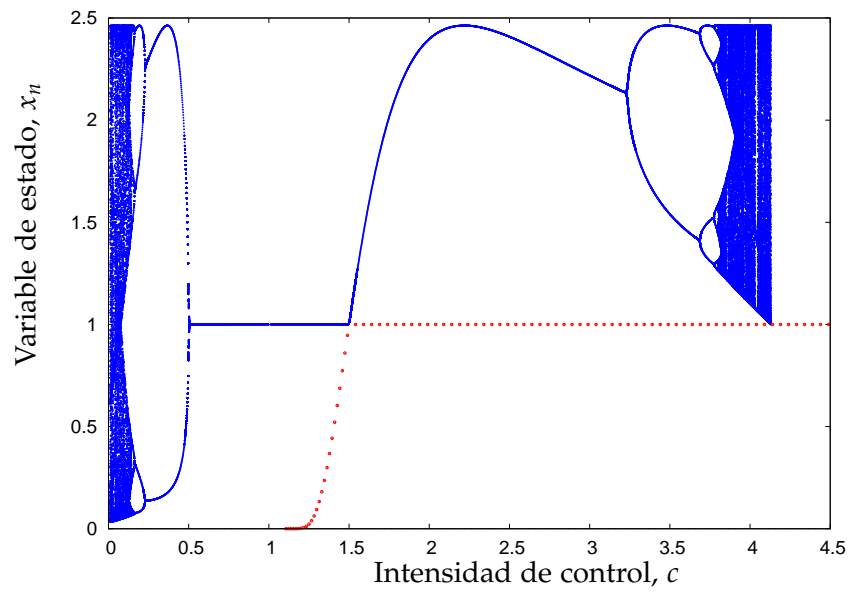


Figura 4.7: Diagrama de bifurcación permitiendo a la intensidad de control  $c$  tomar valores superiores a 1. Como en la figura anterior, tomamos la función de Ricker con  $r = 3$  y  $T = 1$ . Nótese que CN estabiliza un equilibrio positivo también para algunos valores de  $c$  superiores a 1, pero aparece un equilibrio positivo inestable (en color rojo).

## 4.3 ESTABILIZACIÓN LOCAL DE UN EQUILIBRIO POSITIVO

En esta sección mostraremos que la estabilización de un equilibrio positivo al aplicar el control dado por la función  $CN$ , que acabamos de observar para el caso particular de la función de Ricker, es una propiedad general.

*Proposición 4.1:* Supongamos que se verifica la condición (B1) y fijemos  $T \in (0, b]$ . Entonces, existe  $c^* \in [0, 1)$  tal que el sistema controlado definido por la función  $f(CN)$  tiene al menos un equilibrio positivo  $P_c \in (0, b]$  para todo  $c \in (c^*, 1)$ .

Consideremos los puntos fijos del sistema controlado mayores que una constante  $R > 0$ , si  $f'$  está acotada y la intensidad de control  $c$  es suficientemente grande, entonces todos estos puntos fijos son asintóticamente estables.

*Demostración.-* Comenzaremos por la primera afirmación del resultado. Consideraremos tres casos dependiendo del valor de  $f(T) \in [0, b]$  y mostraremos que en todos ellos es posible garantizar la existencia de al menos un equilibrio positivo para el sistema controlado.

*Caso 1:*  $f(T) = b$ .

Nótese que en este caso la condición (B1) implica  $T < b < \infty$ . Además, también por (B1) se tiene que

$$f(CN(b)) \leq b, \quad (4.10)$$

y

$$f(CN(T)) = f(T) = b > T$$

para cualquier  $c \in (0, 1)$ .

En consecuencia, aplicando el Teorema de Bolzano obtenemos la existencia de un equilibrio positivo  $P_c \in (T, b]$  para cualquier  $c \in (0, 1)$ .

*Caso 2:*  $f(T) = 0$ .

En este caso la condición (B1) fuerza a que  $T = b < \infty$ .

Además, podemos seleccionar  $d \in (0, b)$  tal que

$$f(d) = \max_{x \in [0, b]} f(x).$$

Sea  $CN^{-1}: I \rightarrow I$  la función inversa de la función  $CN$ , que está bien definida y verifica

$$\lim_{c \rightarrow 1} CN^{-1}(d) = \lim_{c \rightarrow 1} T \left( \frac{d}{T} \right)^{\frac{1}{1-c}} = 0.$$

Por lo tanto, debe existir  $c^* \in (0, 1)$  tal que

$$CN^{-1}(d) < f(d)$$

para cualquier  $c \in (c^*, 1)$ . Y obtenemos que

$$f(CN(CN^{-1}(d))) = f(d) > CN^{-1}(d), \quad c \in (c^*, 1).$$

Por otro lado, tenemos que en este caso, al igual que en el anterior, se verifica la desigualdad (4.10). Luego, como consecuencia del Teorema de Bolzano, debe existir al menos un equilibrio positivo  $P_c \in (0, b]$  para todo  $c \in (c^*, 1)$ .

*Caso 3:*  $f(T) \in (0, b)$ .

En este último caso consideramos el intervalo  $[\hat{x}, \tilde{x}]$ , donde

$$\hat{x} = \frac{f(T)}{2} \quad \text{y} \quad \tilde{x} = \min\{2f(T), b\}.$$

Claramente, se tiene que  $\hat{x} < f(T) < \tilde{x}$ . Y de la continuidad de  $f$  se obtiene

$$\lim_{c \rightarrow 1} f(CN(\hat{x})) = f(T) = \lim_{c \rightarrow 1} f(CN(\tilde{x})).$$

Por lo tanto existe  $\hat{c} \in (0, 1)$  tal que

$$\hat{x} < f(CN(\hat{x})) \quad \text{y} \quad \tilde{x} > f(CN(\tilde{x})).$$

para cualquier  $c \in (\hat{c}, 1)$ .

Las desigualdades anteriores y el Teorema de Bolzano terminan la prueba de este caso y de la primera afirmación.

Finalmente, la afirmación sobre la estabilidad asintótica de cualquier punto fijo positivo mayor que una constante  $R > 0$  es una consecuencia de la siguiente desigualdad:

$$\begin{aligned} \left| \frac{d}{dx}(f \circ CN)(P) \right| &= \left| f'(CN(P))(1-c) \left( \frac{T}{P} \right)^c \right| \\ &\leq (1-c) \frac{T^c}{R^c} \max_{x \in [R, b]} |f'(x)|. \end{aligned}$$

□



Es interesante notar que las ideas en la demostración de la proposición anterior se pueden emplear para probar un resultado similar para TOC. Ese resultado garantiza la existencia de una intensidad de control  $c^* \in [0, 1)$  tal que el sistema controlado tiene al menos un equilibrio positivo para todo  $c \in (c^*, 1)$ . Por lo tanto, complementa al Lema 1 en (Franco and Liz, 2013) ya que no se asume la siguiente condición considerada allí para garantizar la existencia para todo  $c \in (0, 1)$ :

(C) Existe una constante positiva  $M$  tal que  $f(x) \leq x$  para todo  $x \geq M$ .

El siguiente resultado muestra que añadir la condición (C), también para el control CN, garantiza la existencia de un equilibrio positivo para cualquier intensidad de control.

*Proposición 4.2:* Si en la Proposición 4.1 suponemos que  $b = +\infty$ , la condición (C) se verifica y  $f'(0^+) > 0$ , entonces el sistema controlado tiene al menos un equilibrio positivo  $P_c$  para cualquier intensidad de control  $c \in (0, 1)$ .

*Demostración.*- Claramente,  $CN'(0^+) = +\infty$  para cualquier intensidad de control  $c \in (0, 1)$ . Utilizando la Regla de la cadena, tenemos que la función controlada verifica

$$(f \circ CN)'(0^+) = +\infty.$$

Por lo tanto, existirá un punto  $m$  cerca de cero tal que  $(f \circ CN)(m) > m$ . Ahora, empleando la condición (C) y el Teorema de Bolzano obtenemos el resultado.  $\square$

Muchas de las funciones de producción utilizadas en dinámica de poblaciones son unimodales y tienen solamente dos puntos fijos no negativos. Esencialmente, verifican (B1) y las siguientes dos condiciones:

(B2)  $f$  tiene solamente dos puntos fijos no negativos  $x = 0$  y  $x = K > 0$ ,  $f(x) > x$  para  $0 < x < K$ , y  $f(x) < x$  para  $x > K$ .

(B3)  $f$  tiene un único punto crítico  $d < K$  de tal forma que  $f'(x) > 0$  para todo  $x \in (0, d)$ ,  $f'(x) < 0$  para todo  $x > d$ , y  $f''(x) < 0$  en  $(0, d)$ .

En el siguiente resultado, mostraremos que si la función a controlar verifica las condiciones anteriores junto con la condición (B1), entonces la función controlada también verifica estas condiciones para cualquier intensidad de control  $c \in (0, 1)$ .

