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Escalarizaciones Conjuntistas basadas en la Distancia Orientada con Aplicaciones en Optimización de Multifunciones

Set Scalarizations based on the Oriented Distance with Applications in Set-valued Optimization

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In memory of my parents

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Preface

This memory is submmitted in order to achieve the PhD degree by the National Distance Education University (UNED) in the PhD Programme in Industrial Technologies. Following the current regulation existing in UNED, this work has been developed in English since it is the language customary in mathematical research.

The study of my post-graduate courses in the Department of Applied Mathematics I at UNED under the supervision of Professor Vicente Novo captured my interest in set-valued maps, set relations, monotonicity, optimization and related topics. However, personal circumstances led me to postpone my desire to learn more about these issues. Later, as I still had curiosity to learn, I kept in touch with Professor Vicente Novo to express my desire to work about these topics again and, as kind as always, he informed me about the procedures I had to follow. Once completed, the advisors of my thesis were Vicente Novo and Bienvenido Jiménez, both of them recognized experts in Mathematical optimization, with numerous and relevant research projects published in well-known international journals.

Set-valued optimization problems have extensive applications in many areas. In practice, we need scalarization processes which allow the replacement of one of these problems by a suitable scalar optimization problem in order to obtain the solutions of the original problem by means of solutions of a scalar optimization problem. In this work, we contribute to such aim introducing new set scalarization functions which are extensions of a scalarization function existing in the literature. Moreover, we use them to characterize set relations and minimal solutions of a set optimization problem, and we introduce improvements of results existing in the literature for other scalarizations.

For any suggestion or comment related to the work developed in this memory, the reader can contact me in the following email address:

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Resumen

Los problemas de optimización de multifunciones son una generalización de los problemas de optimización vectorial que admiten múltiples aplicaciones. Para estos problemas, Kuroiwa introdujo en 1997 un criterio conjuntista de solución que es el que se considera en esta tesis. La escalarización permite relacionar las soluciones de un problema de optimización de multifunciones con las soluciones de problemas de optimización escalares. En esta memoria se estudian funciones de escalarización conjuntistas y sus aplicaciones en problemas de optimización de multifunciones. El Capítulo 1 se dedica a la introducción y a exponer los objetivos de la tesis. Además, se fija el marco de trabajo y se dan las notaciones, conceptos y resultados previos.

En el Capítulo 2 se introducen las escalarizaciones conjuntistas existentes en la literatura, extensiones de la función de Gerstewitz y de la distancia orientada de Hiriart-Urruty. En un espacio normado ordenado por un cono convexo no necesariamente sólido, se presentan extensiones conjuntistas de tipo sup-inf de la distancia orientada y se estudian relaciones entre ellas. Además, para dichas funciones de tipo sup-inf, se obtienen resultados de finitud usando cono-propiedad, cono-acotación y una propiedad de cono-acotación con respecto a un conjunto que hemos introducido. Por otro lado, se estudian sus propiedades como, por ejemplo, convexidad, Lipschitz continuidad, monotonía, etc. Mediante esas propiedades, se caracterizan las relaciones conjuntistas inferior y superior de Kuroiwa y sus respectivas relaciones estrictas. Por último, se estudia la monotonía estricta de las funciones de tipo sup-inf antes mencionadas. Se aportan mejoras de resultados existentes para extensiones de la función de Gerstewitz.

En el Capítulo 3 se presentan extensiones conjuntistas nuevas de tipo sup-inf

e inf-sup de la distancia orientada. Para esas funciones, se estudian sus relaciones, algunas caracterizaciones y sus propiedades; además, dichas funciones se usan para dar interesantes caracterizaciones de las seis relaciones conjuntistas de Kuroiwa y de sus respectivas relaciones estrictas, que representan mejoras de resultados similares para la función de Gerstewitz.

En el Capítulo 4 usando las funciones de tipo sup-inf e inf-sup y algunas de sus propiedades, las caracterizaciones de las seis relaciones conjuntistas de Kuroiwa y las caracterizaciones de sus respectivas relaciones estrictas, se obtienen condiciones necesarias y suficientes de soluciones minimales y minimales débiles para seis problemas de optimización de multifunciones.

Finalmente, en el Capítulo 5 se recogen las conclusiones y las futuras líneas de desarrollo que han aparecido durante la elaboración de esta tesis doctoral.

Contents

Chapter 1

Introduction, objectives and preliminaries

1.1 Introduction

In real life it is often necessary to take decisions which usually involve several conflicting objectives. Decision problems in the real world require to choose a best solution among a set of possible alternatives according to a certain optimality criteria. Mathematical optimization can be understood as a mathematical model of decision problems that deals with the problem of finding the best element with regard to some criterion from some set of available alternatives. In optimization problems the aim is minimizing or maximizing an objective function over some feasible set (a constraint set). These problems are faced with conflicting goals which have to be minimized or maximized simultaneously and, therefore, an optimal solution has to be found by a compromise among these goals. These kind of problems are among the most important in engineering and finance when, for example, minimizing production cost, maximizing profits, maximizing safety, minimizing pollution are tried.

An optimization problem is called scalar optimization problem if the objective function is scalar-valued, and is called vector optimization problem if the objective function is vector-valued. A multiobjective optimization problem [119] is a vector optimization problem with an objective function taking its values in the m-dimensional Euclidean space $\mathbb{R}^m$, for $m \geq 2$. These latter problems are called multicriteria decision making problem [24] in economics. The first researches about vector optimization go back to F.Y. Edgeworth (1881) and V. Pareto (1896) who provided the definition of standard optimality concept in multiobjective optimization, although in mathematics this branch of optimization started with the paper of H.W. Kuhn and A.W. Tucker (1951). In engineering and economics, minimal or maximal elements of a set are often called efficient, Edgeworth-Pareto optimal or nondominated.

The main difference between scalar optimization and vector optimization lies in the comparison between the values of the objective function. In the first case, this comparison is clear and, in the second case, it is not obvious anymore how to compare the values of the objective function. Therefore, the key lies in the underlying preference orders involved on the objective space. In the scalar case, the functions to be maximized or minimized are valued in $\mathbb{R}$ where a complete order is given. Due to this important feature, it can be decided for each pair of alternatives which of them is preferred and, moreover, a feasible element will be an optimal solution if the objective function takes in that element its smallest (or largest) scalar value. However, this important feature is no longer valid in the vector case since the preference orders are generally not complete. This fact leads to many difficulties in determining concepts of solutions, monotonicity, convexity, etc. As it is not clear how to compare the values of the objective function, for defining optimality for a vector optimization problem, first of all, it is necessary to define how to compare the elements in a real linear space. To overcome the difficulties caused by the noncompleteness of the orders, there exist techniques to convert the vector optimization problems into appropriate scalar optimization problems.

Unlike traditional mathematical programming with a single objective function, an optimal solution in the sense of one that minimizes or maximizes all the objectives simultaneously does not necessarily exist in multiobjective optimization problems, because in decision making problems with multiple objectives there are conflicts among objectives. In order to compare elements in $\mathbb{R}^n$, Edgeworth and Pareto used the natural ordering cone given by the nonnegative orthant $\mathbb{R}^n_+$ which corresponds to the componentwise partial ordering. By using this idea, in vector optimization it is often assumed that the linear space is ordered by the preorder induced by a convex cone K (see, for example, [56,62,103]), compatible with the linear structure of the space, which will be partial order if K is in addition pointed. It means that in vector optimization problems, incomparable elements and indifferent elements can be found. So, an element is an optimal element (or an efficient element) of a set if it is not dominated by any other feasible alternative with respect to the preorder considered in the linear space. In this way, different concepts of optimality in a partially ordered linear space are discussed in the literature. It is worth that in many areas as, for instance, in mathematical economics, it is common to have preferences that are not necessarily preorder since only the reflexive or transitive property is verified.

Optimization problems arise in, for example, functional analysis (the Hahn-Banach theorem, the lemma of Bishop-Phelps, Ekeland's variational principle), multiobjective programming, multi-criteria decision making, statistics (Bayes solutions, theory of tests, minimal covariance matrices), approximation theory (location theory, simultaneous approximation, solution of boundary value problems) and cooperative game theory (cooperative n player differential games and, as a special case, optimal control problems).

In the last decades, vector optimization problems have been extended to problems with set-valued maps, that is, with set-valued objective functions or set-valued constraints. These problems are called set-valued optimization or set optimization problems, noted SOP $\text{Min}\{F(x) : x \in S\}$ where $F : S \subset X \rightrightarrows Y$ is a set-valued map, S is a nonempty set and X, Y are real linear spaces. Optimization problems with a set-valued objective function F provide an important generalization and they allow us to unify scalar as well as vector optimization problems since the notion of set-valued maps subsumes single-valued maps. In recent years, set-valued optimization problems (see, for instance, [5,38,48,51,53,88]) have received an increasing attention due to extensive applications in many areas, since numerous problems that arise in different fields can be modeled as a set-valued optimization problem. This is what happens in, for example, uncertain multiobjective problems and robust optimization, mathematical finance, welfare economics, optimal control, differential inclusions, viability theory, game theory, fuzzy optimization, duality principles in vector optimization, gap functions for vector variational inequalities, image processing, management science, stochastic programming, and so on (see [6, 62, 78]). A detailed introduction to set optimization and its applications can be seen in [71], and applications in finance can be seen in [46, 47].

For a set-valued optimization problem it seems natural that the first thing that has to be done is to decide how to compare elements of the power set of Y, since the values $F(x)$ are now sets, although it has to be said that is not the only criterion of solution as it can be seen below. In set-valued optimization problems, there exists mainly two criteria of solutions that are reduced to the usual notion of efficiency whenever the set-valued F is a vector single-valued function: the vector criterion (see [12, 17, 62, 63, 78, 103]) and the set criterion (see [5, 41, 51, 62, 85, 94, 129]), which was introduced by Kuroiwa in 1997. On the one hand, by considering the vector criterion, it looks for efficient elements in the union of all objective values of F with respect to the order structure on Y, that is, in this criterion the efficient elements are taken in the vector sense and the set-valued optimization problem is reduced to a vector optimization problem. So, (x_0, y_0) is a solution (or minimizer) of the set-valued optimization problem if for $x_0 \in S$ there exists at least one element $y_0 \in F(x_0)$ which is an efficient point of the image set $F(S)$. Therefore, a minimizer (x_0, y_0) only depends on the certain special element y_0 of $F(x_0)$ by which other elements of $F(x_0)$ are ignored, but one efficient element in some sense does not reveal any information about the performance of the remaining elements in that particular solution set. For this reason, although it is of mathematical interest, it cannot be often used in practice. A set-valued optimization problem with vector criterion is usually called set-valued vector optimization problem or vector optimization problem with set-valued maps.

On the other hand, by using the set criterion, the sets $F(x)$ are compared through set relations with the aim to choose the best set in some sense, that is, this criterion is based on comparisons among values of the set-valued objective map F by means of different set order relations which are preference relations that allow to compare the sets $F(x)$ and to decide which is the best set (efficient set) with respect to the set relation that is considered. A set-valued optimization problem with set criteria will be called a set optimization problem, and represents a natural extension of vector optimization problem. The relationships between vector and set criteria have been studied, for instance, in [51, 52, 75]. It is also worth emphasizing that there is another approach criteria of solution newly developed called the lattice criteria (see [71, 101]), based on order structure generated by the inclusion between the conical extensions of sets, that is, in this criterion the lattice structure of the space of conical extensions of the subsets of the image space is considered. This last criterion is used in order to apply set optimization to mathematical finance.

Set relations (see [65, 70, 87, 89]) on the power set of the objective space Y, relying on the ordering structure given in Y, play one of the most essential roles in set optimization problems since they act as preference relations which provide a natural way to compare the values of the set-valued objective map F. Through these binary relations, given two sets, it is possible to decide if one set dominates another set in a certain sense. As there are several set relations on 2^Y, it is possible to define different concepts of optimal solutions in set optimization problems according to the set relation involved. Set relations are concerned with different fields as, for instance, Ekeland's variational principle (see [3, 48]), well-posedness (see [37,43,134]), minimax theorem for vector-valued functions [7], stability [30], optimality conditions for set-valued optimization problems [21], concepts of efficiency for uncertain multiobjective optimization problems [58], fuzzy programming, interval analysis, etc. In particular, the lower and upper set preorders of Kuroiwa have been commonly used to describe, for example, convexity (see [73, 79, 89, 91–93, 95, 96, 106, 111, 120, 129]) and semicontinuity of set-valued maps (see [79, 88, 107]). The set less order relation has been used outside the optimization community to define a directional derivative from a computational point of view based on a set difference (see [21, 64]).

Kuroiwa is credited for the introduction of set relations on the power set of Y and by being the first who used them for defining optimality notions for optimization problems with a set-valued objective function, although these set relations were independently introduced in different fields as, for example, in terms of algebraic structures by Young in 1931 [130], in theory of fixed points of monotonic operators by Nishnianidze in 1984 [112], in interval arithmetic by Chiriaev and Walster in 1998 [15], in theoretical computer science by Brink in 1993 [10], etc. In [4,70,76] new set order relations are defined by using Minkowski difference of sets. In [59] eight types of fuzzy-set relations based on a convex cone are proposed as new comparison criteria of fuzzy sets. In [50, 83] very general definitions of set order relations are introduced, where the involved set describing the domination structure does not need to be convex or a cone. In [80] several variable order relations in a linear topological space are considered and set order relations equipped with a variable order structure are introduced to compare sets and to study set optimization problems endowed with these variable ordering structures, which generalize the concept of variable ordering structures in vector optimization [26].

In the setting of vector optimization theory, scalarization processes (see [57, 60, 114]) means the replacement of a vector optimization problem *VOP* by a suitable scalar optimization problem, in order to obtain the solutions of the original problem *VOP* by means of solutions of a scalar optimization problem. So, by using a scalarization function, scalarization technique allows to characterize and compute solutions of vector optimization problems as solutions of scalar optimization problems, which is of great importance since scalar optimization theory is widely developed in the literature. In vector optimization, order-preserving and order-representing properties to the scalarization function have been used to characterize minimal solutions through scalarization. To be precise, if the scalarization function is monotone (or also called order-preserving) then sufficient optimality conditions can be obtained, and if it is order-representing then necessary optimality conditions can be derived for the problem VOP.

Among the scalarization techniques, the standard linear scalarization is historically the first method proposed and it is based more on analytical aspects than on a geometrical approach to construct the scalarization function, by using the elements of the positive polar cone of the order cone K. It is used in the separation of two convex sets and to obtain necessary optimality conditions by using convexity assumptions. However, without the convexity (or generalized convexity) assumptions, vector optimization problems may not be equivalently characterized by linear scalar problems. A classical approach to the scalarization of a nonconvex problem is the minimization of some type of distance [60] which is a geometrical approach to the scalarization procedure that avoids the convexity requirements typical of the linear case [109].

In the set-valued case, via scalarization, a set-valued optimization problem is replaced by a family of scalar optimization problems which allow to relate qualitatively the solutions of both problems and solve the initial problem by a numerical method applicable for the scalar problems. Usually, these scalar optimization problems are defined by the composition of the objective mapping F with the elements of a parametric family $\{\varphi_p\}_{p\in\mathcal{P}}$ of extended real-valued mappings $\varphi_p : 2^Y \to \mathbb{R} \cup \{\pm\infty\}$, where $\mathcal{P}$ is an index set. So, the scalarization processes relate the solutions of problem SOP with the solutions of the scalar optimization problems $\mathrm{Min}\{(\varphi_p \circ F)(x) : x \in S\}$.

In practice, scalarization functions are used to check whether one set is better than another set by means of set relations, that is, through a scalarization function it is able to decide if a set relation is fulfilled for two sets evaluating one inequality. Recently, in analogy to the scalarization, it has appeared a vectorization [63] approach for set optimization problems which means the replacement of a set optimization problem by a suitable vector optimization problem.

In view of the foregoing, scalarization methods are one of the most essential tools in vector optimization (see [12, 31, 62, 103, 109]) and set-valued optimization (see [5, 14, 41, 51, 89, 107]) from theoretical as well as computational points of view. Among their multiple applications, it is emphasized the characterization of different types of set relations (see [5, 38, 113]) (this is important to obtain numerical methods to solve set optimization problems [81]), scalar representations [51], to obtain optimality conditions (see [5, 51, 94, 95]), nonconvex separation type theorems [5], Gordan's type alternative theorems (see [5, 110, 113]), Takahashi's type minimization theorems (see [4,5]), minimal element theorems (see [4,48]), generalizations of the Ekeland's variational principle (see [4, 27, 48]), well-posedness [43], fuzzy theory [59], equilibrium problems [36], set-valued Ky Fan minimax inequality [95], minimax theorems [91], existence theorems for saddle points (see [91,96]), stability results [44], vector variational inequalities [19], Caristi's fixed point theorem [4], robustness and stochastic programming [77], approximate solutions [39], etc.

In vector optimization, there are two important types of scalarization functions: Gerstewitz's function (see [12, 31, 36]) defined in a topological vector space with a solid cone K, and the oriented distance of Hiriart-Urruty (see [54, 55, 133]) defined in a normed space with a not necessarily solid convex cone K. Some relations between Gerstewitz's function and the oriented distance function of Hiriart-Urruty can be found in [98]. On the one hand, Gerstewitz's function was proposed in 1990 to give separation theorems for not necessarily convex sets, and it is widely used in vector optimization problems and related problems. For example, it is used in scalarization, optimality conditions, stability in vector optimization, vector equilibrium problems, well-posedness, multicriteria decision problems, Fan's type inequalities, robustness and stochastic programming [77], to characterize cone-quasiconvex functions (see [97, 103]), etc. In [12, 31, 36, 51, 71, 124] many of its properties can be seen. This function was also introduced and used in abstract convexity analysis under the name of topical function [117], in mathematical finance under the name of coherent risk measure [71] and in economics under the name of shortage function. This scalarizing function is the smallest strictly monotonic (increasing) function with respect to the ordering structure [103]. It should be noted that certain Minkowski functionals and norms coincide with Gerstewitz's function on a subset of the space.

On the other hand, the oriented distance, or sometimes called signed distance function [105], was introduced in nonlinear analysis by Hiriart-Urruty in 1979, and was used to build necessary optimality conditions for non-smooth optimization problems from the geometric point of view. This function has been applied to study well-posedness [19], stability [109], to give characterizations of different minimal solutions for vector optimization problems (see [4, 18, 32, 109]), to characterize various notions of solutions of a vector optimization problem in terms of minimal solutions of a scalarized problem (see [33, 74, 133]), to study approximation of set-valued functions by single-valued ones [34], to obtain gap functions and error bounds for vector variational inequalities (see [8, 128]), to obtain necessary optimality conditions (see [2, 55, 109, 133]), existence of the Lagrange multipliers for ϵ-Pareto optimality in vector optimization problems [23], to characterize ϵ-weak Pareto minimal point in terms of approximate solutions of the associated scalar optimization problem [123], existence of Lagrange-Kuhn-Tucker multipliers for Pareto multiobjective programming problems [16], to deal with approximate efficient solutions of single-valued and set-valued vector optimization problems [28], to obtain some results about E-optimality in vector optimization with respect to an improvement set E [135], etc. In [11, 26, 45, 55, 100, 129, 133] many of its properties can be seen. In [100] some dual representation results for the Hiriart-Urruty function are given.

In the literature, set extensions of Gerstewitz's function can be found, for example, in [5, 38, 42, 43, 48, 51, 70, 73, 90, 94, 95, 103, 107, 134]. To the best of our knowledge, the first set extension of Gerstewitz's function defined on a topological linear space with a solid cone was introduced by Hamel and Löhne [48] in 2006, based on two types of set order relations introduced in [89], to obtain minimal set theorems and new variants of Ekeland's principle in a topological vector space without strong assumptions as convexity; there, it is considered a fixed set that plays the role of parameter which is crucial in order to characterize minimal solutions of set-valued optimization problems with set orderings through solutions of associated scalar optimization problems.

Hernández and Rodríguez-Marín [51] in 2007, introduced a generalization of Gerstewitz's function presented in [48], they studied its properties in deep and, for the first time in the literature, they characterized minimal and weak minimal solutions of set-valued optimization problems with the lower set less order via solutions of associated scalar optimization problems. In 2009, Zhang et al. [134], by using a generalized version of Gerstewiz's function given in [51] and cone-bounded sets, gave some new properties of this scalarization mapping, and used this function to derive well-posedness properties of set-valued optimization problems with set orderings; specifically, these authors obtained the equivalent relation between the well-posedness of set optimization problems with cone-closed and cone-bounded objectives and the well-posedness of scalar optimization problems. In 2010, Kuwano et al. [94] defined unified scalarizations which included the scalarization given in [51], and based on the approach of Hamel and Löhne and with respect to six kinds of set order relations introduced in [89], twelve types of set scalarization functions were proposed and optimality conditions were presented.

Gutiérrez, Jiménez, Novo and Thibault [42] in 2010, defined a sup-inf type scalarization mapping based on Hamel and Löhne function and via this mapping these authors derived approximate strict minimal element theorems and approximate versions of the Ekeland's variational principle for set orderings. In 2012, Gutiérrez, Miglierina, Molho and Novo [43] derived new properties of the scalarization mappings due to Hamel and Löhne, generalized the well-posedness properties obtained in [134] and characterized pointwise well-posedness of set optimization problems through the well-posedness of a family of scalar optimization problems. In 2015, Gutiérrez, Jiménez and Novo [41] derived general necessary and sufficient optimality conditions for minimal and weak minimal solutions of set-valued optimization problems with set orderings by dealing with abstract scalarization mappings which satisfy certain order preserving and order representing properties. Other authors have emphasized the importance of considering extensions of Gerstewitz's function, and have investigated their properties and its applications in set-valued optimization (see, for instance, [5, 49, 70, 73, 95, 107]).

It should be noted that there are only a few set extensions of the oriented distance function of Hiriart-Urruty defined in a normed space with a convex cone K not necessarily solid (see [18, 38, 45, 129]). In 2006, Crespi et al. [18] introduced a generalized version of the oriented distance function and characterized minimal solutions and weak minimal solutions of set-valued optimization problems with the vector criterion, but this generalized version does not seem to be suitable to discuss set optimization problems, since it does not have appropriate properties. In 2014, Gutiérrez, Jiménez, Miglierina and Molho [38] proposed another extension of oriented distance function, they proved some of their properties and used this function to provide characterizations of some types of minimal elements to a family of sets. Furthermore, Xu and Li in 2016 [129] presented a more general version of the oriented distance function and discussed some of its properties, they established some alternative theorems and obtained some optimality conditions and scalar representations for set-valued optimization problems with the set criterion without any convexity assumptions.

The expression $\sup_{y\in B}\inf_{x\in A} D(x-y,-K)$, without a specific name, was used by Chen et al. [11] in 2017, in order to provide a characterization of the lower set less order relation; moreover, they presented some characterizations of various set order relations using the oriented distance function and, by using the characterizations of set order relations given, necessary and sufficient conditions were derived for four types of optimal solutions of constrained set optimization problem with the set criterion. In 2017, Ansari et al. [3, 11] characterized several set order relations by defining scalarization functions in terms of the oriented distance function by using variable domination structures instead of a convex cone K, and studied the optimality conditions of set-valued optimization problems. In 2018, Ha [45] used the expression $\sup_{y\in B}\inf_{x\in A} D(x-y,-K)$ to define a Hausdorff-type distance between two sets and thus be able to define a directional derivative for a set-valued map that was applied to optimization problems with set-valued maps.

It should be pointed out that, most recently, Gao et al. [29] in 2018, considered co-radiant sets (more general sets than a cone) which are a main tool in the study of approximate solutions in vector optimization problems; besides, they investigated properties of the oriented distance with respect to a co-radiant set as, for instance, these authors showed that this function is not necessarily positively homogeneous with respect to a co-radiant set. Furthermore, Ansari et al. [4,70] in 2018, characterized new set order relations defined by using Minkowski difference of sets by using the oriented distance function.

1.2 Objectives

This memory is mainly concerned with the study of set scalarization functions and their applications in set-valued optimization problems. The main goals focus on to derive minimality and weak minimality conditions for six set optimization problems with the set criterion of solution. To achieve these aims, in the setting of normed spaces with a convex cone K not necessarily solid, six set scalarization functions of type sup-inf and inf-sup which are set extensions of the oriented distance of Hiriart-Urruty are introduced, their main properties are studied and relations among set scalarizations existing in the literature are provided. After that, these six set scalarization functions and their properties are applied to characterize the six set order relations of Kuroiwa [86, 87] and their corresponding strict set order relations. Finally, by applying all the foregoing, the characterization by scalarization of minimal and weak minimal solutions for six set optimization problems with the set criterion are derived.

To be more precise, the thesis is structured in five chapters as follows.

In Section 1.3, the setting of work is fixed, the notations and previous concepts are collected, and the results needed are gathered. Throughout the thesis, it is considered that Y is a normed space which is ordered by a convex cone K not necessarily solid, unless otherwise stated. Moreover, some results are given as, for example, it is proved that the function of Khoshkhabar-amiranloo et al. [73] coincides with the excess of a set over the conic extension of another set, by considering the norm $\|.\|_e$ generated by Minkowski's functional of an order interval.

In Chapter 2, set scalarization functions are investigated. Firstly, in Section 2.1, set scalarization functions between two sets existing in the literature, which are concerned with Gerstewitz's function and set oriented distance function of Hiriart-Urruty, are recalled. Moreover, a new set extension of oriented distance of type sup-inf is introduced and some new relationships among the set scalarization above mentioned are established. Secondly, in Section 2.2, new properties for the sup-inf set oriented distances given in Definition 2.1.14, denoted $\mathbb{D}^{si}$ and $\widehat{\mathbb{D}}^{si}$, are presented. More specifically, by using cone-properness and cone-boundedness, and a new concept of cone-boundedness with respect to a set which has been introduced by us, results about their finitude are presented. Besides, some new fundamental properties as convexity, Lipschitz continuity, positive homogeneity, invariance respect to conic extensions, monotonicity, diagonal null, invariance with respect to closure, etc. are presented. Finally, in Section 2.3, by using the useful properties that have been shown in the former section, new characterization of the lower set less relation $\preccurlyeq^{\forall\exists}$ and the upper set less relation $\curlyeqprec^{\forall\exists}$ of Kuroiwa by means of the set scalarization functions $\mathbb{D}^{si}$ and $\widehat{\mathbb{D}}^{si}$ are provided and, furthermore, if K is a solid convex cone, characterizations for the corresponding strict set relations $\preccurlyeq_s^{\forall\exists}$ and $\curlyeqprec_s^{\forall\exists}$, by requiring assumptions of K-compactness, are discussed. We also deal with strict monotonicity for the functions $\mathbb{D}^{si}$ and $\widehat{\mathbb{D}}^{si}$ with respect to the strict lower set less relation $\preccurlyeq_s^{\forall\exists}$ and the strict upper set less relation $\curlyeqprec_s^{\forall\exists}$. The results stated in this chapter are collected in our works [66, Sections 3,4 and 5] and [67, Sections 3 and 4].

Chapter 3 is concerned with set scalarization functions, which are extensions of the oriented distance, in the above mentioned setting of work. In Section 3.1, six set scalarization of type sup-inf and inf-sup, which are extensions of the oriented distance, denoted by $\mathbb{D}_K^{\alpha}(A, B)$ and $\widehat{\mathbb{D}}_K^{\alpha}(A, B)$ are presented, four of which are new. Relationships among them are presented, characterization of these scalarizations are given and, furthermore, some of their main properties are studied as, for example, finitude under suitable assumptions of cone-properness and cone-boundedness, invariance by conic extensions, monotonicity by considering the six set relations introduced by Kuroiwa, and closure property.

In Section 3.2, new characterizations of six set relations of Kuroiwa are derived by using the six set scalarizations $\mathbb{D}_K^\alpha(A,B)$ and $\widehat{\mathbb{D}}_K^\alpha(A,B)$ which are introduced in the former section. Furthermore, some examples to illustrate the results obtained are provided but especially to emphasize that the assumptions required cannot be removed. The importance of these results lies in the fact that they could be applied in Section 4.1 to analyze minimality conditions for a set optimization problem with the set criterion of solution.

In Section 3.3, by considering a solid convex cone K and under suitable assumptions, strict monotonicity for the six set scalarizations $\mathbb{D}_K^\alpha(A,B)$ and $\widehat{\mathbb{D}}_K^\alpha(A,B)$ by using the six strict set relations of Kuroiwa is investigated. To this purpose, some new important results which deal with inequalities for the functions $\mathbb{D}_K^\alpha(A,B)$ and $\widehat{\mathbb{D}}_K^\alpha(A,B)$ when one of the variables is a sum of two sets are presented; moreover, it should be noted that these results do not exist in the literature for the Gerstewitz's function. The results about strict monotonicity which are above mentioned will be applied in Section 4.2 to derive minimality conditions for a set optimization problem with the set criterion of solution. In the literature, there are very few authors who have researched strict monotonicity (see [5, 41, 51, 94, 96, 107]) and, in all these cases, Gerstewitz's function has been used. The results obtained represent an improvement since they require weakest assumptions.

In Section 3.4, by considering a solid convex cone K, new characterizations of six strict set relations of Kuroiwa are derived by using the six set scalarizations $\mathbb{D}_K^\alpha(A,B)$ and $\widehat{\mathbb{D}}_K^\alpha(A,B)$. Moreover, some examples to illustrate the results obtained are provided with the aim to emphasize that the assumptions required cannot be removed. These results will be used in section 4.2 to deduce weak minimality conditions for a set optimization problem with the set criterion of solution. The results stated in this chapter are collected in our works [68, Sections 3 and 4] and [69, Sections 3 and 4].

In Chapter 4, applications to set optimization problems with the set criterion of solution by means of set relations of Kuroiwa are searched. In Section 4.1, by considering some good properties of the six set scalarizations $\mathbb{D}_K^\alpha(A,B)$ and

$\widehat{\mathbb{D}}_K^{\alpha}(A,B)$ presented in Section 3.1 as, for example, finitude, monotonicity and their performance with respect to equivalent sets, and by applying the characterizations of the set relations of Kuroiwa which are given in Section 3.2, several characterizations by scalarization of minimal solutions for six set optimization problems are derived. In Section 4.2, by considering some nice properties of the six set scalarizations $\mathbb{D}_K^{\alpha}(A,B)$ and $\widehat{\mathbb{D}}_K^{\alpha}(A,B)$ presented in Section 3.1 as, for example, finitude and monotonicity, as well as their strict monotonicity studied in Section 3.3 and by applying the characterizations of the strict set relations of Kuroiwa, which are given in Section 3.4, several characterizations by scalarization of weak minimal solutions for six set optimization problem are achieved. The results stated in this chapter are collected in our papers [68, Section 5] and [69, Section 5].

Finally, in Chapter 5, we collect the conclusions and future lines of research which have arisen during the elaboration and preparation of this memory.

1.3 Preliminaries

Let Y be a real topological linear space, unless otherwise is stated. Given a subset $A \subset Y$, we denote the interior, the closure and the boundary of A by $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{bd} A$, respectively. Let $\mathcal{P}_0(Y)$ be the set of all nonempty subsets of Y. For every $A, B \in \mathcal{P}_0(Y)$ and $\lambda \in \mathbb{R}$, we denote

$$A + B = \{y_1 + y_2 : \, y_1 \in A, y_2 \in B\}, \;\; \lambda A = \{\lambda y : \, y \in A\}.$$

We assume that $A + \emptyset = \emptyset + A = \emptyset$, $\lambda\emptyset = \emptyset$, for all $A \subset Y$, and we consider that $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$.

A subset K of Y is a cone if $\lambda K \subset K$ for all $\lambda \geq 0$ ($0 \in K$), a cone K is convex if $K + K \subset K$, is solid if $\operatorname{int} K \neq \emptyset$, is pointed if $K \cap (-K) = \{0\}$ and is proper if $\{0\} \neq \operatorname{cl} K \neq Y$. If $A \in \mathcal{P}_0(Y)$ and the convex cone K is solid, then $A + \operatorname{int} K = \operatorname{int}(A + K)$ (see [38, 129]).

From now on, we shall consider that Y is an ordered space by the preorder $\leq_K$ (reflexive and transitive) generated by a convex cone K defined as follows:

$$x \leq_K y \Leftrightarrow y - x \in K, \;\; \forall x, y \in Y.$$

Moreover, if K is solid, it is defined $x \leq_{\text{int}\, K} y$ if $y - x \in \text{int}\, K$.

Therefore, by using the preorder $\leq_K$, we can find non-comparable elements and indifferent elements in Y. In the case that K is pointed, there are not any indifferent elements since the relation $\leq_K$ is antisymmetric and, therefore, the preorder is a partial order. To compare elements in $\mathbb{R}^n$, Edgeworth and Pareto used the natural ordering cone given by the nonnegative orthant $\mathbb{R}^n_+$, generally called Pareto order, which corresponds to the componentwise partial ordering, that is, for $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$ we define $x \leq_K y$ if and only if $x_i \leq y_i$ for $i = 1, .., n$. In $\mathbb{R}$, we have that $\mathbb{R}^1_+ = \mathbb{R}_+$.

Recall that given $x, y \in Y$, with $x \leq_K y$, the order interval of extremes x and y is defined as $[x, y]_K = \{z \in Y : x \leq_K z,\ z \leq_K y\} = (x + K) \cap (y - K)$.

In [75], a preference relation based on an arbitrary nonempty proper subset S instead of a solid convex cone K is induced, by considering that $x \leq_S y$ if and only if $y - x \in S$, for all $x, y \in S$.

A subset $A \subset Y$ is K-proper if $A + K \neq Y$. We denote by $\mathcal{P}_{0,K}(Y)$ the set of all nonempty K-proper subsets of Y. It is said that A is K-closed if $A + K$ is closed, is K-bounded if for each neighborhood U of zero in Y there exists a positive number t such that $A \subset tU + K$, and is K-compact if any cover of A of the form $\{U_\alpha + K : \alpha \in I, U_\alpha \text{ is open}\}$ admits a finite subcover.

Remark 1.3.1. (i) Every K-compact set is K-closed and K-bounded (see [5, 51, 71, 73, 89, 103, 122, 129]).

(ii) If A is K-bounded, then A is K-proper.

Recall that $A + K$ (resp., $A - K$) is the conic extension of A with respect to K (resp., $(-K)$).

For $T : \mathcal{P}_0(Y) \to \mathbb{R}$, given $y \in Y$ we will write $T(y)$ instead of $T(\{y\})$.

Now, we focus on the following six set relations introduced in 1997 by Kuroiwa [89] which represent certain six binary relations that are generalizations of an ordering for vectors induced by a convex cone in a vector space. These six set relations have been studied, for example, in [59, 65, 79, 86, 87, 93, 96, 113, 122, 131, 132].

Definition 1.3.2. Let $A, B \in \mathcal{P}_0(Y)$. We consider the following set relations:

(i) $A \preccurlyeq^{\forall\forall} B \Leftrightarrow \forall a \in A, \forall b \in B : a - b \in -K \Leftrightarrow A \subset \bigcap_{b \in B} (b - K) \Leftrightarrow A - B \subset -K$.

(ii) $A \preccurlyeq^{\exists\forall} B \Leftrightarrow \exists a \in A, \forall b \in B : a - b \in -K \Leftrightarrow A \cap \bigcap_{b \in B} (b - K) \neq \emptyset$.

(iii) $A \preceq^{\forall\exists} B \Leftrightarrow \forall b \in B, \exists a \in A : a - b \in -K \Leftrightarrow B \subset A + K$.

(iv) $A \preceq^{\exists\forall} B \Leftrightarrow \exists b \in B, \forall a \in A : a - b \in -K \Leftrightarrow \bigcap_{a \in A} (a + K) \cap B \neq \emptyset$.

(v) $A \preccurlyeq^{\forall\exists} B \Leftrightarrow \forall a \in A, \exists b \in B : a - b \in -K \Leftrightarrow A \subset B - K$.

(vi) $A \preccurlyeq^{\exists\exists} B \Leftrightarrow \exists a \in A, \exists b \in B : a - b \in -K \Leftrightarrow A \cap (B - K) \neq \emptyset$.

These relations correspond to [94, Definition 2.1] although there, they are denoted, respectively, $A \preccurlyeq^{(1)} B, \ldots, A \preccurlyeq^{(6)} B$. Two more definitions could be added:

- $A \preceq^{\forall\forall} B \Leftrightarrow \forall b \in B, \forall a \in A : a - b \in -K$, and
- $A \preceq^{\exists\exists} B \Leftrightarrow \exists b \in B, \exists a \in A : a - b \in -K$,

but it is clear that they coincide with $\preccurlyeq^{\forall\forall}$ and $\preccurlyeq^{\exists\exists}$, respectively.

If it is necessary to indicate the cone K, we will write $A \preceq^{\alpha}_{K} B$ (resp., $A \preccurlyeq^{\alpha}_{K} B$), instead of $A \preceq^{\alpha} B$ (resp., $A \preccurlyeq^{\alpha} B$), where $\alpha \in \{\forall\forall, \exists\forall, \forall\exists, \exists\exists\}$.

We observe that the set relations $\preceq^{\forall\exists}$ and $\preccurlyeq^{\forall\exists}$ are, respectively, the lower set less preorder and the upper set less preorder, which are widely studied in the literature (see, for example, [51, 65, 87, 129] and the references there). In [65, 71], the weakest relation $\preccurlyeq^{\exists\exists}$ is named possibly less order relation, and in some papers (see, for instance, [11, 65, 71, 81]) a slight modification of the strong relation $\preccurlyeq^{\forall\forall}$ is called certainly less order relation.

When K is a solid cone, we can define the corresponding strict set relations as follows.

Definition 1.3.3. Let $A, B \in \mathcal{P}_0(Y)$ and K be solid. The relations $\preceq^{\alpha}_{s}$ (respectively, $\preccurlyeq^{\alpha}_{s}$) are the corresponding strict set relations to $\preceq^{\alpha}$ (respectively, $\preccurlyeq^{\alpha}$), and they are defined as in Definition 1.3.2 but changing K by $\operatorname{int} K$.

For instance,

- $A \preceq^{\forall\exists}_{s} B \Leftrightarrow \forall b \in B, \exists a \in A : a - b \in -\operatorname{int} K \Leftrightarrow B \subset A + \operatorname{int} K$,
- $A \preccurlyeq^{\forall\exists}_{s} B \Leftrightarrow \forall a \in A, \exists b \in B : a - b \in -\operatorname{int} K \Leftrightarrow A \subset B - \operatorname{int} K$.

If it is necessary to indicate the cone K, we will write $A \curlyeqprec^{\alpha}_{s,K} B$ or $A \preccurlyeq^{\alpha}_{s,K} B$. The relations $\curlyeqprec^{\forall\exists}_s$ and $\preccurlyeq^{\forall\exists}_s$ have been used, for example, in [5, 51, 107, 129].

We denote

$$\mathcal{R} = \{\preccurlyeq^{\forall\forall}, \preccurlyeq^{\exists\forall}, \curlyeqprec^{\forall\exists}, \curlyeqprec^{\exists\forall}, \preccurlyeq^{\forall\exists}, \preccurlyeq^{\exists\exists}\} \text{ and } \mathcal{R}_s = \{\preccurlyeq^{\forall\forall}_s, \preccurlyeq^{\exists\forall}_s, \curlyeqprec^{\forall\exists}_s, \curlyeqprec^{\exists\forall}_s, \preccurlyeq^{\forall\exists}_s, \preccurlyeq^{\exists\exists}_s\}.$$

If $\precsim$ denotes a set relation of $\mathcal{R}$, then $\precsim_s$ the corresponding strict set relation of $\mathcal{R}_s$.

If we consider a set relation $\precsim^{\alpha}_K$ with $\alpha \in \mathcal{R}$, we can define an equivalent relation with respect to α, denoted by $\sim^{\alpha}$, as follows. If $A, B \in \mathcal{P}_0(Y)$, it is said that A and B are α-equivalent, denoted by $A \sim^{\alpha} B$, if and only if $A \precsim^{\alpha}_K B$ and $B \precsim^{\alpha}_K A$. In this case, they belong to the same equivalence class and we write $B \in [A]^{\alpha}$.

In the literature, there exist other set relations (see [4, 11, 65, 70]), some of them have been defined by using Minkowski difference of sets. In [59] eight types of fuzzy-set relations based on a convex cone are proposed as new comparison criteria of fuzzy sets. In [50, 75, 83] very general definitions of set order relations are introduced, where the involved set describing the domination structure does not need to be convex or a cone. Recently, more general lower set less relation have been defined by considering an arbitrary nonempty proper set S instead of a solid convex cone K [75]. In [80] set order relations equipped with a variable order structure are introduced to compare sets and to study set optimization problems endowed with these variable ordering structures, which generalize the concept of ordering structures in vector optimization [26].

To illustrate Definition 1.3.2 we give a simple example.

Example 1.3.4. Consider $Y = \mathbb{R}$, $K = \mathbb{R}_+$ and the intervals $A = [a_i, a_s]$ and $B = [b_i, b_s]$ with $a_i \leq a_s$ and $b_i \leq b_s$. It is derived the following characterization in this particular case (see also [65, Example 3.2]):

$$\begin{aligned}
&A \preccurlyeq^{\forall\forall} B \;\Leftrightarrow\; a_s \leq b_i, \qquad && A \curlyeqprec^{\exists\forall} B \;\Leftrightarrow\; a_s \leq b_s,\\
&A \preccurlyeq^{\exists\forall} B \;\Leftrightarrow\; a_i \leq b_i, \qquad && A \preccurlyeq^{\forall\exists} B \;\Leftrightarrow\; a_s \leq b_s,\\
&A \curlyeqprec^{\forall\exists} B \;\Leftrightarrow\; a_i \leq b_i, \qquad && A \preccurlyeq^{\exists\exists} B \;\Leftrightarrow\; a_i \leq b_s.
\end{aligned}$$

The next example illustrates Definition 1.3.3.

Example 1.3.5. Let $K \subset Y$ be a solid cone and let us consider the order intervals $A = [a_i, a_s]_K$ and $B = [b_i, b_s]_K$ with $a_i \leq_K a_s$ and $b_i \leq_K b_s$. It can be checked the following characterization (see also [65, Example 3.2]):

$$\begin{aligned} A \preccurlyeq_s^{\forall\forall} B &\Leftrightarrow a_s \leq_{\operatorname{int} K} b_i, & A \curlyeqprec_s^{\exists\forall} B &\Leftrightarrow a_s \leq_{\operatorname{int} K} b_s, \\ A \preccurlyeq_s^{\exists\forall} B &\Leftrightarrow a_i \leq_{\operatorname{int} K} b_i, & A \preccurlyeq_s^{\forall\exists} B &\Leftrightarrow a_s \leq_{\operatorname{int} K} b_s, \\ A \curlyeqprec_s^{\forall\exists} B &\Leftrightarrow a_i \leq_{\operatorname{int} K} b_i, & A \preccurlyeq_s^{\exists\exists} B &\Leftrightarrow a_i \leq_{\operatorname{int} K} b_s. \end{aligned}$$

The following properties and relationships between the set relations given in Definitions 1.3.2 and 1.3.3 are immediate or well-known.

The following remark can be found in [94].

Remark 1.3.6. (i) The five first relations of $\mathcal{R}$ and $\mathcal{R}_s$ are transitive.

(ii) The relations $\curlyeqprec^{\forall\exists}$, $\preccurlyeq^{\forall\exists}$ and $\preccurlyeq^{\exists\exists}$ are reflexive.

The next properties can be seen, for example, in [94, 113].

Proposition 1.3.7. *Let $A, B \in \mathcal{P}_0(Y)$. Then*

(i) $A \preccurlyeq^{\forall\forall} B \Rightarrow A \preccurlyeq^{\exists\forall} B \Rightarrow A \curlyeqprec^{\forall\exists} B \Rightarrow A \preccurlyeq^{\exists\exists} B$.

(ii) $A \preccurlyeq^{\forall\forall} B \Rightarrow A \curlyeqprec^{\exists\forall} B \Rightarrow A \preccurlyeq^{\forall\exists} B \Rightarrow A \preccurlyeq^{\exists\exists} B$.

(iii) Parts (i) and (ii) are also true for the corresponding strict set relations.

(iv) $A \precsim_s B \Rightarrow A \precsim B$, *for all* $\precsim \in \mathcal{R}$.

Lemma 1.3.8. *Let $\precsim \in \mathcal{R}$ and $A, B \in \mathcal{P}_0(Y)$. Then*

(i) $A \precsim B \Rightarrow A + y \precsim B + y$, *for all* $y \in Y$.

(ii) $A \precsim B \Rightarrow tA \precsim tB$, *for all* $t > 0$.

Next, we collect some basic and useful properties that show a certain duality between two pairs of relations and the corresponding strict relations.

Lemma 1.3.9. *Let $A, B \in \mathcal{P}_0(Y)$. Then*

(i) $A \curlyeqprec_K^{\forall\exists} B \Leftrightarrow B \curlyeqprec_{-K}^{\forall\exists} A$.

(ii) $A \curlyeqprec_K^{\exists\forall} B \Leftrightarrow B \preccurlyeq_{-K}^{\exists\forall} A$.

(iii) $A \preccurlyeq_s^{\forall\exists} B \Leftrightarrow B \curlyeqprec_{s,-K}^{\forall\exists} A$.

(iv) $A \curlyeqprec_s^{\exists\forall} B \Leftrightarrow B \preccurlyeq_{s,-K}^{\exists\forall} A$.

It is also worth emphasizing that the lower set less preorder and the upper set less preorder are not antisymmetric and, hence, for these two set relations there exist indifferent sets. That is, given $A, B \in \mathcal{P}_0(Y)$, if $A \preccurlyeq^{\forall\exists} B$ and $B \preccurlyeq^{\forall\exists} A$ then this does not imply that A is equal to B (the same can be said for $\curlyeqprec^{\forall\exists}$). In this case, it is well known that $A \sim^{\preccurlyeq^{\forall\exists}} B$ if and only if $A + K = B + K$ and $A \sim^{\curlyeqprec^{\forall\exists}} B$ if and only if $A - K = B - K$ [71]. These two statements could be false for the relations $\sim^{\preccurlyeq_s^{\forall\exists}}$ and $\sim^{\curlyeqprec_s^{\forall\exists}}$, that is, could be fulfilled $A + \operatorname{int} K = B + \operatorname{int} K$ (or $A - \operatorname{int} K = B - \operatorname{int} K$) although $A \not\sim^{\preccurlyeq_s^{\forall\exists}} B$ (or $A \not\sim^{\curlyeqprec_s^{\forall\exists}} B$).

Now, recall the definition of $\precsim$-monotonicity for a function being $\precsim$ a set relation.

Definition 1.3.10. Let $T : \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$ and $\precsim \in \mathcal{R}$.

(a) T is $\precsim$-increasing (resp., $\precsim$-decreasing) if for all $A, B \in \mathcal{P}_0(Y)$, $A \precsim B$ implies $T(A) \leq T(B)$ (resp., $T(A) \geq T(B)$).

(b) T is strictly $\precsim_s$-increasing (resp., $\precsim_s$-decreasing) if for all $A, B \in \mathcal{P}_0(Y)$, $A \precsim_s B$ implies $T(A) < T(B)$ (resp., $T(A) > T(B)$).

It is said that $g : Y \to \mathbb{R}$ is K-increasing (resp., K-decreasing) if for all $y_1, y_2 \in Y$, $y_1 \leq_K y_2$ implies $g(y_1) \leq g(y_2)$ (resp., $g(y_1) \geq g(y_2)$). Note that g is K-increasing if and only if g is $(-K)$-decreasing.

In the following lemma, we relate increasing and decreasing maps with respect to (in short, w.r.t.) "dual" relations.

Lemma 1.3.11. *Let $T : \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$.*

(i) T is $\preccurlyeq_{-K}^{\forall\exists}$-increasing $\Leftrightarrow$ T is $\curlyeqprec_K^{\forall\exists}$-decreasing.

(ii) T is $\preccurlyeq_{-K}^{\forall\exists}$-decreasing $\Leftrightarrow$ T is $\curlyeqprec_K^{\forall\exists}$-increasing.

Both equivalences are true if we replace $\preccurlyeq_{-K}^{\forall\exists}$ with $\preccurlyeq_{-K}^{\exists\forall}$ and $\curlyeqprec_K^{\forall\exists}$ with $\curlyeqprec_K^{\exists\forall}$.

A relation $\precsim$ in $\mathcal{P}_0(Y)$ can be seen as a subset of $\mathcal{P}_0(Y) \times \mathcal{P}_0(Y)$, which is defined by $(A, B) \in \precsim$ if and only if $A \precsim B$.

Definition 1.3.12. Let $\precsim_1$ and $\precsim_2$ two set relations defined on $\mathcal{P}_0(Y)$. It is said that $\precsim_1$ implies $\precsim_2$ if for all $A, B \in \mathcal{P}_0(Y)$, $A \precsim_1 B$ implies $A \precsim_2 B$, that is, if as subsets one has $\precsim_1 \subset \precsim_2$.

Now, we are going to use the last definition to relate monotonicity for two relations.

Lemma 1.3.13. *(i) If $\precsim_1$ implies $\precsim_2$ and T is $\precsim_2$-increasing (resp., decreasing), then T is $\precsim_1$-increasing (resp., decreasing).*

(i) If $\precsim_1$ implies $\precsim_2$ and T is strictly $\precsim_2$-increasing (resp., decreasing), then T is strictly $\precsim_1$-increasing (resp., decreasing).

Next, we recall the Gerstewitz's scalarization function introduced in 1990 (see [5,12,31,36,38,103,126,127]) which is defined, in contrast with other scalarization functions existing in the literature, in a topological vector space with a solid convex cone K.

Definition 1.3.14. Let $K \subset Y$ be a convex closed and solid cone, and $e \in \operatorname{int} K$. The Gerstewitz's function $h_{\inf} : Y \times Y \to \mathbb{R}$, is defined as follows:

$$h_{\inf}(x, y) = \inf\{t \in \mathbb{R} : x \leq_K te + y\},$$

Moreover, it can be defined $h_{\sup}(x, y) = \sup\{t \in \mathbb{R} : te + y \leq_K x\}$ and it is shown that $h_{\sup}(x, y) = -h_{\inf}(-x, -y)$.

In [12, 31, 36, 43, 51, 71, 99, 102, 103, 110, 121, 124] many of its properties can be seen. These properties were used to deal with various problems as, for instance, existence and continuity of solutions, optimality conditions, gap functions, duality, vector variational principles, well-posedness, vector minimax inequalities and vector equilibrium problems.

In [12] it is proved that for all $z \in Y$ it is verified that $h_{\inf}(z, 0) = \max_{\xi \in K_e} \langle \xi, z \rangle$, where $K_e = \{\xi \in K^+ : \langle \xi, e \rangle = 1\}$ with $e \in \operatorname{int} K$ and K^+ the positive polar cone of K in the dual space Y^*.

When Y is the l-dimensional Euclidean space $\mathbb{R}^l$, $K = \mathbb{R}^l_+$, $e = (e_1, ..., e_l)^T$ and $x = (x_1, ..., x_l)^T$, then we have that $h_{\inf}(x, y)$ can be rewritten for $y = (y_1, ..., y_l)^T$ as follows

$$h_{\inf}(x, y) = \max\{(x_i - y_i)/e_i : 1 \leq i \leq l\}.$$

The function $h_{\inf}(x, y)$ assigns the smallest value t such that the property $x \in te + y - K$ is fulfilled, it was used to give nonconvex separation theorems and, moreover, it has extensive applications in vector optimization. Gertewitz's function was also introduced and used in abstract convexity analysis under the name of topical function [117], in mathematical finance under the name of coherent risk measure [71] and in economics under the name of shortage function. This scalarizing function is the smallest strictly monotonic (increasing) function with respect to the ordering structure [103]. It should be noted that certain Minkowski functionals and norms coincide with Gerstewitz's function on a subset of the space.