*Proposición 4.3:* Sea  $f$  una función verificando las condiciones (B1)-(B3) con  $b = +\infty$  y para la que  $f'(0^+)$  existe. Entonces, la función  $f \circ CN$  también verifica (B1)-(B3).

Además, si  $T = K$ , entonces el equilibrio positivo  $K$  de  $f \circ CN$  es asintóticamente estable si la intensidad del control verifica

$$c \in \left( \max\left\{ \frac{|f'(K)| - 1}{|f'(K)|}, 0 \right\}, 1 \right).$$

*Demostración.*- En primer lugar notemos que (B2) y (B3) implican que  $f'(0^+) > 1$ . Por lo tanto, gracias a la Proposición 4.2, solamente necesitamos mostrar la unicidad del punto fijo positivo.

De  $CN'(x) > 0$  para  $x \in (0, b)$  y la condición (B3) obtenemos

$$(f \circ CN)'(x) > 0, \quad x \in (0, d^*),$$

y

$$(f \circ CN)'(x) < 0, \quad x \in (d^*, b),$$

con  $d^* = CN^{-1}(d)$  el único punto crítico de  $f \circ CN$ .

Además, sabemos que

$$(f \circ CN)'(0^+) = +\infty.$$

Por otro lado, utilizando que  $f''(x) < 0$  para  $x \in (0, d)$  y que  $(f \circ CN)''(x) < 0$  para  $x \in (0, b)$  resulta inmediato mostrar que

$$(f \circ CN)''(x) < 0, \quad x \in (0, d^*).$$

Estas propiedades implican que  $f \circ CN$  no puede tener más que un punto fijo positivo. Efectivamente, si no fuese así la función  $f \circ CN - id$  tendría que alcanzar un mínimo relativo. Y, consecuentemente, existiría un punto  $x^*$  tal que  $(f \circ CN)'(x^*) = 1$  y  $(f \circ CN)''(x^*) \geq 0$ . Lo cual no es posible.

Si  $T = K$ , sabemos que el equilibrio positivo  $K$  de  $f$  también lo es de  $f \circ CN$  para cualquier intensidad de control.

Linealizando obtenemos que

$$|(f \circ CN)'(K)| = \left| f'(CN(K))(1-c) \left( \frac{K}{K} \right)^c \right| = (1-c) |f'(K)|.$$

Luego para que  $|(f \circ CN)'(K)| < 1$  debe ocurrir que

$$c > \frac{|f'(K)| - 1}{|f'(K)|},$$

y como  $c$  debe ser mayor que cero obtenemos la expresión que aparece en el enunciado.  $\square$

#### 4.4 ESTABILIDAD GLOBAL EN UN CASO PARTICULAR

En esta sección investigaremos si la estabilidad local producida por el control CN es también global. Un estudio general como el realizado para la estabilidad local resulta complicado. Por ese motivo, nos centraremos en un caso particular. Concretamente, tomaremos que el sistema no controlado viene dado por la función de Ricker

$$f(x) = x \exp(r(1-x)), \quad r > 0,$$

y que la población objetivo coincide con el punto fijo de la función de Ricker, esto es,  $T = 1$ . Debemos notar que el modelo de Ricker se utiliza para modelar la dinámica de muchas especies diferentes (Sah and Dey, 2014) y por lo tanto el caso particular que hemos escogido es de especial interés en biología matemática.

Muchas funciones de producción, incluida la función de Ricker anterior (Thunberg, 2001), verifican además de las condiciones (B1)-(B3) que su derivada Schwarziana es negativa. Esta última propiedad fue utilizada por Singer (1978) para demostrar que la estabilidad local implica la estabilidad global para estos sistemas. Como ya hemos comentado anteriormente, la linealidad de muchos controles es crucial demostrar resultados de estabilidad global basados en el signo de la derivada Schwarziana porque las funciones controladas heredan el signo de la función a controlar. Sin embargo, al aplicar el control no lineal que hemos presentado aquí la derivada Schwarziana de la función que define al sistema controlado puede cambiar de signo aunque la derivada Schwarziana ligada al sistema no controlado tenga signo

constante. Esto puede observarse en las Figuras 4.8 y 4.9 donde representamos las derivadas Schwarzianas para una función de Ricker con parámetro  $r = 3$  e intensidades de control  $c = 0,5$  y  $c = 0,6$  respectivamente.

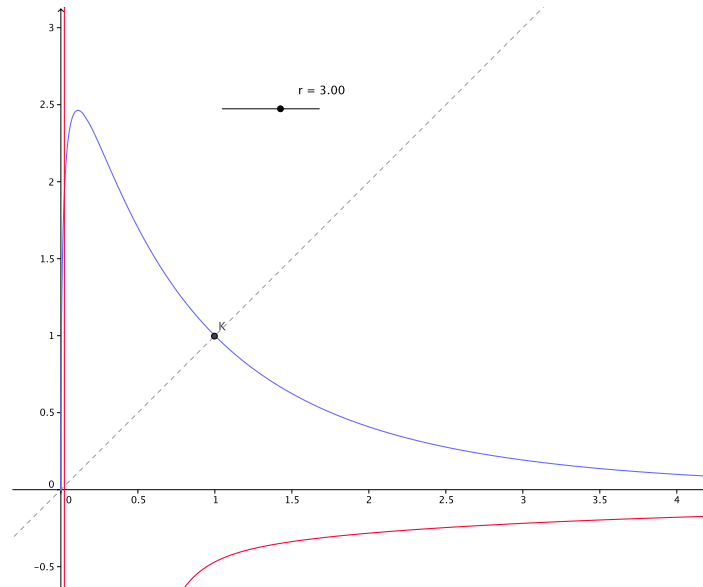


Figura 4.8: Derivada Schwarziana cambiando de signo cerca del origen. Representamos las gráficas de la función de Ricker controlada  $x^{1-c}e^{r(1-x^{1-c})}$  con  $r = 3$  (curva de color azul) y su derivada Schwarziana (color rojo). La intensidad de control es  $c = 0,5$ .

Nótese que al seleccionar  $c = 0,5$  estamos tomando la intensidad de control límite a partir de la cual la Proposición 4.3 garantiza que se produce la estabilidad asintótica del equilibrio positivo  $K = 1$ .

En las Figuras 4.8 y 4.9 podemos observar que la derivada Schwarziana parece ser únicamente positiva cerca del origen tanto para  $c = 0,5$  como para  $c = 0,6$ . Esto sería deseable ya que se sabe que lo realmente importante para funciones unimodales es tener derivada Schwarziana negativa desde el punto donde la función alcanza el máximo (El-Morshedy and López, 2008, Corollary 2.9)<sup>3</sup>.

Puesto que la Proposición 4.3 garantiza que el control CN respeta las condiciones (B1)-(B3), podríamos utilizar directamente el resulta-

<sup>3</sup> Recordamos al lector que este resultado aparece recogido en el capítulo anterior como Lema 3.1.

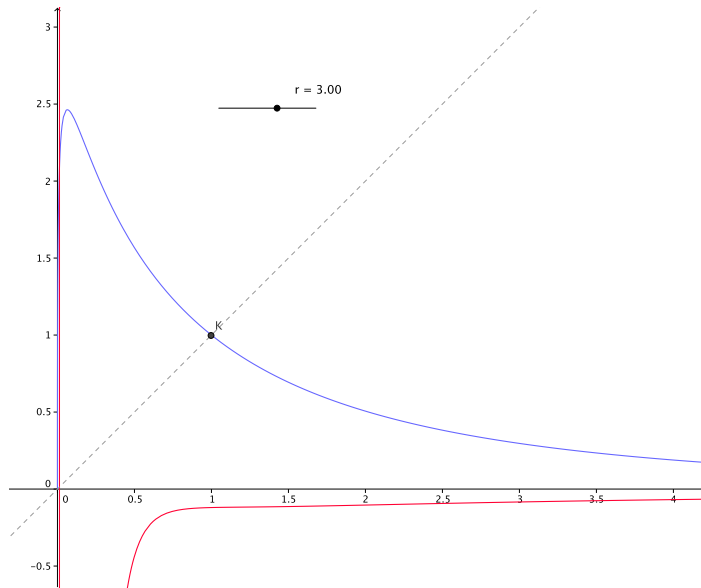


Figura 4.9: Derivada Schwarziana cambiando de signo cerca del origen. Representamos las gráficas de la función de Ricker controlada  $x^{1-c}e^{r(1-x^{1-c})}$  con  $r = 3$  (curva de color azul) y su derivada Schwarziana (color rojo). La intensidad de control es  $c = 0,6$ .

do de (El-Morshedy and López, 2008) para obtener que la estabilidad local implica la local para el control no lineal.

Sin embargo, la derivada Schwarziana de la función controlada no siempre mantiene el signo negativo a la derecha del punto en el que se alcanza el máximo de la función controlada. Esto se puede deber a dos motivos. Bien una intensidad de control alta o bien un valor demasiado grande del parámetro  $r$  en la función de Ricker.

La Figura 4.10 se corresponde con la primera situación. En ella tomamos la función de Ricker con  $r = 3$  (para la que ya observamos previamente que las intensidades de control  $c = 0,5$  y  $c = 0,6$  no daban problemas en las Figuras 4.8 y 4.9) y aumentamos la intensidad del control hasta  $c = 0,7$ . Este aumento de la intensidad de control no afecta a la estabilidad local del equilibrio. Pese a ello, sí afecta al comportamiento del signo de la derivada Schwarziana. Así, observamos que la derivada Schwarziana posee un máximo relativo positivo que se alcanza a la derecha del punto en el que la función controlada alcanza su máximo. Impidiendo, por lo tanto, la utilización del Lema 3.1 para garantizar la estabilidad global del equilibrio positivo.

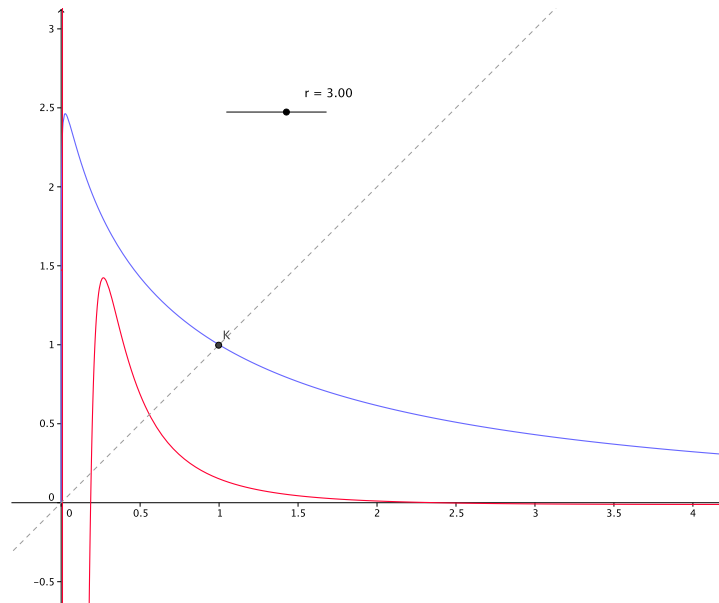


Figura 4.10: La derivada Schwarziana (curva de color rojo) de la función controlada (curva de color azul) no mantiene el signo negativo a la derecha del punto en donde se alcanza el máximo de la función debido a que la intensidad de control es demasiado grande. Como en las figuras anteriores tomamos la función de Ricker con  $r = 3$ , pero ahora la intensidad de control es mayor, concretamente  $c = 0,7$ .

La Figura 4.11 ilustra la segunda situación. En ella tomamos la función de Ricker con  $r = 3,8$  y la intensidad límite  $c_{3,8} = \frac{2-3,8}{1-3,8}$  a partir de la cual el equilibrio positivo es asintóticamente estable. Es decir, estamos considerando el valor más bajo de la intensidad de control capaz de estabilizar el equilibrio positivo. Para el caso  $r = 3$ , vimos que valores de  $c$  cercanos a esta valor límite mantenían el signo negativo de la derivada Schwarziana. Sin embargo, al aumentar el valor de  $r$ , observamos también que la derivada Schwarziana posee un máximo relativo positivo que se alcanza a la derecha del punto en el que la función controlada alcanza su máximo.

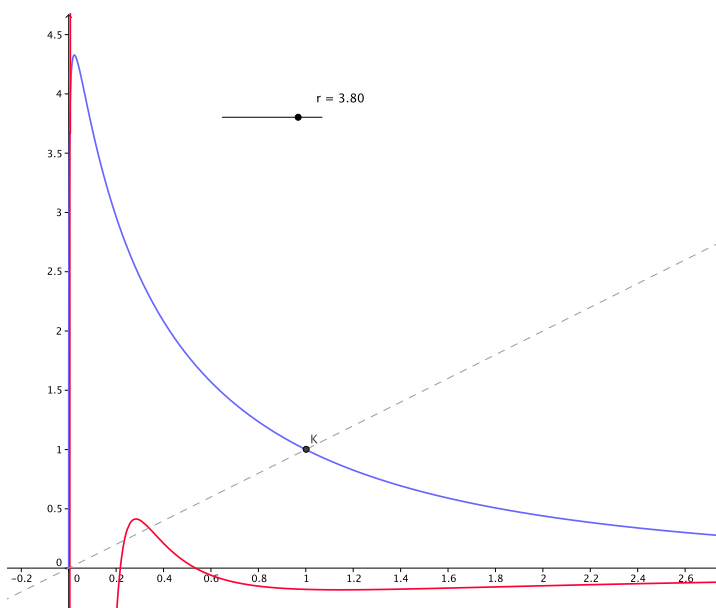


Figura 4.11: La derivada Schwarziana de la función controlada no mantiene el signo negativo a la derecha del punto en donde se alcanza el máximo de la función debido a que el parámetro  $r$  es demasiado grande. Función de Ricker con  $r = 3,8$  e intensidad de control límite que garantiza la estabilidad local del equilibrio positivo,  $c_{3,8} \approx 0,642$ .

Los casos considerados hasta ahora parecen indicar que el parámetro  $r$  de la función de Ricker juega un papel en que se mantenga el signo negativo de la derivada Schwarziana a la derecha del máximo de la función controlada para intensidades cercanas a  $c_r$ . Numericamente hemos visto que es así y hemos calculado el valor crítico del parámetro  $r$  hasta el que la derivada Schwarziana mantiene signo

constante negativo a la derecha del máximo de la función controlada cuando la intensidad del control es  $c_r$ . Este valor es  $r_* \approx 3,69$ . En consecuencia, parece factible utilizar el resultado de (El-Morshedy and López, 2008) para demostrar la estabilidad global del equilibrio positivo si  $2 < r < 3,69$ . Pero solamente para valores de  $c$  cercanos a  $c_r$  y no en general.

La estimación anterior puede ser mejorada en dos direcciones diferentes. Por un lado, consiguiendo un intervalo mayor del parámetro  $r$  para el cual aplicar el control con intensidad  $c_r$  implique la estabilidad global del sistema controlado. Por otro lado, garantizando que incrementar la intensidad de control a partir de ese valor  $c_r$  no afecta a la estabilidad global del sistema controlado. Para conseguirlo utilizaremos una técnica presentada por Paul Cull; véanse (Cull, 2007; Cull et al., 2005) y las referencias citadas en allí.

Paul Cull reduce el estudio de la estabilidad global de un sistema discreto a encontrar otro sistema con estabilidad global que lo *envuelva*. Así, se dice que la función  $\phi: (0, b) \rightarrow [0, b)$  envuelve a la función  $f$  (que suponemos verifica (B1)-(B3)) si y solamente si

- $\phi(x) > f(x)$  para  $x \in (0, K)$ ;
- $\phi(x) < f(x)$  para  $x > K$  tal que  $\phi(x) > 0$  y  $f(x) > x$ .

El siguiente resultado establece que si somos capaces de encontrar una función envolviendo a una función controlada para cierta intensidad de control, entonces también es posible para cualquier intensidad de control mayor.

*Proposición 4.4:* Supongamos que las condiciones (B1)-(B3) se verifican y que una función decreciente  $\phi$  envuelve a la función controlada para cierta intensidad de control  $c^*$ , entonces  $\phi$  envuelve a la función controlada para cualquier intensidad de control  $c \in [c^*, 1)$ .

*Demostración.-* Gráficamente, el resultado se deduce de que la gráfica de  $f(x^{1-c})$  se construye a partir de la gráfica  $f$  estirándola horizontalmente hacia 0 e infinito, dejando a  $K = 1$  fijo.



Analíticamente, tenemos que para cada  $x \in (0, 1)$  y  $c \in (c^*, 1)$  existe un único  $y \in (1, x)$  tal que  $y^{1-c^*} = x^{1-c}$ . Dado que  $\phi$  es decreciente, obtenemos

$$f(x^{1-c}) = f(y^{1-c^*}) > \max\{\phi(y), 0\} \geq \max\{\phi(x), 0\}.$$

Y de forma similar obtendríamos que  $f(x^{1-c}) < \phi(x)$  para  $x \in (0, 1)$ .  $\square$

Para envolver a la función de Ricker controlada que estamos estudiando, consideraremos la familia de transformaciones fraccionarias lineales

$$\phi(x, \alpha) := \frac{1 - \alpha x}{\alpha - (2\alpha - 1)x}, \quad \alpha \in [0, 1).$$

Esta familia envuelve a muchas funciones de producción utilizadas en dinámica de poblaciones. Un hecho que ha sido utilizado para demostrar que la estabilidad local garantiza la estabilidad global en esos casos (Cull, 2007; Cull et al., 2005). A continuación listamos algunas de las propiedades que verifican todas las funciones de la familia anterior:

- $\phi(1) = 1$  y  $\phi'(1) = -1$ ,
- $\phi(1/\alpha) = 0$ ,
- $\phi(x) > 0$  y  $\phi'(x) < 0$  para  $x \in (0, \frac{1}{\alpha})$ .