In [75], a generalized version of Gerstewitz's function was given based on an arbitrary nonempty proper set S instead of a solid convex cone K.

In 2006, Hamel and Löhne [48] introduced the next function which is an extension to set optimization of Gerstewitz's function, by using a solid convex cone K. If $e \in \operatorname{int} K$, it is defined $h^l_{\inf} : \mathcal{P}_0(Y) \times \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$ as follows:

$$h^l_{\inf}(A, B) = \inf\{t \in \mathbb{R} : A \preccurlyeq^{\forall\exists} te + B\}. \tag{1.1}$$

Note that A is not a K-proper set if and only if $h^l_{\inf}(A, B) = -\infty$ and, moreover, if B is not a K-proper set and A is a K-proper set, then $h^l_{\inf}(A, B) = +\infty$ (see [38, 41]).

This scalarizing function measure how far a set needs to be moved towards a specific direction to fulfil each set relation above while the other set is fixed, and were used to obtain minimal set theorems and new variants of Ekeland's principle in a topological vector space without strong assumptions as convexity. Hamel and Löhne considered a fixed set that plays the role of parameter which is crucial in order to characterize minimal solutions of set-valued optimization problems with set orderings through solutions of associated scalar optimization problems.

In the literature, expressions using sup-inf of the Gerstewitz's function can be found in several papers (see [38, 41, 49, 50, 81–84]).

In the next theorem (see [38, Theorem 5.5]) the function (1.1) is expressed by means of sup-inf (see [38, 41, 51]).

Theorem 1.3.15. *Given $A, B \in \mathcal{P}_0(Y)$, then*

$$h^l_{\inf}(A, B) = \sup_{y \in B} \inf_{x \in A} \inf\{t \in \mathbb{R} : te + y \in x + K\}.$$

In [38, 41, 94] this function has been used to obtain necessary and sufficient optimality conditions in set optimization problems with the set criterion.

A consequence of Theorem 1.3.15 is the next corollary (see [38, 41, 51]).

Corollary 1.3.16. *Given $A, B \in \mathcal{P}_0(Y)$ and $y \in Y$, then*

(i) $h^l_{\inf}(A, y) := \inf\{t \in \mathbb{R} : te + y \in A + K\} = \inf_{x\in A} \inf\{t \in \mathbb{R} : te + y \in x + K\}$.

(ii) $h^l_{\inf}(A, B) = \sup_{y\in B} h^l_{\inf}(A, y)$.

It should be noted that in the literature for Hamel and Löhne's function, we can find other versions in [95, 96, 113, 131] by using the set relations given in Definition 1.3.2 and, moreover, we can see that there exist other versions type supremun in [79, 93–95]. Moreover, we can find other versions of this function by using a new set order relations defined by using Minkowski difference of sets (see [4, 70]).

There are several authors who have emphasized the importance of considering extensions of Gerstewitz's function, and have investigated their properties (see [5, 38, 41–43, 49, 51, 70, 73, 94, 95, 107, 134]) and its applications in set-valued optimization problems (see, for instance, [5, 38, 41, 51, 70, 73, 94, 95, 107]).

In Khoshkhabar-amiranloo et al. [73] and in Sach [118] the nonlinear scalarization functions $\varphi^l_{e,B}, \varphi^u_{e,B} : \mathcal{P}_0(Y) \to \mathbb{R}_+ \cup \{+\infty\}$, with $B \in \mathcal{P}_0(Y)$, are given as follows:

$$\varphi^l_{e,B}(A) = \inf\{t \geq 0 : A \preccurlyeq^{\forall\exists} te + B\},$$
$$\varphi^u_{e,B}(A) = \inf\{t \geq 0 : A \precsim^{\forall\exists} te + B\}.$$

Note that

$$\varphi^l_{e,B}(A) = \max\{0, h^l_{\inf}(A, B)\}. \tag{1.2}$$

Using (1.2) and Corollary 1.3.16(ii), it follows that

$$\varphi^l_{e,B}(A) = \sup_{y\in B} \varphi^l_{e,y}(A). \tag{1.3}$$

Similarly,

$$\varphi^u_{e,B}(A) = \sup_{x\in A} \varphi^u_{e,B}(x).$$

Moreover, it is verified that $\varphi^u_{e,B}(A) = \varphi^l_{e,-A}(-B)$.

These functions are used in [73] to characterize some optimal solutions to a set optimization problem with the set criterion.

Let $y \in Y$ and $A, B \subset Y$, U_0 the unit open ball, $\overline{U}_0$ the unit closed ball in Y and $U_y(\varepsilon)$ is the open ball with center y and radius $\varepsilon > 0$. Recall that the distance of y to a set A is given by $d(y, A) = \inf_{x\in A} \|x - y\|$, being $+\infty$ if $A = \emptyset$. The distance between two sets is defined by $d(A, B) = \inf_{x\in A} d(x, B) = \inf_{x\in A, y\in B} d(x, y)$. The excess of the set B over the set A is defined by $\rho_B(A) = \sup_{y\in B} d(y, A)$, being $+\infty$ if $A = \emptyset$. We observe that $d(A, B) \leq \rho_A(B)$.

Recall some properties of the distance of a point to a set.

Proposition 1.3.17. *Let $x, y \in Y$ and $A \subset Y$. Then*

(i) $d(y, -A) = d(-y, A)$ and $d(x, A) = d(y + x, y + A)$.

(ii) $d(x - y, Y \backslash (-A)) = d(y, Y \backslash (x + A))$.

(iii) $d(y, A + K) = \inf_{x\in A} d(y, x + K)$.

In 1979, Hiriart-Urruty [55] introduced the next nonlinear scalarization function which was used to deal with non-smooth optimization problems from the geometric point of view.

Definition 1.3.18. Let $A \subset Y$. The oriented distance $D(\cdot, A) : Y \to \mathbb{R} \cup \{\pm\infty\}$ is defined as follows:

$$D(y, A) = d(y, A) - d(y, Y \backslash A) = \begin{cases} d(y, A) & \text{if } y \in Y \backslash A \\ -d(y, Y \backslash A) & \text{if } y \in A. \end{cases}$$

It is considered that $D(y, \emptyset) = +\infty$ and $D(y, Y) = -\infty$. It holds that $D(y, A) \leq d(y, A)$. This function has been used in vector optimization to give necessary optimality conditions.

It should be emphasized that the main advantage of the oriented distance function $D(y, A)$, in contrast to Gerstewitz's scalarization function, is that $D(y, A)$ does not require that A be a solid set.

In the sequel, we collect the basic properties of this function (see, for instance, [3, 23, 55, 71, 129, 133]).

Lemma 1.3.19. *Let* $A, B \in \mathcal{P}_0(Y)$*,* $A \neq Y$ *and* $y \in Y$*. Then*

(i) $D(x, A) \in \mathbb{R}$ *for all* $x \in Y$ *and* $D(\cdot, A)$ *is Lipschitz of rank* 1.

(ii) If A *is convex, then* $D(\cdot, A)$ *is convex.*

(iii) $D(y, A) < 0 \Leftrightarrow y \in \operatorname{int} A$.

(iv) $D(y, A) = 0 \Leftrightarrow y \in \operatorname{bd} A$.

(v) $D(y, A) > 0 \Leftrightarrow y \notin \operatorname{cl} A$.

(vi) If A *is a cone, then* $D(\cdot, A)$ *is positively homogeneous.*

(vii) If $A \subset B$ *then* $D(y, B) \leq D(y, A)$.

(viii) If K *is a closed convex cone, then* $D(\cdot, -K)$ *is* K*-increasing.*

If, in addition, K *is solid, then*

$$y_1 \leq_{\operatorname{int} K} y_2 \Rightarrow D(y_1, -K) < D(y_2, -K).$$

(ix) If K *is a convex cone, then* $D(y_1 + y_2, -K) \leq D(y_1, -K) + D(y_2, -K)$*, for all* $y_1, y_2 \in Y$.

(x) If $\operatorname{int} A \neq \emptyset$ *and* $\operatorname{int} \operatorname{cl} A = \operatorname{int} A$*, then* $D(y, \operatorname{cl} A) = D(y, A)$.

(xi) If $\operatorname{int} A \neq \emptyset$ *and* $\operatorname{cl} \operatorname{int} A = \operatorname{cl} A$*, then* $D(y, \operatorname{int} A) = D(y, A)$.

(xii) If the convex cone K *is solid, then* $D(y, A + \operatorname{int} K) = D(y, A + K)$.

(xiii) $D(-y, -A) = D(y, A)$.

(xiv) $D(y, A) = D(x + y, x + A)$*, for all* $x \in Y$.

(xv) $D(y, Y \backslash A) = -D(y, A)$.

Proof. We only have to prove parts (x), (xi) and (xii), the remainder can be found in [55, 71, 133].

(x) This part has the same proof that [71, Remark 5.3.1].

(xi) Firstly, using the basic property $d(y, M) = d(y, \operatorname{cl} M)$ and the hypothesis, we have

$$d(y, \operatorname{int} A) = d(y, \operatorname{cl} \operatorname{int} A) = d(y, \operatorname{cl} A) = d(y, A).$$

Secondly, $d(y, Y \backslash \operatorname{int} A) = d(y, \operatorname{cl}(Y \backslash A)) = d(y, Y \backslash A)$. Now, the conclusion is clear taking into account Definition 1.3.18.

(xii) From [9, Lemma 2.5] it follows that $\operatorname{cl} \operatorname{int}(A + K) = \operatorname{cl}(A + \operatorname{int} K) = \operatorname{cl}(A + K)$. Now, part (xii) is obtained from part (xi) with $A + K$ instead of A. □

Remark 1.3.20. From part (iii) it follows that $D(y, A) = d(y, A) \geq 0$ whenever $\operatorname{int} A = \emptyset$.

It is observed that the oriented distance function $D(\cdot, A)$ has good properties by depending on the properties of the set A which has been considered. It is well-known that the main properties between Gerstewitz's function and the oriented distance function are very similar. Some relations between Gerstewitz's function and the oriented distance function of Hiriart-Urruty can be found in [98].

In [29,32–34,100] some dual representation results for Hiriart-Urruty function were given. In these papers, it is said that when A is a convex set, it holds that

$$D(y, A) = \sup_{\|\xi\|=1} (\langle \xi, y \rangle - \sup_{a \in A} \langle \xi, a \rangle).$$

Moreover, in the case that $A = -K$ is a convex cone, it follows that

$$D(y, -K) = \sup_{\|\xi\|=1, \xi \in K^+} \langle \xi, y \rangle,$$

where $K^+ = \{\xi \in Y^* : \langle \xi, y \rangle \geq 0, y \in K\}$ is the positive polar cone corresponding to K and Y^* is the dual space of Y.

It should be noted that there are only a few set extensions of the oriented distance function of Hiriart-Urruty defined in a normed space with a convex cone not necessarily solid (see [18, 38, 45, 129]).

In the setting of the order variable structures, some expressions of type sup-inf by using oriented distance function can be found in the literature [3].

The next example illustrates Definition 1.3.18.

Example 1.3.21. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$ and $y = (y_1, y_2) \in \mathbb{R}^2$. Then, it holds

$$D(y, -\mathbb{R}^2_+) = \begin{cases} \sqrt{(y_1)^2 + (y_2)^2} & \text{if } y_1, y_2 \geq 0 \\ y_2 & \text{if } y_1 \leq 0, y_2 \geq 0 \\ \max\{y_1, y_2\} & \text{if } y_1, y_2 \leq 0 \\ y_1 & \text{if } y_1 \geq 0, y_2 \leq 0. \end{cases}$$

In the following, we give some results that will be needed later on.

Lemma 1.3.22. *Let $A \in \mathcal{P}_0(Y)$, $y \in Y$ and $r \geq 0$. If $y \notin A + r\overline{U}_0$, then $d(y, A) = r + d(y, A + r\overline{U}_0)$.*

Proof. Let $R = r + d(y, A + r\overline{U}_0)$. We have to prove: 1) $d(y, a) \geq R$, for all $a \in A$ and 2) for all $\varepsilon > 0$, there exists $a \in A$ such that $d(y, a) < R + \varepsilon$.

1) Let $a \in A$ and take $u = r\frac{y-a}{\|y-a\|}$. It is clear that $z = a + u \in A + r\overline{U}_0$ since $\|u\| = r$. Then

$$d(a, y) = d(a, z) + d(z, y) \geq d(a, z) + d(y, A + r\overline{U}_0) = r + d(y, A + r\overline{U}_0) = R.$$

2) Given $\varepsilon > 0$, let $r' = d(y, A + r\overline{U}_0)$. Then, there exists $z \in A + r\overline{U}_0$ such that $d(y, z) < r' + \frac{\varepsilon}{2}$. Moreover, it is verified that $d(z, A) \leq r$ and, therefore, there exists $a \in A$ such that $d(z, a) < r + \frac{\varepsilon}{2}$. Consequently, $d(y, a) \leq d(y, z) + d(z, a) \leq r' + \frac{\varepsilon}{2} + r + \frac{\varepsilon}{2} = R + \varepsilon$. □

Remark 1.3.23. In general, if $\operatorname{int} A \neq \emptyset$, $\operatorname{int} A \subset S \subset \operatorname{cl} A$ and $\operatorname{cl}\operatorname{int} A = \operatorname{cl} A$, then $d(y, S) = d(y, \operatorname{int} A) = d(y, \operatorname{cl} A)$.

Lemma 1.3.24. *Let $A \in \mathcal{P}_0(Y)$, $y \in Y$ and $r \geq 0$. It is verified that $y \in \operatorname{cl}(A + rU_0)$ if and only if $d(y, A) \leq r$. This is equivalent to say that $\operatorname{cl}(A + rU_0) = \{y \in Y : d(y, A) \leq r\}$.*

Proof. ($\Rightarrow$) If $r = 0$, it is clear. So, let $r > 0$ and $A_r := \{y \in Y : d(y, A) \leq r\}$. Then, we have

$$A + rU_0 \subset A + r\overline{U}_0 \subset A_r,$$

since $d(a + u, a) = \|u\| \leq r$, for all $u \in r\overline{U}_0$ and all $a \in A$. As the function $d(\cdot, A)$ is continuous, we deduce that A_r is closed and, consequently, $\operatorname{cl}(A + rU_0) \subset A_r$.

($\Leftarrow$) Suppose that $d(y, A) \leq r$ and $y \notin \operatorname{cl}(A + rU_0)$. Then, there exists $\varepsilon > 0$ such that $U_y(\varepsilon) \cap (A + rU_0) = \emptyset$, and thus $d(y, A + rU_0) \geq \varepsilon$. Therefore, by Lemma 1.3.22, we derive that $d(y, A) = r + d(y, A + rU_0) \geq r + \varepsilon > r$, and this is a contradiction. □

Corollary 1.3.25. *Let $A \in \mathcal{P}_0(Y)$. If $0 \leq r' < r$, then $\operatorname{cl}(A + r'U_0) \subset A + rU_0$.*

Proof. Let $y \in \operatorname{cl}(A + r'U_0)$. By Lemma 1.3.24, we have that $d(y, A) \leq r'$. Assume that $y \notin A + rU_0$. If $y \in A + r\overline{U}_0$, then $d(y, A) = r$ and this is a contradiction. If $y \notin A + r\overline{U}_0$, by Lemma 1.3.22, it is verified that $d(y, A) = r + d(y, A + r\overline{U}_0) \geq r$, and this is a contradiction too. □

Lemma 1.3.26. *For $y_1, y_2 \in Y$ and for all $A \in \mathcal{P}_0(Y)$, we have that $d(y_1, A) - d(y_2, A) \leq d(y_1, y_2)$.*

In the rest of this chapter, we assume that $K \subset Y$ is a closed convex pointed and solid cone. So, if $e \in \operatorname{int} K$ then the order interval associated with K is given by $[-e, e]_K = (-e + K) \cap (e - K)$.

Next, we recall the definition of norm $\|.\|_e$ generated by Minkowski's functional of an order interval with $e \in \operatorname{int} K$ (see [35, 38, 61, 62, 103, 109, 133]).

Definition 1.3.27. Let $y \in Y$ and $e \in \operatorname{int} K$. The norm $\|.\|_e$ generated by Minkowski's functional is defined as follows:

$$\|y\|_e = \inf\{t > 0 : (1/t)y \in [-e, e]_K\}.$$

In [38], with this norm it is proved that for all $y \in Y$ and $A \subset Y$,

$$d(y, A) = \inf\{t \in \mathbb{R}_+ : y \in t[-e, e]_K + A\}. \tag{1.4}$$

Moreover, we have that $t[-e, e]_K + K = -te + K$, for all $t \in \mathbb{R}_+$ and, therefore,

$$t[-e, e]_K + A + K = -te + A + K, \forall t \in \mathbb{R}_+. \tag{1.5}$$

In the next proposition, we obtain expressions for the distance between a point and the conic extension to a set.

Proposition 1.3.28. *If $y \in Y$, $A \subset Y$ and we consider the norm $\|.\|_e$, then we have*

$$d(y, A + K) = \inf\{t \in \mathbb{R}_+ : y \in -te + A + K\} = \varphi^l_{e,y}(A), \tag{1.6}$$

$$d(y, A - K) = \inf\{t \in \mathbb{R}_+ : y \in te + A - K\} = \varphi^u_{e,A}(y). \tag{1.7}$$

Proof. Firstly, the first equality in (1.6) is a consequence of (1.4) and (1.5), and the second one is clear. Secondly, we have that $t[-e, e]_K - K = te - K$, for all $t \in \mathbb{R}_+$ and, consequently, $t[-e, e]_K + A - K = te + A - K$. Now, the equality (1.7) follows as a consequence of (1.4). □

Remark 1.3.29. (a) If $x \notin -\operatorname{int} K$, then $h_{\inf}(x,0) \geq 0$ and, therefore, $h_{\inf}(x,0) = d(x,-K) = \inf\{t \in \mathbb{R}_+ : x \in te - K\} = \varphi^u_{e,0}(x)$.

(b) We have that $d(x-y,-K)$ coincides with $\varphi^l_{e,y}(x)$ and $\varphi^u_{e,y}(x)$. Indeed, taking into account (1.6), $\varphi^l_{e,y}(x) = d(y, x+K) = d(y-x, K) = d(x-y,-K)$, and applying (1.7), $\varphi^u_{e,y}(x) = d(x, y-K) = d(x-y,-K)$ (we have also used Proposition 1.3.17(i)).

After that, it is proved that the function $\varphi^l_{e,B}(A)$ (respectively, $\varphi^u_{e,B}(A)$), coincides with the excess of B (respectively, A) over $A+K$ (respectively, over $B-K$).

Theorem 1.3.30. *If $A, B \subset Y$ and we consider the norm $\|.\|_e$, then*

(i) $\varphi^l_{e,B}(A) = \rho_B(A+K)$.

(ii) $\varphi^u_{e,B}(A) = \rho_A(B-K)$.

Proof. We prove only part (i) since the proof of part (ii) is similar. By the characterization for the function $\varphi^l_{e,B}(A)$ given in (1.3), Proposition 1.3.28 and definition of $\rho_B(A+K)$, we have that

$$\varphi^l_{e,B}(A) = \sup_{y\in B} \varphi^l_{e,y}(A) = \sup_{y\in B} d(y, A+K) = \rho_B(A+K).$$

□

The results stated in this chapter are collected in [66, Section 2].

Chapter 2

Set scalarization functions

In this chapter, we have gathered some set scalarization functions available in the literature, which are extensions of Gerstewitz's function either of the oriented distance function of Hiriart-Urruty. Set extensions of Gerstewitz's function can be found in (see [5, 38, 42, 43, 48, 51, 70, 73, 94, 95, 103, 107, 134]) and, to the best of our knowledge, the first set extension of Gerstewitz's function defined on a topological linear space with a solid convex cone K was introduced by Hamel and Löhne [48] in 2006, based on two set order relations introduced in [89], such as the lower set less preorder and the upper set less preorder of Kuroiwa. It should be noted that there are only a few set extensions of the oriented distance function of Hiriart-Urruty defined in a normed space with a convex cone K not necessarily solid (see [38, 45, 129]). We recall the set extension of oriented distance function of Hiriart-Urruty introduced by Ha [45] and we present a new set scalarization function of type sup-inf which is an extension of the oriented distance function of Hiriart-Urruty too. It is worth noting that the generalized oriented distance functions are simpler in structure and easier to calculate than generalized Gerstewitz's function (see [37, 129]).

The results stated in this chapter are collected in [66, Sections 3,4 and 5] and [67, Sections 3 and 4].

2.1 Relations among set scalarizations

We recall set scalarization functions between two sets existing in the literature, which are concerned with Gerstewitz's function and set oriented distance of Hiriart-Urruty (see [18, 38, 129]). We call set oriented distances to the set extensions of the oriented distance function, which are generalizations to sets of the oriented distance function of Hiriart-Urruty and defined in a normed space with a convex cone K not necessarily solid. Moreover, a new set extension of oriented distance of type sup-inf is introduced and some new relationships among the set scalarization mentioned above are established.

In Crespi et al. [18] it is introduced the next scalarization function $\mathcal{D} : \mathcal{P}_0(Y) \times 2^Y \to \mathbb{R} \cup \{\pm\infty\}$.

Definition 2.1.1. Let $A \in \mathcal{P}_0(Y)$ and $B \subset Y$. The oriented distance function of Crespi et al. for A and B is given as follows:

$$\mathcal{D}(A, B) = \inf_{x \in A} D(x, B). \tag{2.1}$$

If $B = \emptyset$, then $\mathcal{D}(A, \emptyset) = +\infty$. As $D(x, B) \leq d(x, B)$ if we take infima in $x \in A$, we obtain that $\mathcal{D}(A, B) \leq d(A, B)$. If $\operatorname{int} B = \emptyset$, then $\mathcal{D}(A, B) = d(A, B)$ according to Remark 1.3.20

This function is used in [18] to characterize minimal and weak minimal solutions to a set optimization problem with the vector criterion.

In the next proposition we obtain a characterization for the distance of Crespi et al.

Theorem 2.1.2. *If $A \in \mathcal{P}_0(Y)$ and $B \subset Y$, then*

$$\mathcal{D}(A, B) = d(A, B) - \rho_A(Y \backslash B) = \begin{cases} d(A, B) & \text{if } A \cap B = \emptyset \\ -\rho_A(Y \backslash B) & \text{if } A \cap B \neq \emptyset. \end{cases}$$

Proof. First of all, if $A \cap B = \emptyset$, then $x \notin B$, for all $x \in A$ and, in this case, $D(x, B) = d(x, B)$, so we have $\mathcal{D}(A, B) = \inf_{x \in A} d(x, B) = d(A, B)$ and we conclude.

Secondly, by Definition 2.1.1 we have that $\mathcal{D}(A,B) = \inf_{x\in A} D(x,B)$. Now, if we suppose that $A \cap B \neq \emptyset$, then $\inf_{x\in A} D(x,B) = \inf_{x\in A\cap B} D(x,B)$. Furthermore, it is verified that $\inf_{x\in A\cap B} D(x,B) = \inf_{x\in A\cap B}[-d(x,Y\backslash B)]$ by Definition 1.3.18. Since $\inf(-f) = -\sup f$, we deduce that $\inf_{x\in A\cap B}[-d(x,Y\backslash B)] = -\sup_{x\in A\cap B} d(x,Y\backslash B)$. Finally, as $\sup_{x\in A\cap B} d(x,Y\backslash B) = \sup_{x\in A} d(x,Y\backslash B)$, if we apply the definition of excess, it results that

$$-\sup_{x\in A\cap B} d(x,Y\backslash B) = -\sup_{x\in A} d(x,Y\backslash B) = -\rho_A(Y\backslash B),$$

and we conclude. □

Now, the following oriented distance functions between two sets are introduced. As the previous one, it is not required that the cone K is solid.

Definition 2.1.3. Let $B \in \mathcal{P}_0(Y)$. The functions $\delta_B, \Delta_B, \widehat{\Delta}_B : 2^Y \to \mathbb{R} \cup \{\pm\infty\}$ are given as follows:

$$\begin{aligned} \delta_B(A) &= \rho_B(A) - d(B, Y\backslash A), \\ \Delta_B(A) &= \delta_B(A+K), \\ \widehat{\Delta}_B(A) &= \delta_B(A-K). \end{aligned} \tag{2.2}$$

For $A=\emptyset$, we have $\delta_B(\emptyset) = +\infty$, and if $A = Y$, then $\delta_B(Y) = -\infty$.

The function Δ_B is the oriented distance between two sets given by Gutiérrez et al. [38].

If $A=\emptyset$, then $\Delta_B(\emptyset) = +\infty$ and, moreover, $\Delta_B(A) = -\infty$ if and only if A is not K-proper [38]. Analogously, if $A=\emptyset$, then $\widehat{\Delta}_B(\emptyset) = +\infty$ and, furthermore, $\widehat{\Delta}_B(A) = -\infty$ if and only if A is not $(-K)$-proper.

Remark 2.1.4. (a) We have that $\Delta_y(x) = D(x-y, -K)$, for all $x, y \in Y$. Indeed, $\Delta_y(x) = \rho_y(x+K) - d(y, Y\backslash(x+K)) = d(x-y,-K) - d(x-y, Y\backslash(-K))$, for all $x,y \in Y$ (we have used Proposition 1.3.17(ii)).

(b) We have that $\Delta_y(A) = D(y, A+K)$, for all $y \in Y$ and $A \subset Y$.

In Xu and Li [129] the nonlinear scalarization function $\mathfrak{D}_A : \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$ is introduced as follows.

Definition 2.1.5. Let $A \subset Y$ and $B \in \mathcal{P}_0(Y)$. The oriented distance of Xu and Li for A and B is given as follows:

$$\mathfrak{D}_A(B) = \sup_{y \in B} D(y, A). \tag{2.3}$$

If $A = \emptyset$, then $\mathfrak{D}_\emptyset(B) = +\infty$, and $\mathfrak{D}_Y(B) = -\infty$ if $A = Y$. We have $\mathfrak{D}_A(B) \leq \rho_B(A)$. If $\operatorname{int} A = \emptyset$, then $\mathfrak{D}_A(B) = \rho_B(A)$ according to Remark 1.3.20.