La expansión de Taylor centrada  $x = 1$  de la diferencia entre la función controlada, con intensidad de control  $c_r = \frac{2-r}{1-r}$ , y  $\phi$  es:

$$\frac{(2r^2 - 3r + 2)\gamma - r}{(2r^2 - 4r + 2)\gamma - 2r^2 + 4r - 2} (x - 1)^2 + \dots$$

Por lo tanto, el único valor de  $\gamma$  para el cual  $\phi$  podría envolver a  $x^{1-c_r} e^{r(1-x^{1-c_r})}$  es

$$\gamma = \frac{r}{2r^2 - 3r + 2}.$$

Numéricamente, hemos observado que  $\phi$  envuelve a la función controlada para el valor crítico de la intensidad de control  $c_r$  si  $r \in (2, r^*)$  con  $r^* \approx 4,02$ . La Figura 4.12 se corresponde con uno de estos casos y podemos apreciar como la transformación fraccionaria lineal envuelve a la función controlada.

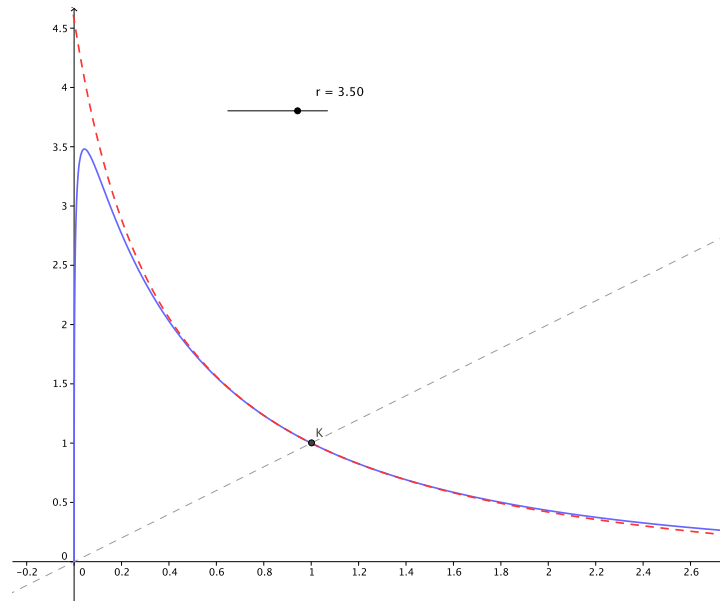


Figura 4.12: Gráficas de la función controlada (curva de color azul) y la transformación fraccionaria lineal (curva de color rojo) que la envuelve. Hemos tomado  $r = 3,5$  y la intensidad de control límite  $c_{3,5}$ .

Por lo tanto, los cálculos numéricos permiten conjeturar que para la función de Ricker controlada (4.8) el equilibrio positivo  $K = 1$  es globalmente estable siempre que lo sea localmente estable si  $r$  es menor que  $r^*$ .

Para  $r$  mayores que  $r^*$  no existe una transformación fraccionaria lineal  $\phi$  capaz de envolver a la función controlada tal y como ilustramos en la Figura 4.13. En consecuencia, conjeturar si el equilibrio positivo atrae a todas las órbitas no triviales resulta mucho más complicado.

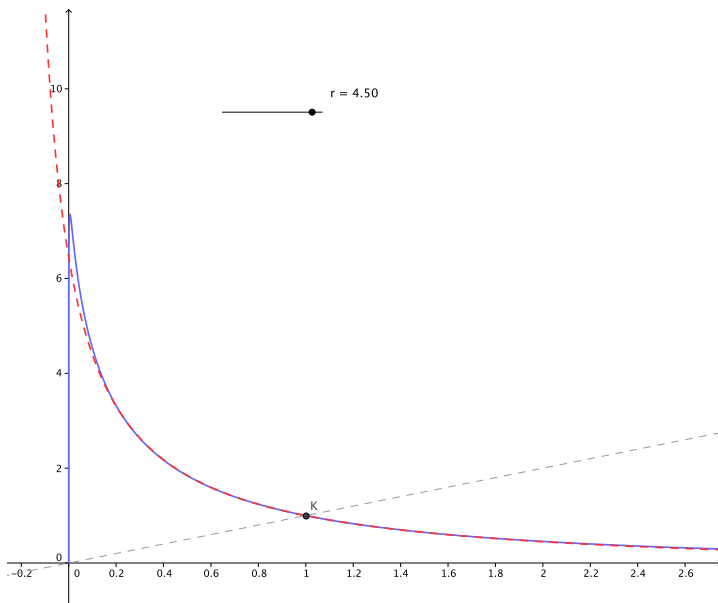


Figura 4.13: Gráficas de la función controlada (curva color azul) y la única transformación fraccionaria lineal (curva color rojo) que podría envolverla —pero que no lo hace. Hemos tomado  $r = 4,5$  y la intensidad de control límite  $c_{4,5}$ .



Parte II

SISTEMAS BIDIMENSIONALES



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## MIGRACIÓN Y CONTROL

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### 5.1 INTRODUCTION

The increasing fragmentation of the natural environment causes more and more species to inhabit heterogeneous regions. These regions are often connected, that is, individuals can migrate from one region to another. Understanding the role of these connections in population dynamics is an important problem in ecology (Nathan, 2001). In addition to a purely scientific interest of this problem, there are also practical implications. For instance, facilitating/hindering the movement of individuals across different regions has been used as an environmental control tool (e.g. ecological corridors, dispersal barriers (Earn et al., 2000; Goldberg and Lande, 2007; Tillmann, 2005)). Hence, to design and manage these strategies, it is necessary to understand how the modification of the dispersal rate affects population dynamics. The main aim of this chapter is to contribute to this understanding by numerically exploring the effect of dispersal on the mean total population size and its fluctuation range.

Several mathematical models have been proposed to capture the movement of individuals in a spatially-separated population. One of the simplest among them assumes that the dynamics in each sub-region, when it is isolated, can be described by a discrete population map; then, dispersal is incorporated by means of a nonnegative square matrix  $C$  in which each element  $c_{ij}$  represents the percentage of population moving from patch  $j$  to patch  $i$ —see the next section for more details. Despite its formal simplicity, this deterministic model has shown an exceptional versatility to explain experiments about the

effects of dispersal on stability, synchrony, or extinction risk (Dey and Joshi, 2006; Gonzalez et al., 1998). Nevertheless, we should stress that the movement of individuals highly matters in aspects not captured by this model as evolution (Parvinen, 2014) or the spread of diseases (Juher and Mañosa, 2014; Saldaña, 2010). For simplicity, many authors have focused on the case of just two single-species subpopulations linked by dispersal—also known as a two-coupled map lattice (Ben Zion et al., 2010; Dey et al., 2014, 2015; Doebeli, 1995; Franco and Ruiz-Herrera, 2015; Gonzalez et al., 1998; Gyllenberg et al., 1993; Hastings, 1993; Kendall and Fox, 1998; Lloyd, 1995). In this case, connecting the two subpopulations can be done by unidirectional (one subpopulation acts as the donor whereas the other is the recipient) or bidirectional dispersal (Dey et al., 2014). Here, we consider a two-coupled map lattice with spatial heterogeneity; i.e. local dynamics in each subregion are different.

Dispersal can largely influence population dynamics in the model we are considering. Linking subpopulations with *complicated* or even *chaotic* local dynamics (Thunberg, 2001) can create an overall stable equilibrium or cycle (Dey et al., 2014; Gyllenberg et al., 1993). Besides, increasing dispersal enhances synchrony (Earn et al., 2000), which is considered a dangerous effect for the species because having low population density in both patches simultaneously puts at risk the viability of the whole population. Finally, connecting heterogeneous subregions can create a “rescue effect” (Franco and Ruiz-Herrera, 2015; Hastings and Botsford, 2006; Gotelli, 1991): a subregion where the population would not survive if isolated is permanently inhabited with the help of another subregion. Most of these properties have been shown for specific cases with symmetric dispersal, but the potential interactions between local dynamics and asymmetric dispersal remain almost unexplored (Dey et al., 2014).