Remark 2.1.6. We have that $\mathcal{D}(B, A) \leq \mathfrak{D}_A(B)$ since $\mathcal{D}(B, A) = \inf_{y \in B} D(y, A)$ and $\mathfrak{D}_A(B) = \sup_{y \in B} D(y, A)$. Thus, $\mathcal{D}(B, A) \leq \mathfrak{D}_A(B) \leq \rho_B(A)$.

In [129] the function $\mathfrak{D}_A$ is applied to obtain alternative theorems, some optimality conditions and scalar representations to set optimization problems with the set criterion.

After that, we present a relation between the oriented distances of Crespi et al. (2.1) and Xu and Li (2.3).

Theorem 2.1.7. *If $A \subset Y$ and $B \in \mathcal{P}_0(Y)$, then we have*

$$\mathfrak{D}_A(B) = -\mathcal{D}(B, Y \backslash A) = \rho_B(A) - d(B, Y \backslash A) = \delta_B(A).$$

Proof. Taking into account that $\sup f = -\inf(-f)$ and Lemma 1.3.19(xv), we have

$$\begin{aligned}\mathfrak{D}_A(B) &= \sup_{y \in B} D(y, A) = -\inf_{y \in B}[-D(y, A)] = -\inf_{y \in B} D(y, Y \backslash A) \\ &= -\mathcal{D}(B, Y \backslash A).\end{aligned}$$

The second equality follows from Theorem 2.1.2, and the last equality is the definition of δ_B.

If $A = \emptyset$, then $\mathfrak{D}_A(B) = -\mathcal{D}(B, Y \backslash A) = \delta_B(A) = +\infty$, and if $A = Y$, it is verified that $\mathfrak{D}_A(B) = -\mathcal{D}(B, Y \backslash A) = \delta_B(A) = -\infty$. □

By the previous Theorem 2.1.7, it follows that if $B \cap (Y \backslash A) \neq \emptyset$, then $\mathfrak{D}_A(B) = \rho_B(A)$.

An interesting consequence of Definition 2.1.3 and Theorem 2.1.7 is the next corollary, where the oriented distance of Gutiérrez et al. is related to the distances of Crespi et al. and Xu and Li.

Corollary 2.1.8. *If $A \subset Y$ and $B \in \mathcal{P}_0(Y)$, then we have*

$$\mathfrak{D}_{A+K}(B) = \Delta_B(A) = -\mathcal{D}(B, Y\backslash(A+K)),$$
$$\mathfrak{D}_{A-K}(B) = \widehat{\Delta}_B(A) = -\mathcal{D}(B, Y\backslash(A-K)).$$

Let us see a characterization of the distance $\Delta_B(A)$.

Proposition 2.1.9. *Let $A \subset Y$ and $B \in \mathcal{P}_0(Y)$. Then, we have*

$$\Delta_B(A) = \sup_{y\in B} D(y, A+K) = \sup_{y\in B} \Delta_y(A).$$

Proof. Applying Corollary 2.1.8, Definition 2.1.5 and Remark 2.1.4(b) it results that $\Delta_B(A) = \mathfrak{D}_{A+K}(B) = \sup_{y\in B} D(y, A+K) = \sup_{y\in B} \Delta_y(A)$. □

In Gutiérrez et al. [38, Theorem 6.15] the next theorem, which is essential for our purposes, is proved.

Theorem 2.1.10. *Let Y be a normed space with the norm $\|.\|_e$, $K \subset Y$ is a closed pointed and solid cone, and $e \in \operatorname{int} K$. If $B \in \mathcal{P}_0(Y)$, then*

$$h^l_{\inf}(A, B) = \Delta_B(A),\ \forall A \subset Y.$$

By the previous Theorem 2.1.10 and Remark 2.1.4(a), using the norm $\|.\|_e$, we derive that

$$D(x-y, -K) = h_{\inf}(x, y),\ \forall x, y \in Y. \tag{2.4}$$

The next relation between the function $h^l_{\inf}(A, B)$ and $\mathfrak{D}_{A+K}(B)$ is a consequence of Corollary 2.1.8 and Theorem 2.1.10.

Corollary 2.1.11. *Let $A \subset Y$, $B \in \mathcal{P}_0(Y)$ and $K \subset Y$ be a closed pointed and solid cone. If we consider the norm $\|.\|_e$, then we have*

$$\mathfrak{D}_{A+K}(B) = h^l_{\inf}(A, B).$$

Next, by using the norm $\|.\|_e$, the function $h^l_{\inf}(A, y)$ is related to $D(y, A+K)$.

Proposition 2.1.12. *If $A \in \mathcal{P}_0(Y)$, $y \in Y$ and we consider the norm $\|.\|_e$, then*

(i) $h^l_{\inf}(A, y) = \inf_{x\in A} D(x-y, -K) = \mathcal{D}(A-y, -K)$.

(ii) $D(y, A+K) = h^l_{\inf}(A, y) = \inf_{x\in A} D(y, x+K)$.

Proof. (i) By Corollary 1.3.16(i) and equality (2.4), we have

$$h^l_{\inf}(A, y) = \inf_{x \in A} h_{\inf}(x, y) = \inf_{x \in A} D(x - y, -K).$$

The second equality is deduced by Definition 2.1.1.

(ii) In Theorem 2.1.10, particularized to the case $B = \{y\}$, it results that $h^l_{\inf}(A, y) = \Delta_y(A)$, for all $A \subset Y$. Then, by applying part (i), it follows that

$$\Delta_y(A) = h^l_{\inf}(A, y) = \inf_{x \in A} D(x - y, -K).$$

Finally, by Remark 2.1.4(b), we have that $\Delta_y(A) = D(y, A + K)$ and, consequently,

$$D(y, A + K) = \inf_{x \in A} D(x - y, -K) = \inf_{x \in A} D(y, x + K)$$

by parts (xiii) and (xiv) of Lemma 1.3.19. □

Remark 2.1.13. As $x + K \subset A + K$, for all $x \in A$, then by Lemma 1.3.19(vii) it follows that $D(y, A + K) \leq D(y, x + K)$, for all $x \in A$, being $y \in Y$ and, therefore, for any norm it follows that

$$D(y, A + K) \leq \inf_{x \in A} D(y, x + K).$$

In general, for $y \in A + K$ this inequality is strict (see Example 2.1.23).

Now, in the following definition, we study two set extensions of type sup-inf of the oriented distance function of Hiriart-Urruty. First of all, we deal with a set scalarization function of type sup-inf, denoted $\mathbb{D}^{si}_K(A, B)$, introduced by Ha [45] and, secondly, we present another function, denoted $\widehat{\mathbb{D}}^{si}_K(A, B)$, which is an extension of the oriented distance function of Hiriart-Urruty too, and that was introduced by us. Moreover, we investigate some of their properties and their relationships to other scalarization functions which are available in the literature. The first function, was used by Ha to define a Hausdorff-type distance between two sets and thus be able to define a directional derivative for a set-valued map and, afterwards, it was applied to optimization problems with set-valued maps.

So, in the following definition, we are going to provide the two set scalarization functions of type sup-inf mentioned above, that can be regarded as an oriented distance between two sets, which are set extensions of the oriented distance function of Hiriart-Urruty.

Definition 2.1.14. The function $\mathbb{D}^{si}_K : \mathcal{P}_0(Y) \times \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$ is defined as follows

$$\mathbb{D}^{si}_K(A,B) = \sup_{y\in B} \inf_{x\in A} D(x-y,-K). \tag{2.5}$$

We also define $\widehat{\mathbb{D}}^{si}_K(A,B) = \sup_{x\in A} \inf_{y\in B} D(x-y,-K)$.

Without a specific name, the expression $\sup_{y\in B} \inf_{x\in A} D(x-y,-K)$ appears in [11] in order to provide a characterization of the lower set less order relation.

If the cone K is understood, for the sake of simplicity, we write $\mathbb{D}^{si}(A,B)$ and $\widehat{\mathbb{D}}^{si}(A,B)$ instead of $\mathbb{D}^{si}_K(A,B)$ and $\widehat{\mathbb{D}}^{si}_K(A,B)$. Similar expressions for the Gerstewitz's function appear in other papers as, for example, in [81, 83].

Remark 2.1.15. (a) If $A = \{x\}$ and $B = \{y\}$, then we have $\mathbb{D}^{si}(x,y) = D(x-y,-K)$.

(b) Taking into account Lemma 1.3.19(xiii), we have the next relation between the generalized oriented distances $\mathbb{D}^{si}$ and $\widehat{\mathbb{D}}^{si}$:

$$\widehat{\mathbb{D}}^{si}_K(A,B) = \mathbb{D}^{si}_{-K}(B,A) = \mathbb{D}^{si}_K(-B,-A). \tag{2.6}$$

(c) By definition, it follows that

$$\mathbb{D}^{si}(A,B) = \sup_{y\in B} \mathbb{D}^{si}(A,y), \tag{2.7}$$

Moreover, we have that $\widehat{\mathbb{D}}^{si}(A,B) = \sup_{x\in A} \widehat{\mathbb{D}}^{si}(x,B)$.

(d) It is verified that

$$\mathbb{D}^{si}(A,y) \le \mathbb{D}^{si}(A,B) \le \mathbb{D}^{si}(x,B),\ \forall x \in A, \forall y \in B.$$

The first inequality is obvious in view of (2.7). To prove the second one, let $a_0 \in A$ and $r_0 = \mathbb{D}^{si}(a_0,B) = \sup_{b\in B} D(a_0-b,-K)$. Therefore, $\inf_{a\in A} D(a-b,-K) \le D(a_0-b,-K) \le r_0$, for all $b \in B$. Then, we have that

$$\mathbb{D}^{si}(A,B) = \sup_{b\in B} \inf_{a\in A} D(a-b,-K) \le r_0.$$

Let us give some properties of the scalarization function $\mathbb{D}^{si}(A,B)$ defined above.

Proposition 2.1.16. *Let $A, B \in \mathcal{P}_0(Y)$ and $y \in Y$. Then, we have*

$$\mathbb{D}^{si}(A, y) \leq d(y, A + K) \quad \text{and} \quad \mathbb{D}^{si}(A, B) \leq \rho_B(A + K).$$

Proof. Indeed, for the first inequality, one has $D(x-y, -K) \leq d(x-y, -K)$ and it is enough to take infima with $x \in A$ and to use Proposition 1.3.17(iii):

$$\mathbb{D}^{si}(A, y) = \inf_{x\in A} D(x-y, -K) \leq \inf_{x\in A} d(x-y, -K) = \inf_{x\in A} d(y, x+K) = d(y, A+K).$$

For the second inequality, using the first one, taking suprema with $y \in B$ and in view of equation (2.7), we have that $\mathbb{D}^{si}(A, B) \leq \rho_B(A + K)$. □

Proposition 2.1.17. *Let $A \in \mathcal{P}_0(Y)$ and $y \in Y$. If $y \notin A + K$ or $\mathbb{D}^{si}(A, y) \geq 0$ or if K is not solid, then*

$$\mathbb{D}^{si}(A, y) = d(y, A + K).$$

Proof. In the case of $y \notin A + K$, by definition of $\mathbb{D}^{si}$, we have that

$$\mathbb{D}^{si}(A, y) = \inf_{x\in A}[d(y, x + K) - d(y, Y\backslash(x + K))]$$

and it is clear that $d(y, Y\backslash(x + K)) = 0$, for all $x \in A$, and therefore $\mathbb{D}^{si}(A, y) = \inf_{x\in A} d(y, x + K) = d(y, A + K)$ by Proposition 1.3.17(iii). On the other hand, if K is not solid, by Remark 1.3.20 we have $D(x - y, -K) = d(x - y, -K)$, for all $x, y \in Y$ and, consequently,

$$\mathbb{D}^{si}(A, y) = \inf_{x\in A} D(x - y, -K) = \inf_{x\in A} d(y, x + K) = d(y, A + K).$$

□

We present two examples in order to illustrate the above definition.

Example 2.1.18. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = \{(x, y) \in \mathbb{R}^2 : \; y = -x, \; x \leq 0\}$ and $B = \{b\}$ with $b = (0, 0)$. Then, $\mathbb{D}^{si}(A, b) = \inf_{a\in A} D(a - b, -K) = 0$.

Example 2.1.19. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = \{(x, y) \in \mathbb{R}^2 : \; y = x, \; x \geq 0\}$ and $B = \{(x, y) \in \mathbb{R}^2 : \; y = \frac{1}{x}, x < 0\}$. As $b \notin A + K$, for all $b \in B$, by Proposition 2.1.17 we have that $\mathbb{D}^{si}(A, b) = d(b, A + K)$ and, then, $\mathbb{D}^{si}(A, B) = \sup_{b\in B} \mathbb{D}^{si}(A, b) = \sup_{b\in B} d(b, A+K) = \sup_{b\in B} \sqrt{b_1^2 + b_2^2} = +\infty$ with $b = (b_1, b_2)$.

As a consequence of Remark 2.1.15(c) and Proposition 2.1.17, we have the next result.

Proposition 2.1.20. *If $A, B \in \mathcal{P}_0(Y)$ and K is not solid, then*

(i) $\mathbb{D}^{si}(A, B) = \rho_B(A + K)$.

(ii) $\mathbb{D}^{si}(A, B) \geq 0$.

Hence it is a necessary condition for $\mathbb{D}^{si}(A, B) < 0$ that $\operatorname{int} K \neq \emptyset$. Moreover, if K is not solid and $B \subset \operatorname{cl}(A + K)$ then $\mathbb{D}^{si}(A, B) = 0$.

In the next proposition, the distances (2.1), (2.2) and (2.5) are related, considering any norm in Y.

Proposition 2.1.21. *If $A, B \in \mathcal{P}_0(Y)$, then we have*

$$\mathcal{D}(B, A + K) \leq \Delta_B(A) \leq \mathbb{D}^{si}(A, B) \leq \rho_B(A + K). \tag{2.8}$$

Proof. The first inequality is obtained by Corollary 2.1.8 and Remark 2.1.6. To prove the second one, by Remark 2.1.13 and Lemma 1.3.19 parts (xiii) and (xiv), we have that

$$D(y, A + K) \leq \inf_{x \in A} D(y, x + K) = \inf_{x \in A} D(x - y, -K).$$

Therefore, taking suprema with $y \in B$ we have that

$$\sup_{y \in B} D(y, A + K) \leq \sup_{y \in B} \inf_{x \in A} D(x - y, -K) = \mathbb{D}^{si}(A, B)$$

and so the inequality is proved by Proposition 2.1.9. The third inequality is Proposition 2.1.16. □

In the following, we derive a relation by the previous Theorem 2.1.21.

Corollary 2.1.22. *If $A \in \mathcal{P}_0(Y)$ and $B = \{y\}$, then we have*

$$D(y, A + K) \leq \mathbb{D}^{si}(A, y) \leq d(y, A + K).$$

It is worth noting that this result can also be obtained by using Remark 2.1.13 and Proposition 2.1.16.

The following example shows that the equality in (2.8), in general, do not hold.

Example 2.1.23. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$ and $B = \{b_1, b_2\}$, where $b_1 = (1, 1)$ and $b_2 = (1, 0)$. We have:

- $\mathcal{D}(B, A + K) = \inf_{b \in B} D(b, A + K) = \min_{i=1,2}\{-d(b_i, Y \setminus (A + K))\} = \min\{-\sqrt{2}, -\sqrt{2}/2\} = -\sqrt{2}$.

- $\Delta_B(A) = \rho_B(A + K) - d(B, Y \setminus (A + K)) = -d(b_2, Y \setminus (A + K)) = -\sqrt{2}/2$.

- $\mathbb{D}^{si}(A, B) = \sup_{b \in B} \inf_{a \in A} D(a - b, -K) = \max_{i=1,2} \inf_{a \in A} D(a, b_i - K)$.

  The infimum in $a \in A$ of $D(a, b_i - K)$ is attained at the point a_i where A cuts to the line $y = x$ for $i = 1$ and to the line $y = x - 1$ for $i = 2$. So $a_1 = (0, 0)$ ant $a_2 = (1/2, -1/2)$. Therefore, $\mathbb{D}^{si}(A, B) = \max_{i=1,2}\{-d(a_i, Y \setminus (b_i - K))\} = \max\{-1, -1/2\} = -1/2$.

- $\rho_B(A + K) = \sup_{b \in B} d(b, A + K) = 0$.

In consequence, (2.8) is satisfied with strict inequalities: $-\sqrt{2} < -\sqrt{2}/2 < -1/2 < 0$.

Now, we show that the four scalarization functions $\mathbb{D}^{si}$, $\mathfrak{D}$, Δ and $h^l_{\inf}$ are the same if we consider the norm $\|.\|_e$.

Theorem 2.1.24. *If we consider the norm $\|.\|_e$ in Y, then*

$$\mathbb{D}^{si}(A, B) = \mathfrak{D}_{A+K}(B) = \Delta_B(A) = h^l_{\inf}(A, B), \ \forall A, B \in \mathcal{P}_0(Y).$$

Proof. By Corollary 2.1.8, we have that $\mathfrak{D}_{A+K}(B) = \Delta_B(A)$. By Theorem 2.1.10, we obtain that $\Delta_B(A) = h^l_{\inf}(A, B)$. Applying Corollary 1.3.16 parts (ii) and (i), and equation (2.4), it results that

$$\begin{aligned}\Delta_B(A) = h^l_{\inf}(A, B) &= \sup_{y \in B} h^l_{\inf}(A, y) = \sup_{y \in B} \inf_{x \in A} h_{\inf}(x, y) \\ &= \sup_{y \in B} \inf_{x \in A} D(x - y, -K) = \mathbb{D}^{si}(A, B).\end{aligned}$$

□

In the next theorem, we relate the functions $\mathbb{D}^{si}$, Δ and $\mathfrak{D}$, considering any norm in Y.

Theorem 2.1.25. *Let* $A \in \mathcal{P}_{0,K}(Y)$ *and* $B \in \mathcal{P}_0(Y)$ *such that* $B\cap(Y\backslash(A+K)) \neq \emptyset$*. Then*

$$\mathfrak{D}_{A+K}(B) = \Delta_B(A) = \mathbb{D}^{si}(A,B) = \rho_B(A+K).$$

Proof. Applying Corollary 2.1.8, we obtain the following

$$\Delta_B(A) = \mathfrak{D}_{A+K}(B) = \sup_{y\in B}[d(y,A+K) - d(y,Y\backslash(A+K))].$$

As $B \cap (Y\backslash(A+K)) \neq \emptyset$, it results that

$$\sup_{y\in B}[d(y,A+K) - d(y,Y\backslash(A+K))] = \sup_{y\in B} d(y,A+K) = \rho_B(A+K).$$

If we take $y \notin A+K$, then $y \notin x+K$, for all $x \in A$ and, therefore, we obtain $d(y,x+K) = D(y,x+K)$. Hence, by Propositions 2.1.17 and 1.3.17(iii), we have that

$$d(y,A+K) = D(y,A+K) = \inf_{x\in A} D(y,x+K).$$

Then, if $B \cap (Y\backslash(A+K)) \neq \emptyset$, we obtain that

$$\begin{aligned}\Delta_B(A) &= \sup_{y\in B} d(y,A+K) = \sup_{y\in B\cap(Y\backslash(A+K))} d(y,A+K)\\ &= \sup_{y\in B\cap(Y\backslash(A+K))} \inf_{x\in A} D(y,x+K) = \sup_{y\in B}\inf_{x\in A} D(y,x+K) = \mathbb{D}^{si}(A,B).\end{aligned}$$

□

Now, we derive an interesting result as a consequence of the previous Theorem 2.1.25.

Corollary 2.1.26. *If* $A \in \mathcal{P}_0(Y)$ *and* $y \notin A+K$*, then* $\mathbb{D}^{si}(A,y) = \Delta_y(A) = \mathfrak{D}_{A+K}(y) = d(y,A+K) = D(y,A+K)$*.*

Remark 2.1.27. In Theorem 2.1.25, we require that the set A is K-proper because otherwise condition $B \cap (Y\backslash(A+K)) \neq \emptyset$ is not true for any B.

Let us observe that it has been proved in Example 2.1.23 that, in general, $\mathfrak{D}_{A+K}(B) = \Delta_B(A) \neq \mathbb{D}^{si}(A,B)$ whenever $B \subset A+K$.

2.2 Properties for set oriented distances of type sup-inf

In the previous section, we have presented two set extensions of type sup-inf in Definition 2.1.14, denoted $\mathbb{D}^{si}$ and $\widehat{\mathbb{D}}^{si}$, to the oriented distance function of Hiriart-Urruty, the first one, introduced by Ha [45] and, the second one, introduced by us. In this section, we investigate new properties for the functions $\mathbb{D}^{si}$ and $\widehat{\mathbb{D}}^{si}$. More specifically, by using cone-properness and cone-boundedness, and a new concept of cone-boundedness with respect to a set which have been introduced, results about their finitude are presented. Besides, some new fundamental properties as convexity, Lipschitz continuity, positive homogeneity, invariance respect to conic extensions, monotonicity, diagonal null, invariance with respect to closure, etc. are presented.

Now, we start with a theorem which establishes a condition in order to the function $\mathbb{D}^{si}$ can be smaller or equal to a non-negative real number.

Theorem 2.2.1. *Let $A, B \in \mathcal{P}_0(Y)$ and $r \geq 0$. Then*

$$\mathbb{D}^{si}(A, B) \leq r \Leftrightarrow B \subset \mathrm{cl}(rU_0 + A + K).$$

Proof. ($\Rightarrow$) Ad absurdum, assume that there exists $y_0 \in B$ with $y_0 \notin \mathrm{cl}(rU_0 + A + K)$. By Lemma 1.3.24, we have that $r_0 = d(y_0, A + K) > r$. In this case, by Corollary 2.1.26, we obtain that $\mathbb{D}^{si}(A, y_0) = d(y_0, A + K) = r_0 > r$. Therefore, $\mathbb{D}^{si}(A, B) = \sup_{y \in B} \mathbb{D}^{si}(A, y) \geq r_0 > r$, which is a contradiction.

($\Leftarrow$) By Lemma 1.3.24, we have that $d(y, A + K) \leq r$, for all $y \in B$. By Proposition 2.1.16, we deduce $\mathbb{D}^{si}(A, y) \leq d(y, A + K) \leq r$, for all $y \in B$. Then, $\mathbb{D}^{si}(A, B) = \sup_{y \in B} \mathbb{D}^{si}(A, y) \leq r$, and we conclude. □

In the following, let us see a proposition which provides an equivalent condition to $\mathbb{D}^{si}(A, y) = -\infty$.

Proposition 2.2.2. *Let $A \in \mathcal{P}_0(Y)$, $y \in Y$ and K solid. Then, $\mathbb{D}^{si}(A, y) = -\infty$ if and only if there exists a sequence $(a_n) \subset A$ such that $d(a_n - y, Y \backslash (-K)) \to +\infty$.*

Proof. It is clear since

$$\mathbb{D}^{si}(A,y) = \inf_{a\in A} D(a-y,-K) = \inf_{a\in A}[d(a-y,-K) - d(a-y, Y\backslash(-K))],$$

and this value is $-\infty$ if $\inf_{a\in A}[-d(a-y, Y\backslash(-K))] = -\infty$, which is equivalent to $\sup_{a\in A} d(a-y, Y\backslash(-K)) = +\infty$. The last equality is true if and only if there exists a sequence $(a_n) \subset A$ such that $d(a_n - y, Y\backslash(-K)) \to +\infty$. □

Next, we give equivalent conditions to $\mathbb{D}^{si}(A,B) = -\infty$.

Proposition 2.2.3. *Let $A \in \mathcal{P}_0(Y)$ and K solid. Then, the next statements are equivalent:*

(i) $\mathbb{D}^{si}(A,y_0) = -\infty$*, for some* $y_0 \in Y$.

(ii) $\mathbb{D}^{si}(A,y) = -\infty$*, for all* $y \in Y$.

(iii) $\mathbb{D}^{si}(A,B) = -\infty$*, for some* $B \in \mathcal{P}_0(Y)$.

(iv) $\mathbb{D}^{si}(A,B) = -\infty$*, for all* $B \in \mathcal{P}_0(Y)$.

Proof. We only see the implication $(i) \Rightarrow (ii)$ since the rest are obvious. As $\mathbb{D}^{si}(A,y_0) = -\infty$, by Proposition 2.2.2, there exists $(a_n) \subset A$ such that $d(a_n - y_0, Y\backslash(-K)) \to +\infty$. By Lemma 1.3.26 applied to $A = Y\backslash(-K)$, $y_2 = a_n - y$ and $y_1 = a_n - y_0$, we have that

$$d(a_n - y, Y\backslash(-K)) \geq d(a_n - y_0, Y\backslash(-K)) - d(y, y_0).$$

Furthermore, as $d(a_n - y_0, Y\backslash(-K)) \to +\infty$, it follows that $d(a_n - y, Y\backslash(-K)) \to +\infty$, for all $y \in Y$. Therefore, by Proposition 2.2.2, we conclude that $\mathbb{D}^{si}(A,y) = -\infty$, for all $y \in Y$. □

In the next lemma, we obtain a characterization for K-proper sets.

Lemma 2.2.4. *Let K be solid and $e \in \operatorname{int} K$. Then, A is K-proper if and only if there exists $t_0 > 0$ such that $A \subset Y\backslash(-t_0 e - \operatorname{int} K)$.*

Proof. ($\Rightarrow$) By contradiction, let us suppose that there exists $t_n \to +\infty$ such that $A \not\subset Y\backslash(-t_n e - \operatorname{int} K)$. Then, for each n,

$$\exists\, a_n \in A \text{ such that } a_n \in -t_n e - \operatorname{int} K. \tag{2.9}$$

Now, we assert that $\bigcup\limits_{n}(a_n + K) = Y$. Otherwise,

$$\exists\, y_0 \in Y \text{ such that } y_0 \notin a_n + K,\ \forall n. \tag{2.10}$$

By (2.9), we have

$$-a_n \in t_n e + \operatorname{int} K,\ \forall n. \tag{2.11}$$

On the other hand, we know that $Y = \bigcup\limits_{t>0}(-te + \operatorname{int} K)$ and, then, there exists $t > 0$ such that $y_0 \in -te + \operatorname{int} K$. Therefore, in view of (2.11), it results that

$$y_0 - a_n \in -te + \operatorname{int} K + t_n e + \operatorname{int} K \subset (t_n - t)e + \operatorname{int} K.$$

For n large enough, $t_n - t > 0$ and, consequently, $y_0 - a_n \in (t_n - t)e + \operatorname{int} K \subset \operatorname{int} K$, which contradicts (2.10). Hence, $A + K = Y$ and this is a contradiction since A is K-proper.

($\Leftarrow$) It is clear because

$$A+K \subset Y\backslash(-t_0e-\operatorname{int} K)+K = -t_0e+Y\backslash(-\operatorname{int} K)+K \subset -t_0e+Y\backslash(-\operatorname{int} K) \neq Y.$$

The second inclusion is true since if $y \notin -\operatorname{int} K$ and $k \in K$, then $y+k \notin -\operatorname{int} K$; otherwise, if $y + k \in -\operatorname{int} K$ then we have that $y \in -k - \operatorname{int} K \subset -\operatorname{int} K$, which is a contradiction. □

Now, we establish a characterization of the lack of K-properness of a set in the case when K is solid.

Proposition 2.2.5. *Let $A, B \in \mathcal{P}_0(Y)$ and let K be solid. Then, A is not K-proper if and only if $\mathbb{D}^{si}(A, B) = -\infty$.*

Proof. ($\Rightarrow$) By Proposition 2.2.3, it is sufficient to prove that $\mathbb{D}^{si}(A, 0_Y) = -\infty$ and, in accordance with Proposition 2.2.2, it is sufficient to find a sequence $(a_n) \subset A$ such that $\sup_n d(a_n, Y\backslash(-K)) = +\infty$. If $e \in \operatorname{int} K$ and A is not K-proper, by Lemma 2.2.4 it results that

$$A \not\subset Y\backslash(-te - \operatorname{int} K),\ \forall t > 0. \tag{2.12}$$

Now, if $t_n \to +\infty$ then, taking into account (2.12), we obtain that there exists $a_n \in A$ such that $a_n \notin Y\backslash(-t_n e - \operatorname{int} K)$, that is,

$$a_n \in -t_n e - \operatorname{int} K. \tag{2.13}$$

Let us see that $\sup_n d(a_n, Y\backslash(-K)) = +\infty$. Assume that $\sup_n d(a_n, Y\backslash(-K)) = \alpha$. Then, for all n,

$$a_n \in Y\backslash(-K) + 2\alpha U_0. \tag{2.14}$$

In view of (2.13) and (2.14), it follows that $a_n = -t_n e - k_n = -y_n + z_n$, with $k_n \in \operatorname{int} K$, $y_n \in Y\backslash K$, $z_n \in 2\alpha U_0$. Then,

$$-e = \frac{1}{t_n}(-y_n + k_n) + t_n^{-1} z_n \in Y\backslash(-K) + t_n^{-1} z_n,$$

since $-y_n + k_n \in -Y\backslash K + \operatorname{int} K \subset -Y\backslash K$. As $t_n^{-1} z_n \to 0$ because $\|z_n\| < 2\alpha$, we obtain that $-e \in \operatorname{cl}[Y\backslash(-K)] = Y\backslash(-\operatorname{int} K)$, that is, $e \in Y\backslash \operatorname{int} K$, which is a contradiction.

($\Leftarrow$) By contradiction, suppose that there exists $y \in Y\backslash(A+K)$. As $\mathbb{D}^{si}(A, B) = -\infty$, by Proposition 2.2.3, we can assume $\mathbb{D}^{si}(A, 0_Y) = -\infty$ and, by Proposition 2.2.2, there exists $(a_n) \subset A$ such that $d(0, a_n + Y\backslash K) \to +\infty$. Since $a_n + K \subset A + K$, or equivalently, $Y\backslash(A + K) \subset Y\backslash(a_n + K) = a_n + Y\backslash K$, it follows that

$$0 = d(y, Y\backslash(A + K)) \geq d(y, a_n + Y\backslash K).$$

By Lemma 1.3.26, we have that

$$0 \geq d(y, a_n + Y\backslash K) \geq d(0, a_n + Y\backslash K) - d(y, 0),$$

and $d(0, a_n + Y\backslash K) \leq \|y\|$, for all n, which is a contradiction because $d(0, a_n + Y\backslash K) \to +\infty$. □

The following result is a direct consequence of Propositions 2.2.5 and 2.1.20(ii).

Corollary 2.2.6. *Let $A, B \in \mathcal{P}_0(Y)$. If A is K-proper, then $\mathbb{D}^{si}(A, B) \neq -\infty$.*

Other authors have obtained similar results to Corollary 2.2.6. For example, in [45, Lemma 3.1] it is required that A is K-bounded, however, we ask weaker assumptions as A is K-proper and, therefore, our result represents an extension.

In the next proposition, we characterize that A is not a K-proper set for the case that K is not solid.

Proposition 2.2.7. *Let $A \in \mathcal{P}_0(Y)$ and assume that K is not solid. Then, we have*

(i) If A is not K-proper, then $\mathbb{D}^{si}(A, B) = 0$, for all $B \in \mathcal{P}_0(Y)$.

(ii) The reverse is true if we suppose that A is K-closed.

Proof. (i) Indeed, by Proposition 2.1.20(i) we have $\mathbb{D}^{si}(A, B) = \rho_B(A + K) = 0$ because $A + K = Y$.

(ii) If we assume $\mathbb{D}^{si}(A, y) = 0, \forall y \in Y$, by Theorem 2.2.1, it follows that $y \in \text{cl}(A + K)$ for all $y \in Y$, that is, $Y \subset \text{cl}(A + K)$ and as A is K-closed we conclude that $Y = A + K$. □

In the following proposition, we characterize the K-boundedness through the function $\mathbb{D}^{si}$.

Proposition 2.2.8. *Let $A, B \in \mathcal{P}_0(Y)$ and let A be K-bounded. Then, B is K-bounded if and only if $\mathbb{D}^{si}(A, B) < +\infty$.*

Proof. ($\Rightarrow$) If B is K-bounded and $a_0 \in A$, we have that $B - a_0$ is also K-bounded. Therefore, for the open unit ball U_0, there exits $r > 0$ such that $B - a_0 \subset rU_0 + K$, and it follows that $B \subset a_0 + rU_0 + K$. By Theorem 2.2.1 we obtain that $\mathbb{D}^{si}(a_0, B) \leq r$ and, by Remark 2.1.15(d), $\mathbb{D}^{si}(A, B) \leq \mathbb{D}^{si}(a_0, B) \leq r < +\infty$.

($\Leftarrow$) Let $\mathbb{D}^{si}(A, B) \leq r$ with $r > 0$. By Theorem 2.2.1 and Corollary 1.3.25, it results that $B \subset \text{cl}(rU_0 + A + K) \subset r_1 U_0 + A + K$ for $r < r_1$. As A is K-bounded, then for U_0 there exists $t > 0$ such that $A \subset tU_0 + K$. Therefore, $B \subset (r_1 + t)U_0 + K$, so B is K-bounded. □

Note that in the '$\Rightarrow$' part, the requirement A is K-bounded is not used.

Remark 2.2.9. Similar results to Proposition 2.2.8 have been proved by other authors. For example, in [51, Theorem 3.6] for the Gerstewitz's scalarization when we consider $G_e(A, B)$ but it is required that B is K-proper, and in [129, Theorem 3.1] for the function $\mathfrak{D}_{A+K}(B)$ of Xu and Li where it is also required that the set B is K-proper. Therefore, Proposition 2.2.8 is an improvement.

Now, let us look at a characterization so that a set be K-bounded.

Proposition 2.2.10. *Let $B \in \mathcal{P}_0(Y)$. The next statements are equivalent:*

(i) B is K-bounded.

(ii) $\mathbb{D}^{si}(y, B) < +\infty$, for all $y \in Y$.

(iii) $\mathbb{D}^{si}(y_0, B) < +\infty$, for some $y_0 \in Y$.

(iv) $\mathbb{D}^{si}(A, B) < +\infty$, for all $A \in \mathcal{P}_0(Y)$.

(v) $\mathbb{D}^{si}(A, B) < +\infty$, for some K-bounded set A.

Proof. $(i) \Rightarrow (ii)$ follows from Proposition 2.2.8 since $\{y\}$ is bounded and, therefore, K-bounded.

$(ii) \Rightarrow (iii)$, $(iv) \Rightarrow (ii)$ and $(v) \Rightarrow (ii)$ are obvious.

$(iii) \Rightarrow (i)$ follows from Proposition 2.2.8.

$(ii) \Rightarrow (iv)$ is an immediate consequence of the next inequality

$$\mathbb{D}^{si}(A, B) \leq \mathbb{D}^{si}(a, B), \ \forall a \in A,$$

given in Remark 2.1.15(d).

$(i) \Rightarrow (v)$ follows from Proposition 2.2.8. □

Next, we provide a condition which ensures that the function $\mathbb{D}^{si}$ is finite.

Corollary 2.2.11. *Let $A, B \in \mathcal{P}_0(Y)$. If A is K-proper and B is K-bounded, then $\mathbb{D}^{si}(A, B) \in \mathbb{R}$.*

Proof. By applying Proposition 2.2.10, we have that $\mathbb{D}^{si}(A, B) < +\infty$ and, by Corollary 2.2.6, $\mathbb{D}^{si}(A, B) \neq -\infty$. Therefore, we conclude that $\mathbb{D}^{si}(A, B) \in \mathbb{R}$. □

We present an example for illustrating the previous Proposition 2.2.10, where the function $\mathbb{D}^{si}$ takes the value $+\infty$, B is not K-bounded but B is K-proper.

Example 2.2.12. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $y_0 = (0, 0)$ and $B = \{(x, y) \in \mathbb{R}^2 : \ y = x^2, x \leq -1\}$. We have that

$$\mathbb{D}^{si}(y_0, B) = \sup_{b \in B} D(y_0 - b, -K) = \sup_{b \in B} D(b, K) = +\infty.$$

By Proposition 2.2.10, it follows that B is not K-bounded although is K-proper.

Remark 2.2.13. Let K be solid. If $A \in \mathcal{P}_0(Y)$ is not K-proper and B is not K-bounded, we obtain

(a) $\mathbb{D}^{si}(A, B) = -\infty$ by Proposition 2.2.5.

(b) $\mathbb{D}^{si}(A, b) = -\infty$, for all $b \in B$ by Proposition 2.2.5.

(c) $\mathbb{D}^{si}(a, b) \in \mathbb{R}$, for all $a \in A$ and all $b \in B$.

(d) $\mathbb{D}^{si}(a, B) = +\infty$, for all $a \in A$ by Proposition 2.2.10.

Next, we introduce the concept of K-boundedness w.r.t. a set.

Definition 2.2.14. Let $A, B \in \mathcal{P}_0(Y)$. We say that B is K-bounded with respect to A (w.r.t. A) if there exits $r > 0$ such that

$$B \subset rU_0 + A + K.$$

We observe that if $A \preccurlyeq^{\forall\exists} B$, then B is K-bounded with respect to A.

Remark 2.2.15. (a) B is K-bounded if and only if B is K-bounded w.r.t. $\{0\}$.

(b) If B_1 is K-bounded w.r.t. A and B_2 is K-bounded, then $B_1 + B_2$ is K-bounded w.r.t. A.

(c) If B is K-bounded w.r.t. A and A is K-bounded, then B is K-bounded.

Now, let us look at a characterization so that a set be K-bounded w.r.t a set.

Proposition 2.2.16. *Let $A, B \in \mathcal{P}_0(Y)$. Then, B is K-bounded w.r.t. A if and only if $\mathbb{D}^{si}(A, B) < +\infty$.*

Proof. ($\Rightarrow$) It is a consequence of former Definition 2.2.14 and Theorem 2.2.1.

($\Leftarrow$) Let $r \geq 0$ such that $\mathbb{D}^{si}(A, B) \leq r$. By Theorem 2.2.1, we have that $B \subset \mathrm{cl}(rU_0 + A + K)$. Choosing $r' > r$, by applying Corollary 1.3.25, we derive $B \subset \mathrm{cl}(rU_0 + A + K) \subset r'U_0 + A + K$, and then B is K-bounded w.r.t. A. $\square$

In the next proposition, we introduce a condition so that the function $\mathbb{D}^{si}$ can take a real value.

Proposition 2.2.17. *Let $A, B \in \mathcal{P}_0(Y)$ and let A be K-proper. Then, B is K-bounded w.r.t. A if and only if $\mathbb{D}^{si}(A, B) \in \mathbb{R}$.*

Proof. (⇒) By applying Proposition 2.2.16, we have that $\mathbb{D}^{si}(A,B) \neq +\infty$ and, by Corollary 2.2.6, we derive that $\mathbb{D}^{si}(A,B) \neq -\infty$.

(⇐) It follows from Proposition 2.2.16. □

In the following proposition, we provide a condition what ensures the equivalence between K-boundedness and K-boundedness w.r.t. a set.

Proposition 2.2.18. *Let $A, B \in \mathcal{P}_0(Y)$ and let A be K-bounded. Then, B is K-bounded if and only if B is K-bounded w.r.t. A.*

Proof. (⇒) By applying Proposition 2.2.8, we have that $\mathbb{D}^{si}(A,B) < +\infty$, and by Proposition 2.2.16 we conclude.

(⇐) It is Remark 2.2.15(c). □

Proposition 2.2.18 extends Remark 2.2.15(a) since now if B is K-bounded, then B is K-bounded w.r.t. to any K-bounded set A.

The following corollary is a consequence of Proposition 2.2.16.

Corollary 2.2.19. *Let $A, B \in \mathcal{P}_0(Y)$. Then, B is not K-bounded w.r.t. A if and only if $\mathbb{D}^{si}(A,B) = +\infty$ and, moreover, in either case we can ensure that A is K-proper.*

Proof. Indeed, it is Proposition 2.2.16 and the final part is true because if A were not K-proper, then $B \subset A + K = Y$, and B would be K-bounded w.r.t. A, which is a contradiction. □

Next, we establish conditions for a set B not to be K-proper.

Theorem 2.2.20. *Let $B \in \mathcal{P}_0(Y)$ and let K be solid. The following statements are equivalent:*

(i) B is not K-proper.

(ii) B is not K-bounded w.r.t. any K-proper set A.

(iii) B is not K-bounded w.r.t. $-Y \backslash K$.

(iv) $\mathbb{D}^{si}(A,B) = +\infty$, for any K-proper set A.

(v) $\mathbb{D}^{si}(-Y \backslash K, B) = +\infty$.

Proof. $(i) \Rightarrow (ii)$ By contradiction, assume that B is K-bounded w.r.t. A for some K-proper set A. Then, $B \subset rU_0 + A + K$, for some $r > 0$. It follows that $Y = B + K \subset rU_0 + A + K + K = rU_0 + A + K$, that is,

$$Y = rU_0 + A + K, \tag{2.15}$$

but this cannot happen if $\operatorname{int} K \neq \emptyset$ and A is K-proper since in this case, by Lemma 2.2.4, we have that $A \subset -t_0 e - Y \backslash \operatorname{int} K$ and, from this, it follows that $A + K \subset -t_0 e - Y \backslash \operatorname{int} K + K \subset -t_0 e - Y \backslash \operatorname{int} K$. Therefore,

$$Y = rU_0 + A + K \subset -t_0 e + rU_0 - Y \backslash \operatorname{int} K. \tag{2.16}$$

Choosing $t_n \to +\infty$, in view of (2.16), we have $-t_n e \in -t_0 e + rU_0 - Y \backslash \operatorname{int} K$, that is, $-e \in \frac{-t_0}{t_n} e + \frac{r}{t_n} U_0 - Y \backslash \operatorname{int} K$, for all n. Taking the limit to $n \to +\infty$, since $-Y \backslash \operatorname{int} K$ is closed and U_0 is bounded, we have that $-e \in -Y \backslash \operatorname{int} K$, which is a contradiction.

$(ii) \Rightarrow (iii)$ We obtain this implication because $Y \backslash (-K)$ is K-proper.

$(iii) \Rightarrow (i)$ If B is not K-bounded w.r.t. $-Y \backslash K$, then there exits sequences $r_n \to +\infty$ and $(b_n) \subset B$ such that

$$b_n \notin r_n U_0 - Y \backslash K. \tag{2.17}$$

It is stated that

$$\bigcup_n (b_n + K) = Y, \tag{2.18}$$

because otherwise,

$$\exists\, y_0 \in Y \text{ such that } y_0 \notin b_n + K, \ \forall n. \tag{2.19}$$

If $e \in U_0 \cap \operatorname{int} K$, then $-r_n e \in r_n U_0$ and, therefore, $-r_n e - Y \backslash K \subset r_n U_0 - Y \backslash K$. By (2.17), we have that $b_n \notin -r_n e - Y \backslash K$, for all n, that is, $b_n \notin -r_n e - Y \backslash K = Y \backslash (-r_n e - K)$. This implies that $b_n \in -r_n e - K$ and, then, $-b_n \in r_n e + K$. From here and by (2.19), we deduce that

$$y_0 - b_n \notin K \text{ e } y_0 - b_n \in y_0 + r_n e + K. \tag{2.20}$$

Now, for n large enough, it is verified that $y_0 + r_n e \in \operatorname{int} K$ since $\frac{1}{r_n} y_0 + e \in \operatorname{int} K$ because $e \in \operatorname{int} K$, $\operatorname{int} K$ is open and $\frac{1}{r_n} y_0 \to 0$. By (2.20), we obtain that

$$y_0 - b_n \in y_0 + r_n e + K \subset \operatorname{int} K + K = \operatorname{int} K,$$

and this contradicts the fact that $y_0 - b_n \notin K$. Thus, we have proved (2.18) and, therefore, $B + K = Y$.

Equivalences $(iii) \Leftrightarrow (v)$ and $(ii) \Leftrightarrow (iv)$ are clear by Proposition 2.2.16. □

Remark 2.2.21. (a) If K is not solid, then the implication $(i) \Rightarrow (ii)$ of Theorem 2.2.20 is false, that is, equality (2.15) may be true when A is K-proper. See Example 2.2.22 and observe that $rU_0 + A + K = Y$ when A is K-proper.

(b) If K is not solid and A is K-proper, as a consequence of Proposition 2.1.20, we have that $\mathbb{D}^{si}(A, B) \in [0, +\infty]$.

Let us consider some illustrative examples. Example 2.2.22 proves Remark 2.2.21(b) and Example 2.2.23 illustrates Proposition 2.2.17.

Example 2.2.22. Let $Y = \mathbb{R}^2$, $K = \{(x, 0) : x \geq 0\}$, $r \geq 0$, $A = \mathbb{R}^2 \setminus \{(x, y) : x \leq 0, 0 \leq y \leq r\}$ and $B = \mathbb{Q} \times \mathbb{R}$. We have that $A + K = A$ and $B + K = Y$, $\sup_{b \in B} d(b, A + K) = r/2$ and $\mathbb{D}^{si}(A, B) = r/2$.

If now $A = \{(x, y) : x \geq 0 \quad \text{or} \quad y \leq 0\}$, then $\mathbb{D}^{si}(A, b) = d(b, A + K) = \min\{-b_1, b_2\}$ for $b \notin A$. Therefore, $\mathbb{D}^{si}(A, B) = +\infty$.

Example 2.2.23. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = \{(x, y) : y = -x + r, x \leq 0\}$ with $r \geq 0$, which is not K-bounded, and $B = \{(x, y) : y = -x, x \leq 0\}$ which is K-bounded w.r.t. A. We have $\mathbb{D}^{si}(A, B) = \sqrt{2}\, r/2$.

If we take $B = \{(x, 0) : x \leq 0\}$, which is not K-bounded w.r.t. A, we have that $\mathbb{D}^{si}(A, B) = +\infty$.

Next, we need an useful result to study properties of the set scalarization function $\mathbb{D}^{si}$. The following theorem is obtained from Theorem 2.2.1 with $r = 0$.

Theorem 2.2.24. *Let $A, B \in \mathcal{P}_0(Y)$. Then*

$$\mathbb{D}^{si}(A, B) \leq 0 \Leftrightarrow B \subset \operatorname{cl}(A + K).$$

Now, we are going to study some properties of the set scalarization function $\mathbb{D}^{si}$. First of all, we will prove the convexity of $\mathbb{D}^{si}(\cdot, B)$.

Proposition 2.2.25. *If $B \in \mathcal{P}_0(Y)$, then $\mathbb{D}^{si}(\cdot, B)$ is convex on $\mathcal{P}_0(Y)$.*

Proof. By Lemma 1.3.19(ii), the oriented distance function $D(\cdot, -K)$ is convex on Y. Then, for all $A_1, A_2 \in \mathcal{P}_0(Y)$, $b \in Y$ and $\lambda \in [0, 1]$, we have

$$\begin{aligned}
\mathbb{D}^{si}(\lambda A_1 + (1-\lambda)A_2, b) &= \inf_{z \in \lambda A_1 + (1-\lambda)A_2} D(z - b, -K) \\
&= \inf_{a_1 \in A_1, a_2 \in A_2} D(\lambda a_1 + (1-\lambda)a_2 - b, -K) \\
&= \inf_{a_1 \in A_1, a_2 \in A_2} D(\lambda(a_1 - b) + (1-\lambda)(a_2 - b), -K) \\
&\leq \inf_{a_1 \in A_1, a_2 \in A_2} [\lambda D(a_1 - b, -K) + (1-\lambda) D(a_2 - b, -K)] \\
&= \inf_{a_1 \in A_1} \lambda D(a_1 - b, -K) + \inf_{a_2 \in A_2} (1-\lambda) D(a_2 - b, -K) \\
&= \lambda \, \mathbb{D}^{si}(A_1, b) + (1-\lambda) \, \mathbb{D}^{si}(A_2, b).
\end{aligned}$$

This tells us that $\mathbb{D}^{si}(\cdot, b)$ is convex on $\mathcal{P}_0(Y)$ for all $b \in Y$.

Therefore, by Remark 2.1.15(c) it results the following

$$\begin{aligned}
\mathbb{D}^{si}(\lambda A_1 + (1-\lambda)A_2, B) &= \sup_{b \in B} \mathbb{D}^{si}(\lambda A_1 + (1-\lambda)A_2, b) \\
&\leq \sup_{b \in B} [\lambda \, \mathbb{D}^{si}(A_1, b) + (1-\lambda) \, \mathbb{D}^{si}(A_2, b)] \\
&\leq \sup_{b \in B} \lambda \, \mathbb{D}^{si}(A_1, b) + \sup_{b \in B} (1-\lambda) \, \mathbb{D}^{si}(A_2, b) \\
&= \lambda \, \mathbb{D}^{si}(A_1, B) + (1-\lambda) \, \mathbb{D}^{si}(A_2, B).
\end{aligned}$$

□

In the following proposition, we state that $\mathbb{D}^{si}$ is positively homogeneous.

Proposition 2.2.26. *If $A, B \in \mathcal{P}_0(Y)$, then $\mathbb{D}^{si}(\lambda A, \lambda B) = \lambda \, \mathbb{D}^{si}(A, B)$, for all $\lambda > 0$, that is, the function $\mathbb{D}^{si}$ is positively homogeneous.*

Proof. First of all, we are going to prove that $\mathbb{D}^{si}(\lambda A, \lambda b) = \lambda \, \mathbb{D}^{si}(A, b)$, for all $\lambda > 0$, $b \in Y$. Indeed, as the oriented distance $D(\cdot, -K)$ is positively homogeneous by Lemma 1.3.19(vi), one has

$$\mathbb{D}^{si}(\lambda A, \lambda b) = \inf_{y \in \lambda A} D(y - \lambda b, -K) = \inf_{a \in A} D(\lambda a - \lambda b, -K)$$
$$= \inf_{a \in A} \lambda D(a - b, -K) = \lambda \, \mathbb{D}^{si}(A, b).$$

On the other hand, by Remark 2.1.15(c) and the previous expression it follows the general case:

$$\mathbb{D}^{si}(\lambda A, \lambda B) = \sup_{y \in \lambda B} \mathbb{D}^{si}(\lambda A, y) = \sup_{b \in B} \mathbb{D}^{si}(\lambda A, \lambda b) = \lambda \, \mathbb{D}^{si}(A, B).$$

□

Next, we study the Lipschitz continuity of functions $\mathbb{D}^{si}(A, \cdot)$ and $\mathbb{D}^{si}(\cdot, B)$.

Theorem 2.2.27. *Given $A \in \mathcal{P}_{0,K}(Y)$, then the function $\mathbb{D}^{si}(A, \cdot) : Y \to \mathbb{R}$ is Lipschitz continuous of rank* 1.

Proof. First of all, we observe that $\mathbb{D}^{si}(A, y) \in \mathbb{R}$ for all $y \in \mathbb{R}$, by Corollary 2.2.11. Let us prove that $D(y - x, -K) \leq \|y - x\|$. The oriented distance function $D(\cdot, A)$ of Hiriart-Urruty is Lipschitz continuous of rank 1 by Lemma 1.3.19(i), that is, $|D(y_1, -K) - D(y_2, -K)| \leq \|y_1 - y_2\|$, for all $y_1, y_2 \in Y$.