To the best of our knowledge, there are few theoretical studies of the effects of dispersal on the total population size using two-coupled map lattices. In (Doebeli, 1995), a population with complicated local dynamics in both subregions was considered. There, the author observed that at an equilibrium stabilised by dispersal, the total pop-



ulation size was much larger than the sum of the unstable equilibrium sizes in each isolated subpopulation. On the other hand, Ives et al. (2004) showed that three factors increase the total population size: weak density dependence, high environmental variability affecting population growth rates, and the lack of synchrony among the fluctuations in populations. But this was done for a stochastic system. Considering a deterministic model, it was shown in (Franco and Ruiz-Herrera, 2015) that the total population size generally has a unimodal response to the increase of dispersal. At low rates of dispersal the total population size increases, whereas at higher rates of dispersal the total population size decreases. However, this was only shown for a population with simple local dynamics in both subregions, i.e. assuming that each subpopulation has an equilibrium that is a global attractor.

In this chapter, we extend and complete the previous results about the effect of dispersal on the total population size by considering situations in which one of the subpopulations has an attracting equilibrium and the other complicated dynamics. Capturing how the total population size responds to dispersal is hard, mainly because measuring such a size when facing complicated dynamics is difficult. Indeed, for an isolated region with Ricker dynamics (Ricker, 1954), it can be proved that the asymptotic mean of the population size converges to the unique equilibrium of the map (even if this equilibrium is unstable) (Gyllenberg et al., 1996), but this is not true for linked regions or other maps. Hence, here the asymptotic mean total population size is numerically approximated. Our results show that the response of the mean total population to dispersal varies not only with the type of connection, but also with the intrinsic dynamics in each subpopulation, making it difficult to give a general recipe for the use of dispersal to manage the size of a spatially-separated population. Nevertheless, in one case (stable recipient and donor with complicated dynamics), increasing dispersal rates seems to increase the mean total population size in all considered configurations.

For metapopulations with complicated dynamics, modifying dispersal rates was proposed as a method for controlling chaos (Doebeli,

1995). Controlling chaos aims to stabilise the population dynamics by creating a stable equilibrium or a limit cycle (Capeáns et al., 2014; Solé et al., 1999; Tung et al., 2014)). This was the main goal of (Doebeli, 1995). Interestingly, the recent paper (Dey et al., 2014) shows that the local dynamics and the type of connection play an important role in the effect of dispersal on such a stabilization. But the stability of a population can be related to three main properties (Grimm and Wissel, 1997): “staying essentially unchanged” (constancy), “returning to the reference state after a temporary disturbance” (resilience), and “persistence through time” (persistence). Another objective of this chapter is to measure the effect of dispersal on constancy and persistence, thus complementing the results in (Dey et al., 2014). We do it by numerically calculating how the fluctuation range responds to changes of dispersal rates in different scenarios.

Finally, we address the question of whether or not multistability affects the behaviour of the mean total population size and its fluctuation range when both subpopulations have complicated local dynamics. It is well known that the dynamics of a two-patch system may depend on initial conditions (even when the dynamics of the subpopulations have global attractors) (Hastings, 1993; Lloyd, 1995). We present two examples in which the response of total population size and its fluctuation range depends on the initial conditions. This suggests that it might be difficult to predict the outcome of control strategies only based on changes of the dispersal rates.

## 5.2 MODEL DESCRIPTION

Our model considers two connected patches ( $A$  and  $B$ ) with discrete dynamics. The post-breeding population size in each patch is determined by a map denoted by  $f_A(N)$  and  $f_B(N)$ , respectively. Dispersal occurs after the breeding season according to the dispersal matrix

$$C = \begin{pmatrix} 1 - m_A & m_B \\ m_A & 1 - m_B \end{pmatrix}.$$

This means that a density-independent constant fraction  $m_A \in [0, 1]$  of the individuals in subpopulation  $A$  moves to subpopulation  $B$  and a density-independent constant fraction  $m_B \in [0, 1]$  of the individuals in subpopulation  $B$  moves to subpopulation  $A$ . Thus, population dynamics are described by system

$$\begin{cases} N_A(t+1) = (1 - m_A)f_A(N_A(t)) + m_B f_B(N_B(t)), \\ N_B(t+1) = m_A f_A(N_A(t)) + (1 - m_B)f_B(N_B(t)), \end{cases} \quad (5.1)$$

where  $N_i(t)$  denotes the population size in patch  $i$  after dispersal at the beginning of generation  $t$ .

Without loss of generality, we assume that the dispersal rates satisfy  $m_A \geq m_B$ . Clearly, if  $m_A = m_B = 0$  both subpopulations are isolated. On the other hand, there are two different scenarios of dispersal: unidirectional dispersal ( $m_A > m_B = 0$ ), in this situation region  $A$  takes the role of donor and region  $B$  of recipient; and bidirectional dispersal ( $m_A \geq m_B \neq 0$ ) which can be symmetric ( $m_A = m_B > 0$ ) or asymmetric ( $m_A > m_B > 0$ ). In this work, we restrict the dispersal rates  $m_A$  and  $m_B$  to the interval  $[0, 0.5]$ . Higher dispersal rates might be categorized as less realistic since individuals in corridors evolve much lower dispersal rates than those in mainland population (Travis and Dytham, 1999).

As in (Dey et al., 2014), we consider system (5.1) for Ricker population maps (Ricker, 1954)

$$f_A(N) = N \exp(r_A(1 - N/25))$$

and

$$f_B(N) = N \exp(r_B(1 - N/25)).$$

We recall that the dynamics of such Ricker maps depend only on the parameter  $r_i$ . There is a global positive equilibrium for  $0 < r_i \leq 2$ , and a cascade of period doubling bifurcations leads to complicated dynamics as  $r_i$  increases from 2.

## 5.3 RESULTS AND DISCUSSION

## 5.3.1 Mean total population size and its fluctuation range

We perform various numerical experiments to capture the behaviour of the mean total population size and its fluctuation range. In all of them, one of the subpopulations has an attracting equilibrium and the other has *complicated* dynamics (Thunberg, 2001). We select  $r_i \in \{1, 1.5, 2\}$  for subpopulations with an attracting equilibrium. Whereas,  $r_i \in \{3.65, 3.125, 2.825\}$  is selected for patches with complicated dynamics. This choice of the parameters is primarily motivated by (Dey et al., 2014). There, Dey et al. report different responses in the reduction of the complexity of the population dynamics for subpopulations with  $r_A \in (3.3, 4)$ ,  $r_A \in (2.95, 3.3)$ , and  $r_A \in (2.7, 2.95)$  when connected to a subpopulation with an attracting equilibrium (Dey et al., 2014). Note that the set  $\{3.65, 3.125, 2.825\}$  is formed by the midpoints of the previous intervals.

The mean total population size is approximated using 100 generations after running the system for 3000 generations to remove transient effects, that is by

$$\sum_{t=3001}^{3100} \frac{N_A(t) + N_B(t)}{100},$$

with the initial condition chosen pseudo-randomly in the interval  $[0, 50]$  for both subpopulations.

The fluctuation range is approximated using 100 generations after running the system for 3000 generations by

$$\max\{|N_A(t) + N_B(t) - N_A(s) - N_B(s)| : t, s = 3001, \dots, 3100\}.$$

*Unidirectional dispersal*

Since we assume that one subpopulation has an attracting equilibrium and the other has complicated dynamics, it is necessary to consider two possibilities: (i) the donor or (ii) the recipient has complicated dynamics.

Figure 5.14 shows the results obtained when the donor has complicated dynamics. The mean total population size is always favoured

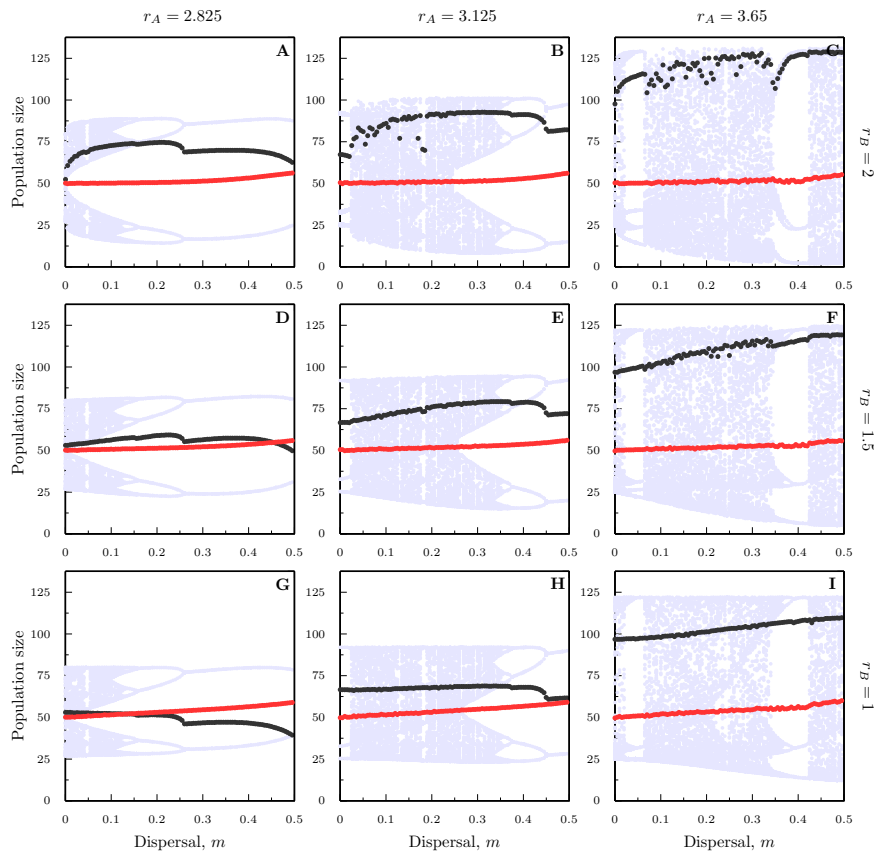


Figure 5.14: Unidirectional dispersal for a donor with complicated dynamics and a recipient with an attracting equilibrium. Panels A-I show nine different configurations with  $r_A$  constant in each column (increasing from left to right  $r_A = 2.825, 3.125, 3.65$ ) and  $r_B$  constant in each row (increasing from bottom to top  $r_B = 1, 1.5, 2$ ). The mean total population size (red) and its fluctuation range (black) overlies the bifurcation diagram, where the system was run for 3000 generations to remove transient effects, and then the total population size in the next 100 generations was plotted (light blue). See main text for a discussion of the results.