If we take $y_1 = y - x$ and $y_2 = 0$, then

$$|D(y - x, -K)| \leq \|y - x\|, \ \forall x, y \in Y. \tag{2.21}$$

For $x, y \in Y$, by applying Lemma 1.3.19(ix) with $y_1 = a - y$ and $y_2 = y - x$, we have that

$$\mathbb{D}^{si}(A, x) = \inf_{a \in A} D(a - x, -K) \leq D(a - x, -K)$$
$$\leq D(a - y, -K) + D(y - x, -K), \ \forall a \in A.$$

From here, $\mathbb{D}^{si}(A, x) - D(y - x, -K) \leq D(a - y, -K)$, for all $a \in A$. Then, $\mathbb{D}^{si}(A, x) - D(y - x, -K) \leq \inf_{a \in A} D(a - y, -K) = \mathbb{D}^{si}(A, y)$. So, if we consider (2.21), it results that

$$\mathbb{D}^{si}(A, x) - \mathbb{D}^{si}(A, y) \leq D(y - x, -K) \leq \|y - x\|. \tag{2.22}$$

Changing x for y, we have that

$$\mathbb{D}^{si}(A, y) - \mathbb{D}^{si}(A, x) \leq D(x - y, -K) \leq \|x - y\|. \tag{2.23}$$

From (2.22) and (2.23), we obtain

$$\left| \mathbb{D}^{si}(A, x) - \mathbb{D}^{si}(A, y) \right| \leq \|x - y\|.$$

Therefore, $\mathbb{D}^{si}(A, \cdot)$ is Lipschitz continuous of rank 1. □

Theorem 2.2.28. *Assume that $B \in \mathcal{P}_0(Y)$ is K-bounded. Then, the function $\mathbb{D}^{si}(\cdot, B) : Y \to \mathbb{R}$ is Lipschitz continuous of rank* 1.

Proof. First, we observe that $\mathbb{D}^{si}(y, B) \in \mathbb{R}$ by Corollary 2.2.11 since B is K-bounded. Let $x, y \in Y$ two arbitrary points. Applying Lemma 1.3.19(ix) and taking into account (2.21), we derive that for all $b \in B$,

$$D(y - b, -K) \leq D(y - x, -K) + D(x - b, -K) \leq \|y - x\| + D(x - b, -K).$$

Taking suprema in $b \in B$, we obtain

$$\sup_{b \in B} D(y - b, -K) \leq \|y - x\| + \sup_{b \in B} D(x - b, -K),$$

that is, $\mathbb{D}^{si}(y, B) \leq \|y - x\| + \mathbb{D}^{si}(x, B)$. From here, it follows that $\mathbb{D}^{si}(y, B) - \mathbb{D}^{si}(x, B) \leq \|y - x\|$. Changing y for x, it results that $\mathbb{D}^{si}(x, B) - \mathbb{D}^{si}(y, B) \leq \|x - y\|$ and, consequently, $|\, \mathbb{D}^{si}(y, B) - \mathbb{D}^{si}(x, B)| \leq \|x - y\|$. So, we have proved that $\mathbb{D}^{si}(\cdot, B)$ is Lipschitz continuous of rank 1. □

In the next results we prove that the function $\mathbb{D}^{si}$ is invariant by conic extensions (K-invariance property) and some related results. Recall that $A + K$ is the conic extension of A.

Lemma 2.2.29. *If $A \in \mathcal{P}_0(Y)$ and $y_0 \in Y$, then*

$$\mathbb{D}^{si}(A, y_0) = \mathbb{D}^{si}(A + K, y_0).$$

Proof. Since the function $D(\cdot, -K)$ is K-increasing by Lemma 1.3.19(viii) and as $a - y_0 \leq_K a + q - y_0$, for all $q \in K$ and $a \in Y$, it follows that

$$D(a - y_0, -K) \leq D(a + q - y_0, -K),$$

and, consequently, $D(a - y_0, -K) \leq \inf_{q\in K} D(a + q - y_0, -K)$. From this, it results that

$$\mathbb{D}^{si}(A, y_0) = \inf_{a\in A} D(a - y_0, -K) \leq \inf_{a\in A} \inf_{q\in K} D(a + q - y_0, -K) = \mathbb{D}^{si}(A + K, y_0).$$

On the other hand, since $A \subset A + K$, then from the definition of infimum, it follows that

$$\mathbb{D}^{si}(A + K, y_0) = \inf_{y\in A+K} D(y - y_0, -K) \leq \inf_{a\in A} D(a - y_0, -K) = \mathbb{D}^{si}(A, y_0),$$

and we conclude. □

Next, by using Lemma 2.2.29, we are going to prove K-invariance property for the function $\mathbb{D}^{si}$.

Proposition 2.2.30. *If $A, B \in \mathcal{P}_0(Y)$, then*

$$\mathbb{D}^{si}(A, B) = \mathbb{D}^{si}(A + K, B) = \mathbb{D}^{si}(A, B + K) = \mathbb{D}^{si}(A + K, B + K).$$

Proof. The third equality follows from the two first ones. The first equality is a consequence of Remark 2.1.15(c) and Lemma 2.2.29.

Now let us prove the equality $\mathbb{D}^{si}(A, B) = \mathbb{D}^{si}(A, B + K)$. First of all, $a - b - q \leq_K a - b$, for all $a, b \in Y$ and all $q \in K$, and as $D(\cdot, -K)$ is K-increasing we deduce that $D(a - b - q, -K) \leq D(a - b, -K)$, for all $a, b \in Y$ and all $q \in K$. From this, $\inf_{a\in A} D(a - b - q, -K) \leq \inf_{a\in A} D(a - b, -K)$, for all $b \in Y$ and all $q \in K$ and, therefore,

$$\sup_{b\in B} \inf_{a\in A} D(a - b - q, -K) \leq \sup_{b\in B} \inf_{a\in A} D(a - b, -K) = \mathbb{D}^{si}(A, B),$$

for all $q \in K$. Then, by definition we have that

$$\begin{aligned}\mathbb{D}^{si}(A, B + K) &= \sup_{b\in B, q\in K} \inf_{a\in A} D(a - b - q, -K) \\ &= \sup_{q\in K} \sup_{b\in B} \inf_{a\in A} D(a - b - q, -K) \leq \mathbb{D}^{si}(A, B). \qquad (2.24)\end{aligned}$$

Secondly, as $B \subset B + K$, by Remark 2.1.15(c) it follows that

$$\mathbb{D}^{si}(A, B) = \sup_{y\in B} \mathbb{D}^{si}(A, y) \leq \sup_{y\in B+K} \mathbb{D}^{si}(A, y) = \mathbb{D}^{si}(A, B + K). \qquad (2.25)$$

By (2.24) and (2.25) we conclude. □

Similar results to Proposition 2.2.30 have been proved by other authors. For example, in Propositions 4.4 and 6.3(a) of Gutiérrez et al. [38] for the functions $\varphi_{e,B}(A)$ and $\Delta_B(A)$, respectively, and in Lemma 3.4(i) of Ha [45] it is demanded that A is K-bounded. Therefore, Proposition 2.2.30 is an extension or an improvement of these results.

The following sufficient conditions of invariance in a variable for the function $\mathbb{D}^{si}$ are an immediate consequence of Proposition 2.2.30.

Proposition 2.2.31. *(i) If $A_1, A_2, B \in \mathcal{P}_0(Y)$ and $A_1 \sim^{\preccurlyeq^{\forall\exists}} A_2$, then we have*

$$\mathbb{D}^{si}(A_1, B) = \mathbb{D}^{si}(A_2, B).$$

(ii) If $B_1, B_2, A \in \mathcal{P}_0(Y)$ and $B_1 \sim^{\preccurlyeq^{\forall\exists}} B_2$, then we have

$$\mathbb{D}^{si}(A, B_1) = \mathbb{D}^{si}(A, B_2).$$

Analogous results to Proposition 2.2.31 are Proposition 4.2(b) of Gutiérrez et al. [38] for the function $\varphi_{e,B}(A)$ and Theorem 3.8 of Hernández and Rodríguez-Marín [51] for the Gerstewitz's scalarization $G_e(A, B)$ but asking that A_1, A_2, B_1 and B_2 are K-proper.

We are going to present an example where we can see that the inverse of Proposition 2.2.31 is false.

Example 2.2.32. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A_1 = \{(x, y) \in \mathbb{R}^2 : y = -x,\ x \leq 0\}$, $A_2 = \{(x, y) \in \mathbb{R}^2 : y = x,\ x \geq 0\}$ and $B = \{b\}$ with $b = (0, 0)$. We have $\mathbb{D}^{si}(A_1, B) = \mathbb{D}^{si}(A_2, B) = 0$ and, however, $A_1 \not\sim^{\preccurlyeq^{\forall\exists}} A_2$.

Let us see a necessary condition so that two sets are $\sim^{\preccurlyeq^{\forall\exists}}$-equivalent.

Proposition 2.2.33. *Let $A, B \in \mathcal{P}_0(Y)$. If $A \sim^{\preccurlyeq^{\forall\exists}} B$, then $\mathbb{D}^{si}(A, B) = \mathbb{D}^{si}(B, A)$.*

Proof. By definition of $\sim^{\preccurlyeq^{\forall\exists}}$-equivalent sets, we have that $A + K = B + K$. By applying Proposition 2.2.30, it follows that

$$\mathbb{D}^{si}(A, B) = \mathbb{D}^{si}(A + K, B + K) = \mathbb{D}^{si}(B + K, A + K) = \mathbb{D}^{si}(B, A).$$

□

In the following proposition, we prove that $\mathbb{D}^{si}$ is diagonal null on K-proper sets.

Proposition 2.2.34. *If $A \in \mathcal{P}_0(Y)$ is K-proper, then $\mathbb{D}^{si}(A, A) = 0$.*

Proof. By Theorem 2.2.24, since $A \subset A+K$, it follows that $\mathbb{D}^{si}(A, A) \leq 0$. Let us suppose that $\mathbb{D}^{si}(A, A) = -r_0 < 0$ with $r_0 > 0$. Then, $\sup_{b\in A} \inf_{a\in A} D(b, a+K) = -r_0$. Therefore, for all $b \in A$, we have that $\inf_{a\in A} D(b, a+K) \leq -r_0$. As $a+K \subset A+K$, for all $a \in A$, by Lemma 1.3.19(vii), we obtain that $D(b, A+K) \leq D(b, a+K)$, for all $a \in A$ and, consequently, $D(b, A + K) \leq \inf_{a\in A} D(b, a + K) \leq -r_0$. Since $D(b, A+K) \leq -r_0 < 0$, we deduce that $D(b, A+K) = -d(b, Y\backslash(A+K)) \leq -r_0$. So, $d(b, Y\backslash(A + K)) \geq r_0 > 0$, for all $b \in A$, which is impossible because A is a K-proper set in the normed space Y. □

In the next proposition, we state a characterization of $\sim^{\preccurlyeq\forall\exists}$-equivalence through $\mathbb{D}^{si}$.

Proposition 2.2.35. *Let $A \in \mathcal{P}_0(Y)$ and $B \in \mathcal{P}_{0,K}(Y)$.*

(i) If $A \sim^{\preccurlyeq\forall\exists} B$, then $\mathbb{D}^{si}(A, B) = 0$ and $\mathbb{D}^{si}(B, A) = 0$.

(ii) The reverse implication is true if we assume that A and B are K-closed.

Proof. (i) By definition of $\sim^{\preccurlyeq\forall\exists}$-equivalent sets, we have that $A + K = B + K$. By applying Propositions 2.2.30 and 2.2.34, it follows that

$$\mathbb{D}^{si}(A, B) = \mathbb{D}^{si}(A + K, B + K) = \mathbb{D}^{si}(B + K, B + K) = \mathbb{D}^{si}(B, B) = 0.$$

By Proposition 2.2.33, we conclude $\mathbb{D}^{si}(B, A) = 0$.

(ii) First of all, by Theorem 2.2.24 we have that $B \subset \mathrm{cl}(A+K) = A+K$ and, consequently, $B+K \subset (A+K)+K = A+K$. Secondly, $A \subset \mathrm{cl}(B+K) = B+K$ and, hence, $A + K \subset (B + K) + K = B + K$. Therefore, $A + K = B + K$. □

Results of type of Propositions 2.2.33, 2.2.34 and 2.2.35(i), but for the Gerstewitz's scalarization G_e, are Theorems 3.8(iv), 3.10(i) and 3.10(ii) in Hernández and Rodríguez-Marín [51], with stronger assumptions in general. Also some of them can be found in Proposition 6.6(a) of Gutiérrez et al. [38] for the function

$\Delta_B(A)$ or in Lemma 3.4(ii) of Ha [45], where it is required that K is solid or the set A has efficient points.

After that, we are going to study $\preccurlyeq^{\forall\exists}$-monotonicity for the function $\mathbb{D}^{si}$.

Proposition 2.2.36. *(i) Let $B \in \mathcal{P}_0(Y)$. The function $\mathbb{D}^{si}(\cdot, B)$ is $\preccurlyeq^{\forall\exists}$-monotone increasing on $\mathcal{P}_0(Y)$.*

(ii) Let $A \in \mathcal{P}_0(Y)$. The function $\mathbb{D}^{si}(A, \cdot)$ is $\preccurlyeq^{\forall\exists}$-monotone decreasing on $\mathcal{P}_0(Y)$.

Proof. (i) Let $A_1, A_2 \in \mathcal{P}_0(Y)$ and assume that $A_1 \preccurlyeq^{\forall\exists} A_2$. Then $A_2 \subset A_1 + K$, and we deduce that $\inf_{z\in A_1+K} D(z - b, -K) \leq \inf_{y\in A_2} D(y - b, -K)$, for all $b \in B$, that is, $\mathbb{D}^{si}(A_1 + K, b) \leq \mathbb{D}^{si}(A_2, b)$. Therefore, $\sup_{b\in B} \mathbb{D}^{si}(A_1 + K, b) \leq \sup_{b\in B} \mathbb{D}^{si}(A_2, b)$, that is, $\mathbb{D}^{si}(A_1+K, B) \leq \mathbb{D}^{si}(A_2, B)$ by Remark 2.1.15(c). Since $\mathbb{D}^{si}(A_1 + K, B) = \mathbb{D}^{si}(A_1, B)$ by Proposition 2.2.30, we have that $\mathbb{D}^{si}(A_1, B) \leq \mathbb{D}^{si}(A_2, B)$.

(ii) Let $B_1, B_2 \in \mathcal{P}_0(Y)$ and assume that $B_1 \preccurlyeq^{\forall\exists} B_2$, or equivalently, $B_2 \subset B_1 + K$. Using this fact, by Remark 2.1.15(c) we have that

$$\mathbb{D}^{si}(A, B_2) = \sup_{y\in B_2} \mathbb{D}^{si}(A, y) \leq \sup_{y\in B_1+K} \mathbb{D}^{si}(A, y) = \mathbb{D}^{si}(A, B_1 + K) = \mathbb{D}^{si}(A, B_1),$$

since the last equality is true by Proposition 2.2.30. □

Results as Proposition 2.2.36 are Proposition 4.2(a) in Gutiérrez et al. [38] for the function $\varphi_{e,B}(A)$ and Theorem 3.8 in Hernández and Rodríguez-Marín [51] for the Gerstewitz's scalarization G_e but demanding that the sets A and B are K-proper.

In the following proposition we show that the function $\mathbb{D}^{si}$ is invariant with respect to the closure.

Proposition 2.2.37. *Let $A, B \in \mathcal{P}_0(Y)$, then*

(i) $\mathbb{D}^{si}(A, B) = \mathbb{D}^{si}(A, \operatorname{cl} B)$.

(ii) $\mathbb{D}^{si}(A, B) = \mathbb{D}^{si}(\operatorname{cl} A, B)$.

Proof. (i) As $\operatorname{cl} B \preccurlyeq^{\forall\exists} B$, it follows that $\mathbb{D}^{si}(A, B) \leq \mathbb{D}^{si}(A, \operatorname{cl} B)$ since the function $\mathbb{D}^{si}(A, \cdot)$ is $\preccurlyeq^{\forall\exists}$-monotone decreasing by Proposition 2.2.36(ii). By contradiction, assume that $\mathbb{D}^{si}(A, B) < \mathbb{D}^{si}(A, \operatorname{cl} B)$. As $\mathbb{D}^{si}(A, \operatorname{cl} B) = \sup_{y\in\operatorname{cl} B} \mathbb{D}^{si}(A, y)$

by Remark 2.1.15(c), there exists $y_0 \in \operatorname{cl} B$ such that

$$\mathbb{D}^{si}(A, B) < \mathbb{D}^{si}(A, y_0) \leq \mathbb{D}^{si}(A, \operatorname{cl} B).$$

Since $y_0 \in \operatorname{cl} B$, there exists a sequence $y_n \in B$ such that $y_n \to y_0$. As by Theorem 2.2.27 the function $y \to \mathbb{D}^{si}(A, y)$ is continuous, we obtain that $\mathbb{D}^{si}(A, y_n) \to \mathbb{D}^{si}(A, y_0)$. Therefore, for n large enough, it follows that $\mathbb{D}^{si}(A, B) < \mathbb{D}^{si}(A, y_n)$, and this is a contradiction because $\mathbb{D}^{si}(A, B) = \sup_{y \in B} \mathbb{D}^{si}(A, y)$ and $y_n \in B$.

(ii) As $\operatorname{cl} A \preccurlyeq^{\forall\exists} A$, it follows that $\mathbb{D}^{si}(\operatorname{cl} A, B) \leq \mathbb{D}^{si}(A, B)$ since the function $\mathbb{D}^{si}(\cdot, B)$ is $\preccurlyeq^{\forall\exists}$-monotone increasing by Proposition 2.2.36(i). By contradiction, assume that $\mathbb{D}^{si}(\operatorname{cl} A, B) < \mathbb{D}^{si}(A, B)$. As $\mathbb{D}^{si}(A, B) = \sup_{b \in B} \mathbb{D}^{si}(A, b)$ by Remark 2.1.15(c), there exists $b \in B$ such that $\mathbb{D}^{si}(\operatorname{cl} A, B) < \mathbb{D}^{si}(A, b) \leq \mathbb{D}^{si}(A, B)$. Since $\mathbb{D}^{si}(\operatorname{cl} A, B) = \sup_{y \in B} \mathbb{D}^{si}(\operatorname{cl} A, y)$, it follows that $\mathbb{D}^{si}(\operatorname{cl} A, b) \leq \mathbb{D}^{si}(\operatorname{cl} A, B) < \mathbb{D}^{si}(A, b)$, with $\mathbb{D}^{si}(\operatorname{cl} A, b) = \inf_{z \in \operatorname{cl} A} D(z - b, -K)$. Therefore, there exists $z_0 \in \operatorname{cl} A$ such that

$$\inf_{z \in \operatorname{cl} A} D(z - b, -K) \leq D(z_0 - b, -K) < \inf_{y \in A} D(y - b, -K).$$

Since $z_0 \in \operatorname{cl} A$, there exists a sequence $z_n \in A$ such that $z_n \to z_0$. As the function $y \to D(y, -K)$ is continuous by Lemma 1.3.19(i), we have that $D(z_n - b, -K) \to D(z_0 - b, -K)$. Therefore, for n large enough, we have that $D(z_n - b, -K) < \inf_{y \in A} D(y - b, -K)$, with $z_n \in A$, which is a contradiction. □

Remark 2.2.38. As a consequence of Propositions 2.2.30 and 2.2.37,

$$\mathbb{D}^{si}(A, B) = \mathbb{D}^{si}(\operatorname{cl}(\operatorname{cl} A + K), \operatorname{cl}(\operatorname{cl} B + K)) = \mathbb{D}^{si}(\operatorname{cl}(A + K), \operatorname{cl}(B + K)),$$

since $\operatorname{cl}(\operatorname{cl} A + K) = \operatorname{cl}(A + K)$ as it can be easily checked. This remark explains why in many results it is necessary to require K-closedness, which is due to the fact that the function $\mathbb{D}^{si}$ does not distinguish A from $\operatorname{cl}(A + K)$.

If K is a solid convex cone, then by using Remark 2.2.38, [9, Lemma 2.5] and Proposition 2.2.37 we derive that $\mathbb{D}^{si}(A, B) = \mathbb{D}^{si}(A + \operatorname{int} K, B + \operatorname{int} K)$, since $A + \operatorname{int} K = \operatorname{int}(A + K)$.

Next, we collect some similar results now for the upper set less order relation $\preccurlyeq^{\forall\exists}$ and the function $\widehat{\mathbb{D}}^{si}$.

Theorem 2.2.39. *Let $A, B, A_1, A_2, B_1, B_2 \in \mathcal{P}_0(Y)$. It is verified the following assertions:*

(i) $\widehat{\mathbb{D}}^{si}(A,B) = \widehat{\mathbb{D}}^{si}(A-K,B) = \widehat{\mathbb{D}}^{si}(A,B-K) = \widehat{\mathbb{D}}^{si}(A-K,B-K)$.

(ii) If $A \sim^{\preccurlyeq^{\forall\exists}} B$, *then* $\widehat{\mathbb{D}}^{si}(A,B) = \widehat{\mathbb{D}}^{si}(B,A)$.

(iii) If $A \in \mathcal{P}_{0,-K}(Y)$, *then* $\widehat{\mathbb{D}}^{si}(A,A) = 0$.

(iv) Let $A \in \mathcal{P}_{0,-K}(Y)$. *If* $A \sim^{\preccurlyeq^{\forall\exists}} B$, *then* $\widehat{\mathbb{D}}^{si}(A,B) = \widehat{\mathbb{D}}^{si}(B,A) = 0$.

The reverse implication is true if A and B are $(-K)$-closed.

(v)(a) If $B_1 \sim^{\preccurlyeq^{\forall\exists}} B_2$, *then* $\widehat{\mathbb{D}}^{si}(A,B_1) = \widehat{\mathbb{D}}^{si}(A,B_2)$.

(b) If $A_1 \sim^{\preccurlyeq^{\forall\exists}} A_2$, *then* $\widehat{\mathbb{D}}^{si}(A_1,B) = \widehat{\mathbb{D}}^{si}(A_2,B)$.

(vi)(a) The function $\widehat{\mathbb{D}}^{si}(A,\cdot)$ *is* $\preccurlyeq^{\forall\exists}$*-monotone decreasing on* $\mathcal{P}_0(Y)$.

(b) The function $\widehat{\mathbb{D}}^{si}(\cdot,B)$ *is* $\preccurlyeq^{\forall\exists}$*-monotone increasing on* $\mathcal{P}_0(Y)$.

(vii) It is verified that $\widehat{\mathbb{D}}^{si}(A,B) = \widehat{\mathbb{D}}^{si}(A,\operatorname{cl} B) = \widehat{\mathbb{D}}^{si}(\operatorname{cl} A,B)$.

Proof. We only prove part (a) of (vi). If $B_1 \preccurlyeq^{\forall\exists} B_2$, then by Lemma 1.3.9 we have that $B_2 \preccurlyeq^{\forall\exists}_{-K} B_1$. Applying Proposition 2.2.36(i), it follows that $\mathbb{D}^{si}_{-K}(B_2,A) \leq \mathbb{D}^{si}_{-K}(B_1,A)$. By Remark 2.1.15(b), it results that $\widehat{\mathbb{D}}^{si}(A,B_2) \leq \widehat{\mathbb{D}}^{si}(A,B_1)$.

The rest of results are proved using the same ideas. □

Some results of the type Theorem 2.2.39 have been proved in other papers. For example, Propositions 3.3(iii), 3.2(i), 3.2(ii) and 3.3(i) in Xu and Li [129] for the function $\mathfrak{D}_{A-K}(B)$ are, respectively, similar to parts (i), (iii), (iv) and (vi)(b) of Theorem 2.2.39. In Araya [5] for the function $\varphi_{e,B}(A)$, parts (iii) and (iv) of Theorem 3.2 are extended and improved, respectively, by parts (vi)(a) and (vi)(b) of Theorem 2.2.39.

2.3 Characterization of lower and upper set relations of Kuroiwa

In this section, by using the useful properties which have been shown in the former section, new characterizations of the lower set less relation $\preccurlyeq^{\forall\exists}$ and the upper set less relation $\preccurlyeq^{\forall\exists}$ of Kuroiwa by means of the set scalarization functions

$\mathbb{D}^{si}$ and $\widehat{\mathbb{D}}^{si}$ given in Definition 2.1.14, are provided. In the case of K is a solid convex cone, characterizations for the corresponding strict set relations $\prec_s^{\forall\exists}$ and $\prec_s^{\forall\exists}$ by requiring assumptions of K-compactness are discussed. We also deal with strict monotonicity for the functions $\mathbb{D}^{si}$ and $\widehat{\mathbb{D}}^{si}$ with respect to the strict lower set less relation $\prec_s^{\forall\exists}$ and the strict upper set less relation $\prec_s^{\forall\exists}$.

In the following theorem, which is an immediate consequence of Theorem 2.2.24, we establish a characterization of the preorder $\preccurlyeq^{\forall\exists}$ using the function $\mathbb{D}^{si}$.

Theorem 2.3.1. *If $A, B \in \mathcal{P}_0(Y)$, then*

$$A \preccurlyeq^{\forall\exists} B \Leftrightarrow \mathbb{D}^{si}(A, B) \leq 0.$$

For '$\Leftarrow$' part it is required that A is K-closed.

Similar results to Theorem 2.3.1 have been proved by other authors. For example, in Theorem 3.10(iii) of Hernández and Rodríguez-Marín [51] for the Gerstewitz's scalarization $G_e(A, B)$, in Proposition 6.6(b) of Gutiérrez et al. [38] for the function $\Delta_B(A)$, in Proposition 4.11 of Gutiérrez et al. [38] for the function $\varphi_B(A)$ and in Proposition 3.8 of Chen et al. [11], where it is assumed that $\inf_{a \in A} D(a - b, -K)$ is achieved for all $b \in B$. In Lemma 3.3 of Ha [45] it is required that A is K-compact and, therefore, our theorem represents a meaningful improvement since it requires weaker hypothesis.

In Theorem 2.3.1, the K-closedness of A cannot be removed as it is showed in the following example.

Example 2.3.2. Consider $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = \{(x, x) : 0 < x \leq 1\}$ and $B = \{(1, 0)\}$. One has $\mathbb{D}^{si}(A, B) = 0$ and, however $A \not\preccurlyeq^{\forall\exists} B$. Note that A is not K-closed.

Next, we state other characterization of the preorder $\preccurlyeq^{\forall\exists}$.

Theorem 2.3.3. *If $A, B \in \mathcal{P}_0(Y)$, then*

$$A \preccurlyeq^{\forall\exists} B \Leftrightarrow \mathbb{D}^{si}(A, y) \leq \mathbb{D}^{si}(B, y), \ \forall y \in Y.$$

For '$\Leftarrow$' part it is required that A is K-closed.

Proof. (⇒) If $A \preccurlyeq^{\forall\exists} B$, then we have the conclusion taking into account that the function $\mathbb{D}^{si}(\cdot, y)$ is $\preccurlyeq^{\forall\exists}$-monotone increasing for each $y \in Y$ due to Proposition 2.2.36(i).

(⇐) By contradiction, let us suppose that $A \not\preccurlyeq^{\forall\exists} B$. Then there exists $b_0 \in B$ such that $b_0 \notin A+K$. As A is K-closed, one has $b_0 \notin \mathrm{cl}(A+K)$, and by Theorem 2.2.24 we deduce that $\mathbb{D}^{si}(A, b_0) > 0$.

On the other hand, from the hypothesis with $y = b_0$, it follows that

$$\mathbb{D}^{si}(A, b_0) \leq \mathbb{D}^{si}(B, b_0) \leq 0$$

since $\mathbb{D}^{si}(B, b_0) = \inf_{b \in B} D(b - b_0, -K) \leq 0$ because for $b = b_0$ we have $D(b_0 - b_0, -K) = 0$ and, therefore, we have obtained a contradiction. □

The following necessary condition is obtained by applying Proposition 2.2.36(ii).

Theorem 2.3.4. *Let $A, B \in \mathcal{P}_0(Y)$. If $A \preccurlyeq^{\forall\exists} B$, then*

$$\mathbb{D}^{si}(y, A) \geq \mathbb{D}^{si}(y, B), \ \forall y \in Y.$$

An immediate consequence of Theorems 2.3.3 and 2.3.4, (taking into account Lemma 1.3.19(xiii) for part (ii)) choosing $y = 0$, is the next result, which states necessary conditions for the fulfillment of $A \preccurlyeq^{\forall\exists} B$.

Corollary 2.3.5. *Let $A, B \in \mathcal{P}_0(Y)$. If $A \preccurlyeq^{\forall\exists} B$, then*

(i) $\inf_{a \in A} D(a, -K) \leq \inf_{b \in B} D(b, -K)$.

(ii) $\sup_{a \in A} D(a, K) \geq \sup_{b \in B} D(b, K)$.

A similar result to Corollary 2.3.5(i), but for the Gerstewitz function $g_{e,K}$, is Theorem 3.7 of Köbis and Köbis [81]. The last part of Example 2.3.6 proves that part (i) of Corollary 2.3.5 is satisfied and however part (ii) does not, that is, parts (i) and (ii) are independent.

The necessary condition of Theorem 2.3.4 is not sufficient as the following example shows.

Example 2.3.6. Consider $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = \{a_1 = (0,0),\ a_2 = (-2,2)\}$ and $B = \{b = (-1,1)\}$. For every $y \in Y$, the function $D(\cdot, y + K)$ is convex on

Y by Lemma 1.3.19(ii), and as $b = \frac{1}{2}a_1 + \frac{1}{2}a_2$, we deduce that

$$D(b, y+K) \leq \tfrac{1}{2}D(a_1, y+K) + \tfrac{1}{2}D(a_2, y+K) \leq \max\{D(a_1, y+K), D(a_2, y+K)\}.$$

Therefore the conclusion of Theorem 2.3.4 is satisfied, however $A \not\preccurlyeq^{\forall\exists} B$.

If we change A to $A = \{a_1\}$, then part (i) of Corollary 2.3.5 is satisfied but part (ii) does not.

In the following theorem, we establish a sufficient condition to obtain $A \preccurlyeq^{\forall\exists} B$.

Theorem 2.3.7. *Let $A, B \in \mathcal{P}_0(Y)$. Then, we have that*

$$\mathbb{D}^{si}(A, b) < 0, \ \forall b \in B \Rightarrow A \preccurlyeq^{\forall\exists} B.$$

In particular, the hypothesis is true if $\mathbb{D}^{si}(A, B) < 0$.

Proof. Let $\mathbb{D}^{si}(A, b) < 0$, for all $b \in B$, that is, $\inf_{a\in A} D(a - b, -K) < 0$, for all $b \in B$. By contradiction, assume that $A \not\preccurlyeq^{\forall\exists} B$, that is, $B \not\subset A + K$. Then, there exists $b_0 \in B$ such that $b_0 \notin A + K$, that is, $b_0 \notin a + K$, for all $a \in A$. Consequently, by Lemma 1.3.19(v) we have that $D(a - b_0, -K) \geq 0$, for all $a \in A$. Thus, we obtain that $\mathbb{D}^{si}(A, b_0) = \inf_{a\in A} D(a - b_0, -K) \geq 0$, which is a contradiction. □

In the next example, we can see that Theorem 2.3.7 works and, however, Proposition 3.8 in Chen et al. [11] does not work.

Example 2.3.8. Consider $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = \{(0, y) : y > 0\}$ and $B = \{b_n = (n, 1/n) : n \in \mathbb{N}\}$. We have $\mathbb{D}^{si}(A, b_n) = -1/n < 0$ for all n, and so we can apply Theorem 2.3.7 and to deduce that $A \preccurlyeq^{\forall\exists} B$. However, Proposition 3.8 in Chen et al. [11] is not applicable because $\inf_{a\in A} D(a - b, -K)$ is not achieved for any $b \in B$.

In the following lemma, we give conditions in order to a function $g : Y \to \mathbb{R}$ achieves its minimum or maximum on a set A.

Lemma 2.3.9. *Let $A \in \mathcal{P}_0(Y)$. If $g : Y \to \mathbb{R}$ is continuous, K-increasing (resp., K-decreasing) and A is K-compact, then g achieves its minimum (resp., maximum) on A.*

Proof. First of all, let us see that g is lower bounded on A.

Indeed, for all $n \in \mathbb{N}$ we define $U_n = \{x \in Y : \ g(x) > -n\}$. The sets U_n are open because g is continuous. Moreover, $U_n + K = U_n$ since if $x \in U_n$, $q \in K$, then $x \leq_K x + q$ and as g is K-increasing, it follows that $g(x+q) \geq g(x) > -n$, that is, $x + q \in U_n$. It is satisfied that $A \subset \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} (U_n + K)$. Since A is K-compact, we can extract a finite subcover of $\{U_n : \ n \in \mathbb{N}\}$ and as $U_n \subset U_{n+1}$ for all n, we can assume that there exists n such that $A \subset U_n$, which means that g is lower bounded on A by $-n$.

Consequently, we have that $L = \inf_{y \in A} g(y)$ is finite. By contradiction, we assume that the function g does not achieve its infimum L on A. Then, for all $x \in A$, we define the positive number

$$\delta(x) := (g(x) - L)/2.$$

For each $x \in A$, we consider the set

$$U_x = \{z \in Y : g(z) > L + \delta(x)\}.$$

It is verified that $U_x = U_x + K$, which is proved as before with U_n. The sets U_x are open because g is continuous. Furthermore, $A \subset \bigcup_{x \in A} U_x = \bigcup_{x \in A} (U_x + K)$ as $x \in U_x$ for all $x \in A$. Since A is K-compact, there exist $x_1, \ldots, x_n \in A$ such that $A \subset \bigcup_{i=1}^{n} U_{x_i}$. Let $\delta = \min\{\delta(x_i) : \ i = 1, \ldots, n\} > 0$. Therefore, for all $x \in A$ there exists x_i such that $x \in U_{x_i}$, and by definition of U_{x_i} it results that

$$g(x) > L + \delta(x_i) \geq L + \delta,$$

which is a contradiction since the infimum is $L = \inf_{x \in A} g(x) \geq L + \delta$.

Finally, we must prove that if g is K-decreasing, then it achieves its maximum, but this can be proved in a similar way and, hence, we omit the proof. □

The next proposition provides conditions so that the function $D(\cdot - y, -K)$ achieves its finite minimum on A and the function $\mathbb{D}^{si}(A, \cdot)$ achieves its finite maximum on B.

Proposition 2.3.10. *Let $A, B \in \mathcal{P}_0(Y)$.*

(i) If A is K-compact, then the function $D(\cdot - y, -K)$ achieves its finite minimum on A, for all $y \in Y$.

(ii) If A is K-proper and B is K-compact, then the function $y \to \mathbb{D}^{si}(A, y) \in \mathbb{R}$, $y \in Y$, achieves its finite maximum on B. In consequence, there exists $b_0 \in B$ such that $\mathbb{D}^{si}(A, B) = \mathbb{D}^{si}(A, b_0)$.

Proof. (i) It is a consequence of Lemma 2.3.9, since we observe that for all $y \in Y$, the function $g(x) = D(x - y, -K)$ is continuous by Lemma 1.3.19(i) and K-increasing by Lemma 1.3.19(viii).

(ii) As A is K-proper, by Theorem 2.2.27 the function $y \to g(y) = \mathbb{D}^{si}(A, y)$ takes its values in $\mathbb{R}$ and is continuous and, moreover, it is K-decreasing by Proposition 2.2.36(ii). So, by Lemma 2.3.9 the supremum $\sup_{b\in B} g(b)$ is achieved, that is, there exists $b_0 \in B$ such that $\sup_{b\in B} \mathbb{D}^{si}(A, b) = \mathbb{D}^{si}(A, b_0)$ and, therefore, by Remark 2.1.15(c), $\mathbb{D}^{si}(A, B) = \mathbb{D}^{si}(A, b_0)$. □

Analogous results to previous Proposition 2.3.10(ii) have been proved in the literature. For example, in Proposition 3.4 of Hernández and Rodríguez-Marín [51] for the Gerstewitz's scalarization function $G_e(A, B)$, and in Theorem 3.2 of Xu and Li [129] for the function $\mathfrak{D}_{A+K}(B)$ of Xu and Li. In both results, in addition to our assumptions, it is required that A is K-closed, and moreover in the first one it is demanded that $G_e(A, B) < \infty$. In Lemma 3.2 of Ha [45] it is required that A is K-bounded and B is K-compact.

Now, we are going to give a characterization for the strict lower set less preorder relation $\preccurlyeq_s^{\forall\exists}$ based on K-compactness.

Theorem 2.3.11. *Let $A, B \in \mathcal{P}_0(Y)$ and let K be solid. Then*

$$A \preccurlyeq_s^{\forall\exists} B \Leftrightarrow \mathbb{D}^{si}(A, B) < 0.$$

For '$\Rightarrow$' part it is required that B is K-compact.

Proof. If A is not K-proper, the result is clear by Proposition 2.2.5 (both parts of the equivalence are true for all $B \in \mathcal{P}_0(Y)$). Thus, assume that A is K-proper.

($\Rightarrow$) $A \preccurlyeq_s^{\forall\exists} B$ if and only if $B \subset A + \operatorname{int} K$, that is, for all $b \in B$ there exists $a_0 \in A$ such that $b \in a_0 + \operatorname{int} K$ and, therefore, $a_0 - b \in -\operatorname{int} K$. By Lemma 1.3.19(iii), it follows that $D(a_0 - b, -K) < 0$ and, then

$$\mathbb{D}^{si}(A, b) = \inf_{a \in A} D(a - b, -K) < 0, \ \forall b \in B.$$

Hence, as B is K-compact and A is K-proper, by Proposition 2.3.10(ii) one has $\mathbb{D}^{si}(A, B) = \mathbb{D}^{si}(A, b_0)$ for some $b_0 \in B$, and so it results that $\mathbb{D}^{si}(A, B) < 0$.

($\Leftarrow$) Assume that $A \not\preccurlyeq_s^{\forall\exists} B$, that is, $B \not\subset A + \operatorname{int} K$. Then, there exists $\bar{b} \in B$ such that $\bar{b} \notin A + \operatorname{int} K$. So, for all $a \in A$ we have that $a - \bar{b} \notin -\operatorname{int} K$ and, therefore, by Lemma 1.3.19(iii) we have that $D(a - \bar{b}, -K) \geq 0$, for all $a \in A$. This implies that $\mathbb{D}^{si}(A, \bar{b}) = \inf_{a \in A} D(a - \bar{b}, -K) \geq 0$. Consequently, by Remark 2.1.15(c) it results that $\mathbb{D}^{si}(A, B) = \sup_{b \in B} \mathbb{D}^{si}(A, b) \geq 0$, which is a contradiction. □

A result as Theorem 2.3.11 is Corollary 3.11(i) of Hernández and Rodríguez-Marín [51] for the Gerstewitz's scalarization $G_e(A, B)$ where it is required that the sets A and B are K-compact.

Corollary 2.3.12. *If B is K-compact, $B \subset \operatorname{int} A$ and K is solid, then* $\mathbb{D}^{si}(A, B) < 0$.

Proof. Since $B \subset \operatorname{int} A \subset \operatorname{int}(A + K) = A + \operatorname{int} K$, we can apply the previous Theorem 2.3.11 and we obtain the conclusion. □

Let us illustrate with an example that even when $B \subset \operatorname{int}(A + K)$, we cannot ensure that $\mathbb{D}^{si}(A, B) < 0$, that is, the conclusion of Theorem 2.3.11 is not true if B is not K-compact.

Example 2.3.13. With the data of Example 2.3.8, where B is not a K-compact set, one has $A \preccurlyeq_s^{\forall\exists} B$ and, however, $\mathbb{D}^{si}(A, B) = \sup_n \{\frac{-1}{n}\} = 0$.

Remark 2.3.14. Let $A \in \mathcal{P}_0(Y)$ and let K be solid. If A is K-compact, then $A \not\preccurlyeq_s^{\forall\exists} A$.

Indeed, if were $A \preccurlyeq_s^{\forall\exists} A$, by Theorem 2.3.11 we derive that $\mathbb{D}^{si}(A, A) < 0$, but by Proposition 2.2.34 it follows that $\mathbb{D}^{si}(A, A) = 0$, which is a contradiction. Let us observe that A is K-proper by Remark 1.3.1 since is K-compact.

Next, we are going to study the properties of strict $\preccurlyeq_s^{\forall\exists}$-monotonicity of the function $\mathbb{D}^{si}$.

Proposition 2.3.15. *(i) Assume that $A_1, A_2, B \in \mathcal{P}_0(Y)$, A_2 and B are K-compact and K is solid. If $A_1 \preccurlyeq_s^{\forall\exists} A_2$, then $\mathbb{D}^{si}(A_1, B) < \mathbb{D}^{si}(A_2, B)$.*

(ii) Assume that $A, B_1, B_2 \in \mathcal{P}_0(Y)$, A and B_2 are K-compact and K is solid. If $B_1 \preccurlyeq_s^{\forall\exists} B_2$, then $\mathbb{D}^{si}(A, B_1) > \mathbb{D}^{si}(A, B_2)$.

Proof. (i) If A_1 is not K-proper, then $\mathbb{D}^{si}(A_1, B) = -\infty$ and $\mathbb{D}^{si}(A_2, B) \in \mathbb{R}$ by Corollary 2.2.11 and Proposition 2.2.5 since B is K-bounded and A_2 is K-proper by Remark 1.3.1 and, therefore, the conclusion is satisfied. Thus, assume that A_1 is K-proper.

By definition of strict $\preccurlyeq_s^{\forall\exists}$-preorder, we have that $A_1 \preccurlyeq_s^{\forall\exists} A_2$ if and only if for all $a_2 \in A_2$, there exists $\widetilde{a}_1 \in A_1$ such that $\widetilde{a}_1 \leq_{\operatorname{int} K} a_2$ and, hence, $\widetilde{a}_1 - b \leq_{\operatorname{int} K} a_2 - b$, for all $b \in B$. By Lemma 1.3.19(viii), it results that $D(\widetilde{a}_1 - b, -K) < D(a_2 - b, -K)$, for all $b \in B$. From this, we have that

$$\mathbb{D}^{si}(A_1, b) = \inf_{a_1 \in A_1} D(a_1 - b, -K) < D(a_2 - b, -K), \ \forall a_2 \in A_2, \forall b \in B.$$

Since A_2 is K-compact, by Proposition 2.3.10(i) it follows that there exists $\bar{a}_2 \in A_2$ such that

$$\mathbb{D}^{si}(A_1, b) < \mathbb{D}^{si}(A_2, b) = D(\bar{a}_2 - b, -K) = \inf_{a_2 \in A_2} D(a_2 - b, -K), \ \forall b \in B. \quad (2.26)$$

Since B is K-compact and A_1 is K-proper, by Proposition 2.3.10(ii) there exists $b_0 \in B$ such that $\mathbb{D}^{si}(A_1, b_0) = \sup_{b \in B} \mathbb{D}^{si}(A_1, b) = \mathbb{D}^{si}(A_1, B)$. By (2.26) and Remark 2.1.15(c), it follows that

$$\mathbb{D}^{si}(A_1, B) = \mathbb{D}^{si}(A_1, b_0) < \mathbb{D}^{si}(A_2, b_0) \leq \sup_{b \in B} \mathbb{D}^{si}(A_2, b) = \mathbb{D}^{si}(A_2, B).$$

Consequently, $\mathbb{D}^{si}(A_1, B) < \mathbb{D}^{si}(A_2, B)$.

(ii) As $B_1 \preccurlyeq_s^{\forall\exists} B_2$, we have that for all $b_2 \in B_2$ there exists $\widetilde{b}_1 \in B_1$ such that $\widetilde{b}_1 \leq_{\operatorname{int} K} b_2$. Then, $a - b_2 \leq_{\operatorname{int} K} a - \widetilde{b}_1$, for all $a \in A$. By Lemma 1.3.19(viii), it follows that $D(a - b_2, -K) < D(a - \widetilde{b}_1, -K)$, for all $a \in A$. Therefore,

$$\mathbb{D}^{si}(A, b_2) = \inf_{a \in A} D(a - b_2, -K) < D(a - \widetilde{b}_1, -K), \ \forall a \in A.$$

By Proposition 2.3.10(i), since A is K-compact,

$$\mathbb{D}^{si}(A, b_2) < \mathbb{D}^{si}(A, \widetilde{b}_1) = \inf_{a \in A} D(a - \widetilde{b}_1, -K),$$

and this implies $\mathbb{D}^{si}(A, b_2) < \mathbb{D}^{si}(A, \widetilde{b}_1) \leq \sup_{b_1 \in B_1} \mathbb{D}^{si}(A, b_1) = \mathbb{D}^{si}(A, B_1)$, for all $b_2 \in B_2$. Since B_2 is K-compact and A is K-proper (see Remark 1.3.1), by Proposition 2.3.10(ii) there exists $\bar{b}_2 \in B_2$ such that $\mathbb{D}^{si}(A, B_2) = \mathbb{D}^{si}(A, \bar{b}_2) < \mathbb{D}^{si}(A, B_1)$. □

Corollary 2.3.16. *If A is a K-compact set of Y, then $\mathbb{D}^{si}(A, \cdot)$ (resp., $\mathbb{D}^{si}(\cdot, A)$) is strictly $\preccurlyeq_s^{\forall\exists}$-monotone decreasing (resp., increasing) on K-compact sets of Y.*

Some results of sort of Proposition 2.3.15 have been proved in several papers. For example, in Theorem 3.9 of Hernández and Rodríguez-Marín [51] for the Gerstewitz's scalarization G_e, where it is required that all sets are K-compact, and in Theorem 3.5(g) of Gutiérrez et al. [41] for the function $\varphi_{e,B}(A)$ where it is assumed that B is K-bounded and A_2 is K-compact.

Now, we translate the obtained results for the $\preccurlyeq^{\forall\exists}$-preorder to the $\preccurlyeq^{\forall\exists}$-preorder in the following theorem. Its proof follows the same ideas that the proof of Theorem 2.2.39, and for this reason is omitted.

Theorem 2.3.17. *Let $A, B, A_1, A_2 \in \mathcal{P}_0(Y)$. Then*

(i) $A \preccurlyeq^{\forall\exists} B$ if and only if $\widehat{\mathbb{D}}^{si}(A, B) \leq 0$. For '$\Leftarrow$' part it is required that B is $(-K)$-closed.

(ii) $A \preccurlyeq^{\forall\exists} B$ if and only if $\widehat{\mathbb{D}}^{si}(y, A) \geq \widehat{\mathbb{D}}^{si}(y, B)$, for all $y \in Y$. For '$\Leftarrow$' part it is required that B is $(-K)$-closed.

(iii) $A \preccurlyeq^{\forall\exists} B$ implies that $\widehat{\mathbb{D}}^{si}(A, y) \leq \widehat{\mathbb{D}}^{si}(B, y)$, for all $y \in Y$.

(iv) If $\widehat{\mathbb{D}}^{si}(a, B) < 0$, for all $a \in A$, then $A \preccurlyeq^{\forall\exists} B$.

(v) If K is solid, then $A \preccurlyeq_s^{\forall\exists} B$ if and only if $\widehat{\mathbb{D}}^{si}(A, B) < 0$. For '$\Rightarrow$' part it is required that A is $(-K)$-compact.

(vi) If K is solid, and A_1 and B are $(-K)$-compact, then $A_1 \preccurlyeq_s^{\forall\exists} A_2$ implies $\widehat{\mathbb{D}}^{si}(A_1, B) < \widehat{\mathbb{D}}^{si}(A_2, B)$ and $\widehat{\mathbb{D}}^{si}(B, A_1) > \widehat{\mathbb{D}}^{si}(B, A_2)$.

Similar results to parts (i), (v) and (vi) of Theorem 2.3.17 are, respectively, Proposition 3.2(iii), Proposition 3.2(iv) (it is required B is $(-K)$-closed and A

is $(-K)$-compact) and Proposition 3.3(ii) (it is required B_1 and B_2 are $(-K)$-compact) in Xu and Li [129] for the function $\mathfrak{D}_{-K}(B)$ of Xu and Li.

Chapter 3

Six set extensions of oriented distance function

This chapter is concerned with set oriented distances which are set scalarization functions extensions of the oriented distance function of Hiriart-Urruty, noted $\mathbb{D}_K^\alpha(A,B)$ and $\widehat{\mathbb{D}}_K^\alpha(A,B)$, four of which are new. Although we are going to define functions of type sup-sup, sup-inf, inf-sup and inf-inf, they will be called scalarization functions of type sup-inf and inf-sup for the sake of simplicity.

It is worth noting that in Theorem 1.3.15 of Gutiérrez, Jiménez, Miglierina and Molho, we have found inspiration to define the new set oriented distances.

The results stated in this chapter are collected in [68, Sections 3 and 4] and [69, Sections 3 and 4].

3.1 Definitions and properties

In this Section, six set scalarizations of type sup-inf and inf-sup which are extensions of the oriented distance, denoted by $\mathbb{D}_K^\alpha(A,B)$ and $\widehat{\mathbb{D}}_K^\alpha(A,B)$, are presented. Relationships among them are presented, characterizations of these scalarizations are given and, furthermore, some of their main properties are studied as, for example, finitude under suitable assumptions of cone-properness and cone-boundedness, invariance by conic extensions, monotonicity by considering the six set relations introduced by Kuroiwa, and closure property.

We start by presenting the six set scalarization functions based on the oriented distance mentioned above. Two of them have been treated in Section 2.1.

Definition 3.1.1. If $A, B \in \mathcal{P}_0(Y)$, we define the following functions from $\mathcal{P}_0(Y) \times \mathcal{P}_0(Y)$ into $\mathbb{R} \cup \{\pm\infty\}$ as follows:

(i) $\mathbb{D}^{ss}(A, B) = \sup_{b\in B} \sup_{a\in A} D(a - b, -K)$.

(ii) $\widehat{\mathbb{D}}^{is}(A, B) = \inf_{a\in A} \sup_{b\in B} D(a - b, -K)$.

(iii) $\mathbb{D}^{si}(A, B) = \sup_{b\in B} \inf_{a\in A} D(a - b, -K)$.

(iv) $\mathbb{D}^{is}(A, B) = \inf_{b\in B} \sup_{a\in A} D(a - b, -K)$.

(v) $\widehat{\mathbb{D}}^{si}(A, B) = \sup_{a\in A} \inf_{b\in B} D(a - b, -K)$.

(vi) $\mathbb{D}^{ii}(A, B) = \inf_{b\in B} \inf_{a\in A} D(a - b, -K)$.

Remark 3.1.2. We can check that $\mathbb{D}^{ss}$ takes its values in $\mathbb{R} \cup \{+\infty\}$ and $\mathbb{D}^{ii}$ in $\mathbb{R} \cup \{-\infty\}$.

We can define two further functions as follows:

- $\widehat{\mathbb{D}}^{ss}(A, B) = \sup_{a\in A} \sup_{b\in B} D(a - b, -K)$ and
- $\widehat{\mathbb{D}}^{ii}(A, B) = \inf_{a\in A} \inf_{b\in B} D(a - b, -K)$,

but it is clear that $\widehat{\mathbb{D}}^{ss}(A, B) = \mathbb{D}^{ss}(A, B) = \sup_{(a,b)\in A\times B} D(a - b, -K)$ and $\widehat{\mathbb{D}}^{ii}(A, B) = \mathbb{D}^{ii}(A, B) = \inf_{(a,b)\in A\times B} D(a - b, -K)$.

If it is necessary to indicate the cone K, we will write $\mathbb{D}^{\alpha}_K(A, B)$ and $\widehat{\mathbb{D}}^{\alpha}_K(A, B)$ instead of $\mathbb{D}^{\alpha}(A, B)$ and $\widehat{\mathbb{D}}^{\alpha}(A, B)$, where $\alpha \in \{ss, is, si, ii\}$. Note that in (ii) and (v) we write $\widehat{\mathbb{D}}$ because they are given in the order $(a \in A, b \in B)$, unlike the other four functions.

The function $\mathbb{D}^{si}$ was introduced by Ha [45], and $\mathbb{D}^{si}$ and $\widehat{\mathbb{D}}^{si}$ have been treated in a previous chapter, where they have been denoted $\mathbb{D}$ and $\widehat{\mathbb{D}}$, respectively. The other four are new.

We denote $\mathcal{D} := \left\{ \mathbb{D}^{ss}, \widehat{\mathbb{D}}^{is}, \mathbb{D}^{si}, \mathbb{D}^{is}, \widehat{\mathbb{D}}^{si}, \mathbb{D}^{ii} \right\}$.