by dispersal, in the sense that it is larger for linked regions than for isolated ones regardless of the dispersal rate. Further, the mean total population size increases with dispersal rate and attains its maximum value for  $m = m_A = 0.5$  in all the cases. Such an increase could be related to the hydra effect. Essentially, this effect consists in an increase of the population size in response to harvesting (see (Abrams, 2009) for a description of the mechanisms producing this effect). Since net emigration acts like harvesting, it leads to an “altered fluctuation” in the donor as the system is forced to a different attractor; see (Hilker and Liz, 2013, Figure 7) for an example of this behaviour in a different setting. Nevertheless, we point out that the increase of the total population size due to dispersal was also observed in (Franco and Ruiz-Herrera, 2015) for the Beverton-Holt map, which does not show the hydra effect under harvesting. Therefore, we conjecture that the hydra effect could be enhancing this response, but it is not the main reason for it.

The numerical simulations show that the fluctuation range is much more dependent on the local dynamics, determined by the intrinsic growth-rate parameters  $r_i$ , than the mean total population size. Observe that for  $r_A = 3.65$  the fluctuation range increases as dispersal rates increase. But for smaller  $r_A$  it essentially has a one-humped behaviour. Interestingly, only in one case (corresponding to panel G) there is a significant interval of dispersal rates able to reduce the fluctuation range with respect to that of the isolated patches ( $m = 0$ ). For the other cases, fluctuations are generically increased by dispersal. This suggests that having a donor with complicated dynamics impedes constancy. Now, observe the behaviour of the total population size in the whole period of 100 generations for each dispersal rates (light blue dots). With the exception of panel G, the minimums of those sizes are generically smaller than for the case  $m = 0$ . Thus, dispersal could be regarded as negative not only for constancy, but also for persistence.

We conclude that unidirectional dispersal with a donor with complicated dynamics is stabilising only for small values of  $r_A$  and  $r_B$ . In such a case, increasing the dispersal rates improves constancy and

persistence at the same time that reduces the complexity of the population dynamics (period-four and two cycles are stabilised by dispersal after a cascade of period halving bifurcations). Moreover, these stabilising effects coexist with an increase of the mean total population size.

Next, let us consider the case of the recipient having complicated dynamics, which is shown in Figure 5.15. First, we observe that dispersal is always able to stabilise a period-two cycle, consequently reducing the complexity of the population dynamics. Second, and in contrast with the previous case, the mean total population size is not always promoted by dispersal. Indeed, the mean total population size is smaller for the connected regions than for the isolated regions in panel G with high dispersal rates. Besides, the responses of the mean total population size to an increase of the dispersal rates are essentially one-humped. Thus, an increase of dispersal could be related to an increase or a decrease in the mean total population size. Again, an hydra effect may be enhancing these responses, since the emigration from the donor acts like harvesting and releases that subpopulation from intraspecific competition pressure (Abrams, 2009). We stress that the population size is measured after dispersal in our experiments, and it might be that the consequences on the mean total population size are different if the population were censused after reproduction instead, as it occurs with the “hidden” hydra effects (Hilker and Liz, 2013).

Contrary to the case of a donor with complicated dynamics, the fluctuation range reduces with respect to that of the isolated patches—consequently constancy improves since a population has greater constancy when it has a lower variation in size over time (Grimm and Wissel, 1997). This happens in all the configurations, although for different dispersal rates. Therefore, unidirectional dispersal with a donor that has an attracting equilibrium could be used as a stabilising strategy focused on constancy as the adaptive limiter control (Franco and Hilker, 2013, 2014; Sah et al., 2013). On the other hand, the way in which connectivity affects persistence strongly depends on the donor’s growth rate parameter. Regardless of the value of  $r_B$ ,

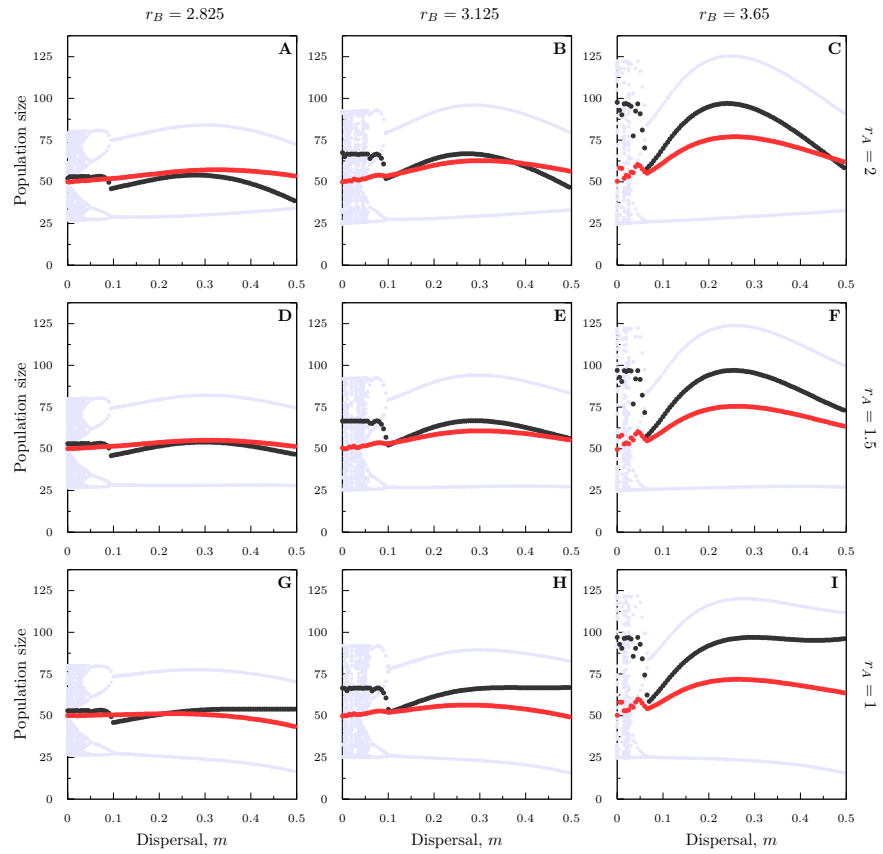


Figure 5.15: Unidirectional dispersal for a donor with an attracting equilibrium and a recipient with complicated dynamics. The mean total population size (red) and its fluctuation range (black) overlaid the bifurcation diagram, where the system was run for 3000 generations to remove transient effects, and then the total population size in the next 100 generations was plotted (light blue). See main text for a discussion of the results.



the minimum total population size (over the considered period of 100 generations) decreases with dispersal for  $r_A = 1$ , stays approximately constant for  $r_A = 1.5$ , and increases for  $r_A = 2$ . In the light of these simulations, the optimal response for stability takes place for donors with large parameter  $r_A$ , since there are intervals of dispersal rates able to improve (at the same time) constancy and persistence.

### *Bidirectional dispersal*

Here we only present simulations for bidirectional symmetric dispersal. This has the advantage of reducing the two parameters related to dispersal to just one, so we can discuss easily the effects of increasing connectivity as in the previous scenario. This restriction should be considered relatively unimportant after studying the two previous unidirectional dispersal cases since, as stated in (Dey et al., 2014), *asymmetric dispersal can be viewed as spanning a continuum between the two extremes of unidirectional and bidirectional symmetric dispersal, respectively*.

Figure 5.16 shows the simulations for bidirectional symmetrical dispersal where subpopulation  $B$  has an attracting equilibrium while subpopulation  $A$  has complicated dynamics. Dispersal reduces the complexity of the population dynamics provided that  $r_A$  and  $r_B$  are small enough. In such a case, a positive equilibrium, a period-two cycle, or a period-four cycle are stabilised after a cascade of period halving bifurcations. The only simulation with a different kind of response is the configuration with  $r_A = 3.65$  and  $r_B = 2$  (panel C). There, increasing dispersal from  $m = 0$  produces a series of period halving bifurcations, followed by the stabilisation of period-four and period-two cycles for intermediate values of  $m$ . Then, a series of period doubling bifurcations appear, followed by complicated dynamics for higher dispersal rates. This latter effect of dispersal on the population dynamics was already reported in two-patch models when both subpopulations have intrinsically complicated dynamics (Ben Zion et al., 2010; Dey et al., 2014; Gyllenberg et al., 1993; Hastings, 1993).

The mean total population size is larger for linked regions than for isolated regions in all configurations. This result agrees with the

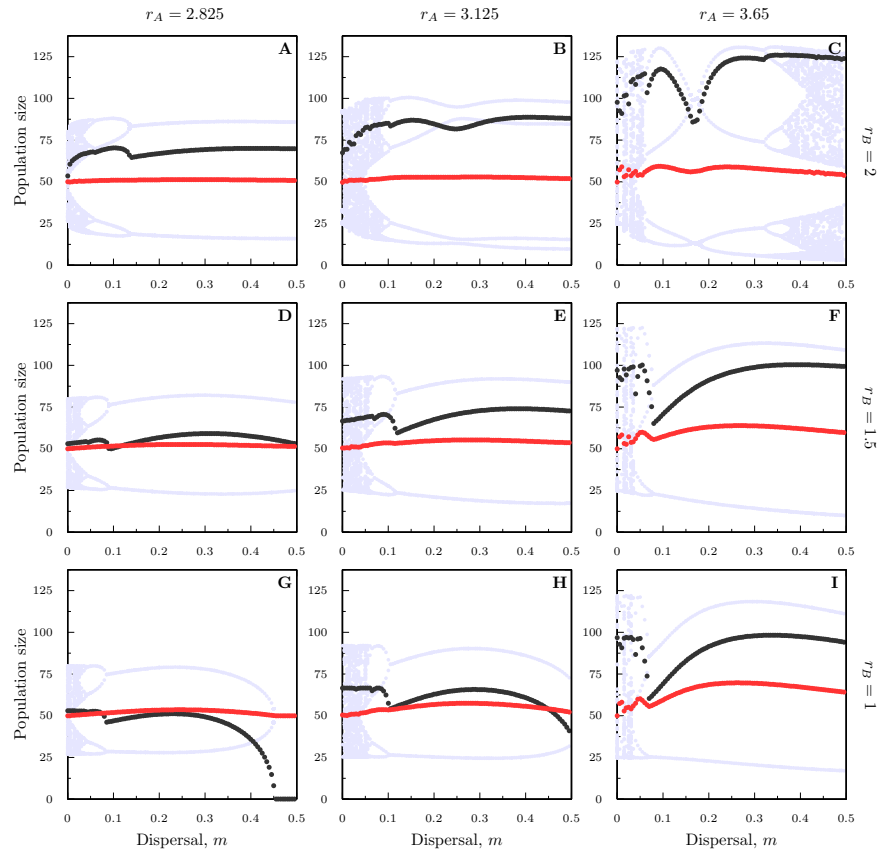


Figure 5.16: Bidirectional symmetric dispersal connecting a region with an attracting equilibrium to a region with complicated dynamics. The mean total population size (red) and its fluctuation range (black) overlies the bifurcation, where the system was run for 3000 generations to remove transient effects, and then the total population size in the next 100 generations was plotted (light blue). See main text for a discussion of the results.

behaviour found in (Franco and Ruiz-Herrera, 2015), where both subpopulations had an attracting equilibrium and the same type of dispersal as here was considered. Also as in (Franco and Ruiz-Herrera, 2015), an increase of the dispersal rate can produce both an increase or decrease of the mean total population size.

The effect of dispersal on the fluctuation range largely depends on the intrinsic growth-rate parameters. Nevertheless, we can obtain some conclusions about how dispersal affects constancy and persistence. With regard to constancy, for relatively small values of  $r_B$  there is a significant interval of dispersal rates able to reduce the fluctuation range with respect to that of the isolated patches. Even more, we observe that a fixed point is stabilised in one case (panel G), so the total population size does not fluctuate. Panel G also shows a positive effect of dispersal on persistence since the minimum values of the total population sizes over the considered period of 100 generations are always greater than the correspondent minimum value for unlinked regions ( $m = 0$ ). But this does not happen for the rest of the cases, since those minimum values are larger than the minimum value for  $m = 0$  for high dispersal rates only (panel H) or they are always smaller. Taking in consideration the mean total population size, persistence and constancy altogether, the situation that has a clearly better response to bidirectional symmetric dispersal than the others is the one with relatively small values of the intrinsic growth-rate of the subpopulations.

### 5.3.2 *Effect of multistability*

It is known that system (5.1) can exhibit multistability, i.e. the dynamics of the system depend on the initial population size (Hastings, 1993; Lloyd, 1995). This occurs both for unidirectional and bidirectional dispersal. In this subsection, we address the question of whether or not multistability affects the behaviour of the mean total population size and its fluctuation range. We are particularly interested in how multistability affects their response to a variation of the migration rates. For this goal we use system (5.1) with parameters

$r_A = 3.0$ ,  $r_B = 4.0$ , which leads to the coexistence of two different attractors both for unidirectional dispersal and bidirectional symmetric dispersal at certain dispersal rates.

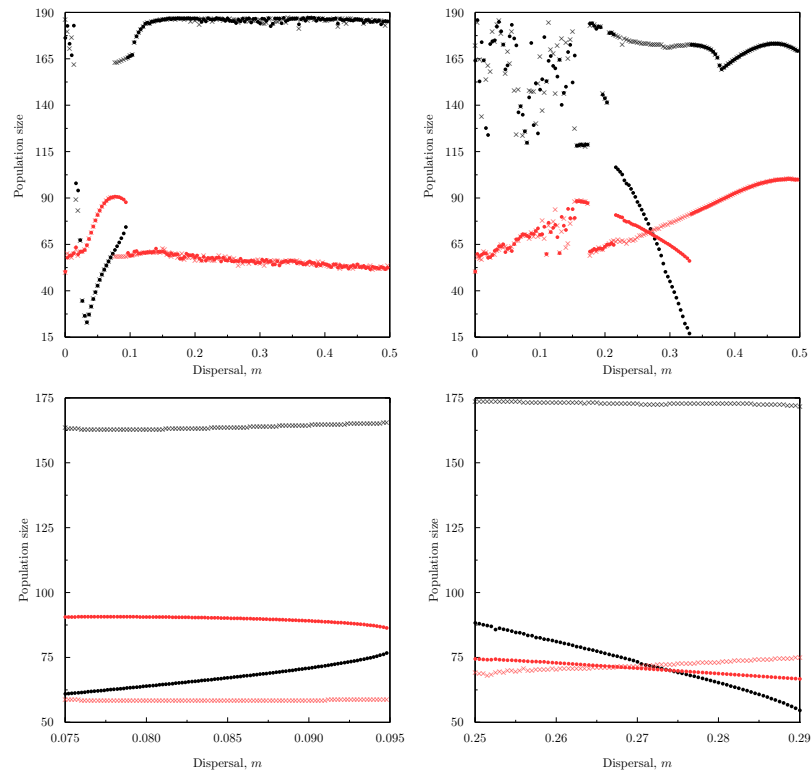


Figure 5.17: In each panel, the mean total population size appears in red and the fluctuation range is in black (both were calculated as in the previous cases). We use two different initial conditions, the crosses and filled circles correspond to each of them. In addition to the different responses to a variation of connectivity, note the differences in value depending on the attractor. Left panels show unidirectional dispersal, and right panels show bidirectional symmetric dispersal. Top panels show the behaviour for  $m \in [0, 0.5]$ , whereas bottom panels highlight specific subintervals of the parameter  $m$ . In all panels,  $r_A = 3.0$  and  $r_B = 4.0$ .

Figure 5.17 illustrates that, depending on the initial condition, the mean total population size and its fluctuation range vary considerably. Further, increasing the migration has different consequences depending on the initial population size both for unidirectional and bidirectional dispersal. For instance, in case of bidirectional dispersal

(right panels), the mean total population size decreases as  $m$  increases for one of the two considered initial conditions, but if we start with the other initial condition, then the mean total population size increases. Similar behaviours can be observed for the fluctuation range. Thus, the coexistence of alternative attractors seems to be a major problem from a control point of view, since depending on the initial condition a modification of migration rates to obtain certain effect could have the opposite one.

#### 5.4 SUMMARY

Many natural populations are formed by heterogeneous groups of semi-independent subpopulations connected by migration. The role of those connections in local and global dynamics is not completely understood. Our numerical study extends and complements existing investigations about the interaction between local subpopulation dynamics and dispersal that were focused on the complexity of the population dynamics (Dey et al., 2014). Under a variety of dispersal scenarios and local dynamics configurations, we have considered the effect of dispersal on the mean total population size and on its fluctuation range. How dispersal affects these values (and consequently constancy and persistence) depends on the type of connection between subpopulations and the dynamics within them.

The main results of the chapter are the following:

1. We find that the mean total population size of a spatially-separated population is clearly affected by the local dynamics and the type of connection involved. In most of the cases under consideration in this work, the mean total population size of connected regions is greater than if those regions are isolated.
2. Both for symmetric bidirectional dispersal and for unidirectional dispersal from a donor with an attracting equilibrium to a recipient with complicated dynamics, the mean total population size generically has a unimodal response. But, for unidirectional dis-

persal from a donor with complicated dynamics to a recipient with an attracting equilibrium, the response is monotone.

3. Connecting regions improves, in certain situations, constancy and persistence. But in other cases, this connection can have a negative effect on those characteristics.
4. We show that multistability affects the previous properties.

### Parte III

## CONCLUSIONES Y LÍNEAS FUTURAS





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## CONCLUSIONES

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En esta memoria hemos abordado el estudio de varios métodos de estabilización para sistemas dinámicos discretos. Estos sistemas son frecuentemente utilizados en el campo de la dinámica de poblaciones y por este motivo hemos tomado modelos provenientes de este campo para ilustrar la aplicabilidad de nuestros resultados y análisis.

En la parte dedicada a los sistemas unidimensionales hemos considerado dos métodos de estabilización distintos. Uno propuesto por otros autores, el control proporcional (PF), y otro que se propone y estudia aquí por primera vez, al que hemos llamado control no lineal (CN).

Hemos probado rigurosamente algunas propiedades del método PF, que generalizan y complementan los resultados existentes en la literatura. Hemos centrado nuestra atención en la estabilización del equilibrio positivo y nuestros resultados cubren tanto los comportamientos locales como globales.

Nuestro primer objetivo ha sido dar una caracterización completa de las funciones que definen las ecuaciones en diferencias para las que el método PF consigue la estabilización local de un equilibrio positivo sin imponer que el parámetro  $\gamma$  sea menor que 1. Destacamos el hecho de que damos una caracterización y no sólo una condición suficiente para una clase de funciones.

Nuestro segundo objetivo principal ha sido presentar condiciones suficientes para asegurar la estabilidad global del equilibrio positivo en el sentido de que todas las órbitas de partida en un estado inicial distinto de cero convergen al equilibrio positivo. Hemos demostrado que para una clase de funciones unimodales, es posible mostrar que

el método PF estabiliza un equilibrio positivo único y que este equilibrio es un atractor global superestable para un valor específico del parámetro sin asumir condiciones de signo constante en la segunda derivada. También se demuestra que es posible extender la atracción global del equilibrio mediante la adición de una condición en el signo de la derivada Schwarziana de la función. Por último, hemos mostrado cómo se puede emplear el método PF con  $\gamma > 1$  para estabilizar hacia un atractor global sistemas con efecto Allee, con lo que hemos visto que este método es útil para modelos de población en los que el mayor peligro es la extinción de la especie cuando el tamaño de la población baja de un determinado umbral.

Hemos presentado el método de control no lineal (CN). Este nuevo método, al igual que en el caso del control orientado a objetivo (TOC), también utiliza dos parámetros que juegan el mismo papel, es decir, uno sirve para fijar el tamaño de la población objetivo y el otro se utiliza para estabilizar el sistema. Además hemos demostrado algunas características ventajosas que presenta el CN: es capaz de estabilizar cualquier función unimodal hacia su equilibrio positivo, un sobrecontrol no produce la extinción de la población, permite eliminar el efecto Allee, y a diferencia del TOC, no cambia la forma de la función controlada. Dado que la aplicación del CN produce el cambio del signo de la derivada Schwarziana de la función controlada, hemos debido utilizar otras herramienta para analizar la estabilidad global, concretamente la de envolvimiento por una función lineal fraccionaria que desarrolló Paul Cull. Con dicha técnica, y restringiéndonos al modelo de Ricker, hemos obtenido resultados que permiten conjeturar la estabilidad global del equilibrio positivo tras la aplicación de CN para ciertos valores del parámetro ligado a la tasa de crecimiento en la función de Ricker.

En la parte dedicada a sistemas bidimensionales hemos estudiado numéricamente el efecto de la migración en el tamaño de la población y sus fluctuaciones. Hemos relacionado este efecto con dos nociones de estabilidad para poblaciones: constancia y persistencia. Los resultados obtenidos muestran que la migración es capaz de mejorar la constancia y la persistencia solamente en ciertas situaciones. Adicio-

nalmente, confirmamos que la multiestabilidad afecta al comportamiento del tamaño medio de la población y su rango de fluctuación. Lo que nos lleva a afirmar que solamente un profundo conocimiento de la dinámica local en cada una de las subregiones que ocupa la población permitiría utilizar una modificación de la tasa de migración natural para controlar la estabilidad y el tamaño de la población.



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## LÍNEAS FUTURAS

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Creemos que sería interesante analizar en más profundidad el método de control CN que hemos propuesto. Este análisis debería realizarse en varias direcciones. Por ejemplo, creemos que sería conveniente comparar sus propiedades con las de otros métodos de control. En este sentido, el trabajo de Tung et al. (2014) podría servir de pauta dado que allí se consideran seis métodos de control entre los que aparece TOC, que como hemos dicho inspiró el diseño del control CN.

El estudio que hemos realizado sobre la estabilidad global del método CN deja muchas preguntas abiertas. Para el modelo de Ricker hemos mostrado que debería ser posible obtener resultados analíticos sobre esa estabilidad global si el parámetro ligado a la tasa de crecimiento (y a la complejidad del sistema) no es demasiado grande. Desde un punto de vista práctico valores muy grandes de ese parámetro pueden considerarse no realistas, pero desde el punto de vista teórico tiene todo el sentido considerarlos. Como hemos ilustrado, para esos valores ni la técnica de envolvimiento de Cull ni la de Singer basada en una derivada Schwarziana con signo constante funcionan. Por lo tanto, será necesario buscar otras alternativas.

Por otro lado, creemos que sería interesante extender el trabajo realizado en la segunda parte a sistemas de dimensión mayor que dos. Al igual que ocurre en el caso de dimensión igual a 2, creemos que la forma en la que la población se dispersa de una región hasta las otras jugará un papel muy importante en la estabilidad del sistema.

También nos gustaría continuar el estudio del método de control proporcional. Opinamos que el caso unidimensional está ya bastante

desarrollado con los resultados presentados aquí y los que aparecen en Liz (2010b). Por este motivo, creemos interesante considerar sistemas dinámicos multidimensionales ligados a poblaciones estructuradas en varias regiones para determinar las ventajas e inconvenientes que presenta el método aplicado de manera zonal, es decir, en regiones concretas, frente a una aplicación en la totalidad de las regiones ocupadas por la población.

Parece que cuando el control se aplica en la totalidad del sistema los resultados son similares a los obtenidos con el caso unidimensional. Sería interesante estudiar el método PF zonal bajo varios criterios para analizar su viabilidad y compararlo con el caso de control en todas las celdas. Uno de los criterios utilizados podría ser el criterio de valor mínimo. Cuanto más alto sea este valor en cualquiera de las fases de la evolución de la población, mejor será el control en cuanto a la posibilidad de sobrevivir ante imprevistos por efectos ambientales. Algunas simulaciones realizadas parecen apuntar a que a medida que aumenta la dispersión, el intervalo del parámetro de control que produce un mejor valor mínimo aumenta en el caso del control zonal.

También proponemos analizar el método desde el punto de vista de la fluctuación producida. Cuanto más bajo sea este valor, mejor será el control en el sentido de que el sistema se acercará más al ideal de "constancia". En este sentido parece que la fluctuación mejora para un alto nivel de dispersión si el porcentaje de celdas controladas no es ni muy grande ni muy pequeño.

Como propusimos anteriormente, también sería interesante desde el punto de vista económico utilizar el criterio de coste.

Como comentábamos, hemos realizado trabajos preliminares en esta línea y creemos posible llegar a responder total o parcialmente las siguientes cuestiones:

- ¿Puede reducirse significativamente el número de regiones sobre las que realizamos el control para estabilizar el sistema?  
¿Puede ahorrar costes el control zonal?
- ¿Puede obtenerse un rango del parámetro de control diferente si el método PF se aplica en regiones concretas frente a una

aplicación en la totalidad de las regiones? ¿Puede evitarse el riesgo de extinción asociado a un sobrecontrol?

- ¿Puede eliminarse en alguna circunstancia la necesidad de medir el tamaño de población previamente a la aplicación del control zonal?
- ¿Qué influencia tiene el control zonal en la multiestabilidad del sistema?





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