There follows, we are going to study some basic properties.

Remark 3.1.3. Let $A, B \in \mathcal{P}_0(Y)$ and $\bar{\mathbb{D}} \in \mathcal{D}$. Then

(i) If K is a non solid cone, then $\bar{\mathbb{D}}(A, B) \geq 0$.

(ii) $\bar{\mathbb{D}}(A + y, B + y) = \bar{\mathbb{D}}(A, B)$, for all $y \in Y$.

(iii) $\bar{\mathbb{D}}_K(A,B) = \bar{\mathbb{D}}_{-K}(-A,-B)$.

Property (iii) follows from Lemma 1.3.19(xiii).

In the following lemma we state inequality relationships between the functions $\bar{\mathbb{D}} \in \mathcal{D}$ given in Definition 3.1.1.

Lemma 3.1.4. *The following inequalities hold:*

(i) $\mathbb{D}^{ii} \leq \mathbb{D}^{si} \leq \widehat{\mathbb{D}}^{is} \leq \mathbb{D}^{ss}$.

(ii) $\mathbb{D}^{ii} \leq \widehat{\mathbb{D}}^{si} \leq \mathbb{D}^{is} \leq \mathbb{D}^{ss}$.

Proof. We only prove the central inequalities, since the rest are easy to check. Let $A, B \in \mathcal{P}_0(Y)$. It is verified that for all $a \in A$ and all $b \in B$,

$$\inf_{a\in A} D(a-b,-K) \leq D(a-b,-K) \leq \sup_{b\in B} D(a-b,-K).$$

Then, $\sup_{b\in B}\inf_{a\in A} D(a-b,-K) \leq \inf_{a\in A}\sup_{b\in B} D(a-b,-K)$ and, therefore, $\mathbb{D}^{si}(A,B) \leq \widehat{\mathbb{D}}^{is}(A,B)$.

Similarly, one has that $\inf_{b\in B} D(a-b,-K) \leq \sup_{a\in A} D(a-b,-K)$ for all $a \in A$ and all $b \in B$. Then, $\sup_{a\in A}\inf_{b\in B} D(a-b,-K) \leq \inf_{b\in B}\sup_{a\in A} D(a-b,-K)$ and, therefore, $\widehat{\mathbb{D}}^{si}(A,B) \leq \mathbb{D}^{is}(A,B)$. □

The following result is easy to prove (it is enough to take into account Lemma 1.3.19(xiii)) and shows a duality between several pairs of functions in $\mathcal{D}$.

Lemma 3.1.5. *Let $A, B \in \mathcal{P}_0(Y)$. The following assertions are true:*

(i) $\widehat{\mathbb{D}}^{si}_K(A,B) = \mathbb{D}^{si}_{-K}(B,A) = \mathbb{D}^{si}_K(-B,-A)$.

(ii) $\widehat{\mathbb{D}}^{is}_K(A,B) = \mathbb{D}^{is}_{-K}(B,A) = \mathbb{D}^{is}_K(-B,-A)$.

(iii) $\mathbb{D}^{ss}_K(A,B) = \mathbb{D}^{ss}_{-K}(B,A) = \mathbb{D}^{ss}_K(-B,-A)$.

(iv) $\mathbb{D}^{ii}_K(A,B) = \mathbb{D}^{ii}_{-K}(B,A) = \mathbb{D}^{ii}_K(-B,-A)$.

Part (i) is Remark 2.1.15(b).

In the next example, we can see that the null diagonal property for $\mathbb{D}^{si}$ given in Proposition 2.2.34 and for $\widehat{\mathbb{D}}^{si}$ given in Theorem 2.2.39(iii) are not valid for the other scalarizations.

Example 3.1.6. Consider $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$.

(a) Let $A = \{(0,0),(1,1)\}$. Then, $\mathbb{D}^{ss}(A,A) = \sqrt{2}$ and $\mathbb{D}^{ii}(A,A) = -1$.

(b) Let $A = \{(0,0),(-1,1)\}$. Then, $\mathbb{D}^{is}(A,A) = \widehat{\mathbb{D}}^{is}(A,A) = 1$.

Remark 3.1.7. In general, one has $\mathbb{D}^{ii}(A,A) \leq 0$, $\mathbb{D}^{ss}(A,A) \geq 0$ and $\mathbb{D}^{is}(A,A) = \widehat{\mathbb{D}}^{is}(A,A) \geq 0$ for every $A \in P_0(Y)$, as it can be checked.

We observe that if K is a non solid cone, then by Remarks 3.1.3(i) and 3.1.7 we deduce that $\mathbb{D}^{ii}(A,A) = 0$.

In order to simplify our development and to deal with it in an easy and systematic way, we introduce the following functions $h^i, \widehat{h}^i : \mathcal{P}_0(Y) \to \mathbb{R} \cup \{-\infty\}$ and $h^s, \widehat{h}^s : \mathcal{P}_0(Y) \to \mathbb{R} \cup \{+\infty\}$.

Definition 3.1.8. Let $A \in \mathcal{P}_0(Y)$. We define:

(i) $h^i(A) = \inf_{a \in A} D(a, -K)$.

(ii) $h^s(A) = \sup_{a \in A} D(a, -K)$.

(iii) $\widehat{h}^i(A) = \inf_{a \in A} D(-a, -K) = \inf_{a \in A} D(a, K)$.

(iv) $\widehat{h}^s(A) = \sup_{a \in A} D(-a, -K) = \sup_{a \in A} D(a, K)$.

If it were necessary to indicate the cone, we would write h^i_K, h^s_K, etc. We denote $\mathcal{H} := \{h^i, h^s, \widehat{h}^i, \widehat{h}^s\}$.

Lemma 3.1.9. *Let $A \in \mathcal{P}_0(Y)$. Then*

(i) $\widehat{h}^i_K(A) = h^i_{-K}(A) = h^i_K(-A)$.

(ii) $\widehat{h}^s_K(A) = h^s_{-K}(A) = h^s_K(-A)$.

We note in particular that if $A = -A$, then we have $\widehat{h}^i_K(A) = h^i_K(A)$ and $\widehat{h}^s_K(A) = h^s_K(A)$. This is the case, for example, when $A = U_0$.

It is clear that the functions $\bar{\mathbb{D}} \in \mathcal{D}$ can be defined through the functions $\bar{h} \in \mathcal{H}$ as follows.

Lemma 3.1.10. *Let $A, B \in \mathcal{P}_0(Y)$. Then*

(i) $\mathbb{D}^{ss}(A,B) = \sup_{b \in B} h^s(A - b)$.

(ii) $\widehat{\mathbb{D}}^{is}(A,B) = \inf_{a \in A} \widehat{h}^s(B - a)$.

(iii) $\mathbb{D}^{si}(A,B) = \sup_{b \in B} h^i(A - b)$.

(iv) $\mathbb{D}^{is}(A,B) = \inf_{b \in B} h^s(A - b)$.

(v) $\widehat{\mathbb{D}}^{si}(A,B) = \sup_{a \in A} \widehat{h}^i(B - a)$.

(vi) $\mathbb{D}^{ii}(A,B) = \inf_{b \in B} h^i(A - b)$.

Now, we are going to study some properties to the basic functions $\bar{h} \in \mathcal{H}$ given in Definition 3.1.8.

Lemma 3.1.11. *Let $A \in \mathcal{P}_0(Y)$. Then*

(i) $h^i(A) = \mathbb{D}^{si}_K(A, 0)$ *and* $\widehat{h}^i(A) = \mathbb{D}^{si}_{-K}(A, 0)$.

(ii) $h^s(A) = \mathbb{D}^{si}_{-K}(0, A)$ *and* $\widehat{h}^s(A) = \mathbb{D}^{si}_K(0, A)$.

In the following proposition, we provide necessary and sufficient conditions so that the functions $\bar{h} \in \mathcal{H}$ are finite.

Proposition 3.1.12. *Let $A \in \mathcal{P}_0(Y)$. Then*

(i) A is K-proper or K is not solid if and only if $h^i(A) \in \mathbb{R}$.

(ii) A is $(-K)$-bounded if and only if $h^s(A) \in \mathbb{R}$.

(iii) A is $(-K)$-proper or K is not solid if and only if $\widehat{h}^i(A) \in \mathbb{R}$.

(iv) A is K-bounded if and only if $\widehat{h}^s(A) \in \mathbb{R}$.

Proof. Parts (i) and (iii) follow from Lemma 3.1.11(i) and Proposition 2.2.5. Parts (ii) and (iv) follow from Lemma 3.1.11(ii) and Proposition 2.2.8. □

The next theorem establishes sufficient conditions so that the functions $\bar{\mathbb{D}}$ are finite.

Theorem 3.1.13. *Let $A, B \in \mathcal{P}_0(Y)$.*

(i) If A is $(-K)$-bounded and B is K-bounded, then $\mathbb{D}^{ss}(A, B) \in \mathbb{R}$.

(ii) If A is K-proper and B is K-bounded, then $\widehat{\mathbb{D}}^{is}(A, B) \in \mathbb{R}$.

(iii) If A is K-proper and B is K-bounded, then $\mathbb{D}^{si}(A, B) \in \mathbb{R}$.

(iv) If A is $(-K)$-bounded and B is $(-K)$-proper, then $\mathbb{D}^{is}(A, B) \in \mathbb{R}$.

(v) If A is $(-K)$-bounded and B is $(-K)$-proper, then $\widehat{\mathbb{D}}^{si}(A, B) \in \mathbb{R}$.

(vi) If either A is K-proper and B is $(-K)$-bounded or A is K-bounded and B is $(-K)$-proper, then $\mathbb{D}^{ii}(A, B) \in \mathbb{R}$.

Proof. (i) First of all, we are going to show that if A is $(-K)$-bounded and B is K-bounded, then $A - B$ is $(-K)$-bounded.

Indeed, as A is $(-K)$-bounded and B is K-bounded, there exist positive numbers t and t' such that $A \subset tU_0 - K$ and $B \subset t'U_0 + K$. Then,

$$A - B = A + (-B) \subset tU_0 - K + t'(-U_0) - K \subset t''U_0 - K$$

with $t'' = t + t' > 0$ since $-U_0 = U_0$ and, therefore, we derive that $A - B$ is $(-K)$-bounded.

Since $\mathbb{D}^{ss}(A, B) = h^s(A - B)$, then by Proposition 3.1.12(ii), we obtain that $\mathbb{D}^{ss}(A, B) \in \mathbb{R}$.

(ii) Firstly, if we take a fixed $b \in B$, then

$$\widehat{\mathbb{D}}^{is}(A, b) = \inf_{a\in A} D(a - b, -K) = h^i(A - b). \tag{3.1}$$

Furthermore, we have that $D(a - b, -K) \leq \sup_{b\in B} D(a - b, -K)$ and by taking infima with $a \in A$, it follows that

$$\widehat{\mathbb{D}}^{is}(A, b) = \inf_{a\in A} D(a - b, -K) \leq \inf_{a\in A} \sup_{b\in B} D(a - b, -K) = \widehat{\mathbb{D}}^{is}(A, B). \tag{3.2}$$

Therefore, in view of (3.1) it follows that $\widehat{\mathbb{D}}^{is}(A, b) = h^i(A-b) \in \mathbb{R}$ by Proposition 3.1.12(i) and, then, by (3.2) we deduce that $\widehat{\mathbb{D}}^{is}(A, B) > -\infty$.

Secondly, by Lemma 3.1.10(ii) we have that $\widehat{\mathbb{D}}^{is}(A, B) = \inf_{a\in A} \widehat{h}^s(B - a)$. If B is K-bounded then $B - a$ is K-bounded too, and by Proposition 3.1.12(iv) we have that $\widehat{h}^s(B - a) < +\infty$, and so $\widehat{\mathbb{D}}^{is}(A, B) < +\infty$.

(iii) This is Corollary 2.2.11.

(iv) This a consequence of Lemma 3.1.5(ii) and part (ii).

(v) This is a consequence of Lemma 3.1.5(i) and part (iii).

(vi) Assume that $\operatorname{int} K \neq \emptyset$ since in another case by Remark 3.1.3(i) we have that $\mathbb{D}^{ii}(A, B) \geq 0$. First of all, we are going to prove that if A is K-proper and B is $(-K)$-bounded, then $A - B$ is K-proper.

Indeed, as A is K-proper then by Lemma 2.2.4 we have that there exists $t_0 > 0$ such that $A \subset -t_0 e - Y \setminus \operatorname{int} K$ with $e \in \operatorname{int} K$ and, moreover, as B is $(-K)$-bounded then for the neighborhood U_0 of zero there exists a positive number t such that $B \subset tU_0 - K$. Moreover, as $e \in \operatorname{int} K$, there exits $t' > 0$ such that $e + t'U_0 \subset \operatorname{int} K$, so $t'U_0 \subset -e + \operatorname{int} K$ and, therefore, $U_0 \subset -t''e + \operatorname{int} K$ where $t'' = 1/t'$. Thus, $tU_0 = -tU_0 \subset -t_1 e + \operatorname{int} K$ with $t_1 = tt'' > 0$ and, consequently,

$$-B \subset -tU_0 + K \subset -t_1 e + \operatorname{int} K + K = -t_1 e + \operatorname{int} K.$$

Then,

$$A - B \subset -t_0 e - Y \setminus \operatorname{int} K - t_1 e + \operatorname{int} K \subset -t_2 e - Y \setminus \operatorname{int} K$$

since $-Y\setminus \operatorname{int} K + \operatorname{int} K \subset -Y\setminus \operatorname{int} K$ and $t_2 := t_0 + t_1$. By applying Lemma 2.2.4 we have that $A - B$ is K-proper.

Now, taking into account that $\mathbb{D}^{ii}(A,B) = h^i(A-B)$, by Proposition 3.1.12(i), we obtain that $\mathbb{D}^{ii}(A,B) \in \mathbb{R}$.

Finally, if A is K-bounded and B is $(-K)$-proper, by symmetry $A - B = (-B) + A$ is also K-proper (the roles of A and B are exchanged in the previous proof). □

The above conditions are only sufficient for finitude. In general, they are not necessary conditions. For example, with the following data: $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = B = \{(x,y) : y = -x\}$, one has $\mathbb{D}^{si}(A,B) = 0$, $\mathbb{D}^{ii}(A,B) = 0$, and A is K-proper but B is not K-bounded.

The following result about $\bar{h} \in \mathcal{H}$ is needed and it is easy to check.

Lemma 3.1.14. *h^i is $\subset$-decreasing and h^s is $\subset$-increasing.*

Next, we are going to study the invariance by conic extensions to the functions $\bar{h} \in \mathcal{H}$.

Lemma 3.1.15. *Let $A \in \mathcal{P}_0(Y)$. Then, it holds that*

(i) $h^i(A) = h^i(A+K)$.

(ii) $h^s(A) = h^s(A-K)$.

Proof. (i) First of all, as $A \subset A + K$, then by applying Lemma 3.1.14 we have that $h^i(A+K) \leq h^i(A)$. Secondly, since $a \leq_K a + q$, for every $q \in K$, $a \in Y$, then by Lemma 1.3.19(viii) we deduce that $D(a,-K) \leq D(a+q,-K)$, for all $q \in K$, $a \in Y$ and, then, by taking infima in $q \in K$ we obtain that $D(a,-K) \leq \inf_{q\in K} D(a+q,-K)$, for all $a \in Y$. Now, taking infima in $a \in A$ we derive that $\inf_{a\in A} D(a,-K) \leq \inf_{a\in A, q\in K} D(a+q,-K)$. So, $h^i(A) \leq h^i(A+K)$ and, consequently, $h^i(A) = h^i(A+K)$.

(ii) The proof is similar, but we consider $a-q \leq_K a$ instead of $a \leq_K a+q$. □

Note that part (ii) is Proposition 3.3(iii) of Xu-Li [129].

The following result provides conditions so that the functions $\bar{h} \in \mathcal{H}$ achieve a minimum or a maximum on a set A.

Lemma 3.1.16. *Let $A \in \mathcal{P}_0(Y)$.*

(i) If A is K-compact, it holds that $h^i(A) = \min_{a\in A} D(a, -K)$ and $\widehat{h}^s(A) = \max_{a\in A} D(a, K)$.

(ii) If A is $(-K)$-compact, then $h^s(A) = \max_{a\in A} D(a, -K)$ and $\widehat{h}^i(A) = \min_{a\in A} D(a, K)$.

This result follows from Lemma 1.3.19, parts (i) and (viii), and Lemma 2.3.9. Let us observe that $D(\cdot, K)$ is $(-K)$-increasing by Lemma 1.3.19(viii), and therefore it is K-decreasing.

In the following proposition, we are going to show some properties for the functions $\bar{h}(A - y)$, where $\bar{h} \in \mathcal{H}$.

Proposition 3.1.17. *Let $A \in \mathcal{P}_0(Y)$. Then*

(i) The functions $y \to h^i(A - y)$ and $y \to h^s(A - y)$ are K-decreasing. Moreover, they are Lipschitz of rank 1, the first one if A is K-proper and the second one if A is $(-K)$-bounded.

(ii) The functions $y \to \widehat{h}^i(A - y)$ and $y \to \widehat{h}^s(A - y)$ are K-increasing. Moreover, they are Lipschitz of rank 1, the first one if A is $(-K)$-proper and the second one if A is K-bounded.

Proof. (i) First, let us see that $h^i(A - \cdot)$ is K-decreasing. Indeed, if $y_1 \leq_K y_2$ then $a - y_2 \leq_K a - y_1$ for all $a \in A$, and by Lemma 1.3.19(viii) we have that $D(a - y_2, -K) \leq D(a - y_1, -K)$, for all $a \in A$. Therefore, by taking infima in $a \in A$ we derive that $\inf_{a\in A} D(a - y_2, -K) \leq \inf_{a\in A} D(a - y_1, -K)$ and, consequently, we obtain that $h^i(A - y_2) \leq h^i(A - y_1)$.

Secondly, we are going to prove that $h^i(A - y)$ is Lipschitz of rank 1 knowing that it is real valued by Proposition 3.1.12(i) since $A - y$ is K-proper because A is K-proper. Indeed, by applying Lemma 1.3.19(ix) we have that

$$D(a - y_1, -K) \leq D(a - y_2, -K) + D(y_2 - y_1, -K), \ \forall a \in A. \tag{3.3}$$

Then, by taking infima in $a \in A$ we deduce that $h^i(A - y_1) \leq h^i(A - y_2) + D(y_2 - y_1, -K)$, that is,

$$h^i(A - y_1) - h^i(A - y_2) \leq D(y_2 - y_1, -K) \leq \|y_2 - y_1\|.$$

The last inequality is true because $D(y, -K) - D(0, -K) \leq \|y\|$ since $D(\cdot, -K)$ is Lipschitz of rank 1 by Lemma 1.3.19(i). Interchanging y_1 and y_2, we conclude.

Similarly to $h^i(A - \cdot)$, we demonstrate that $h^s(A - \cdot)$ is K-decreasing too.

Now, let us see that $h^s(A - \cdot)$ is Lipschitz of rank 1 knowing that it is real valued since $A - y$ is $(-K)$-bounded because A is a $(-K)$-bounded set (see Proposition 3.1.12(ii)). Indeed, if in (3.3) we take suprema in $a \in A$, we derive that $h^s(A-y_1) \leq h^s(A-y_2)+D(y_2-y_1, -K)$ and, therefore, $h^s(A-y_1)-h^s(A-y_2) \leq \|y_2 - y_1\|$. Interchanging y_1 and y_2, we conclude.

(ii) It is enough to take into account that $\widehat{h}^i(A - y) = h^i((-A) - (-y))$ by Lemma 3.1.9(i) and to apply that the function $y \to -y$ is K-decreasing and $h^i((-A) - \cdot)$ is K-decreasing by part (i), to obtain that $\widehat{h}^i(A - \cdot)$ is K-increasing. Moreover, $\widehat{h}^i(A-y) = h^i((-A)-(-y))$ is Lipschitz of rank 1 if A is $(-K)$-proper by part (i).

Finally, a similar reasoning is valid for $\widehat{h}^s(A - y)$. □

The following result is a consequence of Lemma 2.3.9 and Propositions 3.1.17 and 3.1.12 (and Lemma 3.1.16 for the "resp" parts). Remember also that if g is K-increasing, then g is $(-K)$-decreasing.

Corollary 3.1.18. *Let $A, B \in \mathcal{P}_0(Y)$.*

(i) If A is $(-K)$-bounded (resp., $(-K)$-compact) and B is K-compact, then

$$\mathbb{D}^{ss}(A, B) = \max_{b\in B} h^s(A - b) \ \textit{(resp., } \mathbb{D}^{ss}(A, B) = \max_{b\in B} \max_{a\in A} D(a - b, -K)\textit{)}.$$

(ii) If B is K-bounded (resp., K-compact) and A is K-compact, then

$$\widehat{\mathbb{D}}^{is}(A, B) = \min_{a\in A} \widehat{h}^s(B - a) \ \textit{(resp., } \widehat{\mathbb{D}}^{is}(A, B) = \min_{a\in A} \max_{b\in B} D(a - b, K)\textit{)}.$$

(iii) If A is K-proper (resp., K-compact) and B is K-compact, then

$$\mathbb{D}^{si}(A, B) = \max_{b\in B} h^i(A - b) \ \textit{(resp., } \mathbb{D}^{si}(A, B) = \max_{b\in B} \min_{a\in A} D(a - b, -K)\textit{)}.$$

(iv) If A is $(-K)$-bounded (resp., $(-K)$-compact) and B is $(-K)$-compact, then

$$\mathbb{D}^{is}(A, B) = \min_{b\in B} h^s(A - b) \ \textit{(resp., } \mathbb{D}^{is}(A, B) = \min_{b\in B} \max_{a\in A} D(a - b, -K)\textit{)}.$$

(v) If B is $(-K)$-proper (resp., $(-K)$-compact) and A is $(-K)$-compact, then

$$\widehat{\mathbb{D}}^{si}(A,B) = \max_{a\in A} \widehat{h}^i(B-a) \text{ (resp., } \widehat{\mathbb{D}}^{si}(A,B) = \max_{a\in A}\min_{b\in B} D(a-b,K)\text{)}.$$

(vi) If A is K-proper (resp., K-compact) and B is $(-K)$-compact, then

$$\mathbb{D}^{ii}(A,B) = \min_{b\in B} h^i(A-b) \text{ (resp., } \mathbb{D}^{ii}(A,B) = \min_{b\in B}\min_{a\in A} D(a-b,-K)\text{)}.$$

In the next lemma, we are going to study monotonicity properties of the functions $\bar{h} \in \mathcal{H}$.

Lemma 3.1.19. *The following statements are true:*

(i) h^i is $\preccurlyeq^{\forall\exists}$-increasing.

(ii) h^s is $\precsim^{\forall\exists}$-increasing.

(iii) $\widehat{h}^i$ is $\precsim^{\forall\exists}$-decreasing.

(iv) $\widehat{h}^s$ is $\preccurlyeq^{\forall\exists}$-decreasing.

Proof. (i) If $A_1 \preccurlyeq^{\forall\exists} A_2$, then $A_2 \subset A_1 + K$. By using Lemmas 3.1.14 and 3.1.15(i), we have $h^i(A_1) = h^i(A_1+K) \le h^i(A_2)$.

(ii) If $A_1 \precsim^{\forall\exists} A_2$, then $A_1 \subset A_2 - K$. By using Lemmas 3.1.14 and 3.1.15(ii), we have $h^s(A_1) \le h^s(A_2 - K) = h^s(A_2)$.

(iii) and (iv) are proved in a similar way or they may be reduced to parts (i) and (ii) by using Lemma 3.1.9. □

The second statement in Lemma 3.1.19 is Proposition 3.3(i) in Xu-Li [129].

Now, our goal is to show K-invariance for the functions $\bar{\mathbb{D}} \in \mathcal{D}$.

Theorem 3.1.20. *Let $A, B \in \mathcal{P}_0(Y)$. Then*

(i) $\mathbb{D}^{ss}(A,B) = \mathbb{D}^{ss}(A-K,B) = \mathbb{D}^{ss}(A,B+K) = \mathbb{D}^{ss}(A-K,B+K)$.

(ii) $\widehat{\mathbb{D}}^{is}(A,B) = \widehat{\mathbb{D}}^{is}(A+K,B) = \widehat{\mathbb{D}}^{is}(A,B+K) = \widehat{\mathbb{D}}^{is}(A+K,B+K)$.

(iii) $\mathbb{D}^{si}(A,B) = \mathbb{D}^{si}(A+K,B) = \mathbb{D}^{si}(A,B+K) = \mathbb{D}^{si}(A+K,B+K)$.

(iv) $\mathbb{D}^{is}(A,B) = \mathbb{D}^{is}(A-K,B) = \mathbb{D}^{is}(A,B-K) = \mathbb{D}^{is}(A-K,B-K)$.

(v) $\widehat{\mathbb{D}}^{si}(A,B) = \widehat{\mathbb{D}}^{si}(A-K,B) = \widehat{\mathbb{D}}^{si}(A,B-K) = \widehat{\mathbb{D}}^{si}(A-K,B-K)$.

(vi) $\mathbb{D}^{ii}(A,B) = \mathbb{D}^{ii}(A+K,B) = \mathbb{D}^{ii}(A,B-K) = \mathbb{D}^{ii}(A+K,B-K)$.

Proof. The third equality in all parts follows from the other two equalities.

(i) Using Lemma 3.1.15(ii), we have

$$\mathbb{D}^{ss}(A-K,B) = h^s(A-K-B) = h^s(A-B) = \mathbb{D}^{ss}(A,B), \text{ and}$$
$$\mathbb{D}^{ss}(A,B+K) = h^s(A-B-K) = h^s(A-B) = \mathbb{D}^{ss}(A,B).$$

(iv) Using Lemma 3.1.15(ii), we obtain

$$\mathbb{D}^{is}(A,B) = \inf_{b\in B} h^s(A-b) = \inf_{b\in B} h^s(A-b-K) = \mathbb{D}^{is}(A-K,B).$$

Now, let us prove that $\mathbb{D}^{is}(A,B) = \mathbb{D}^{is}(A,B-K)$. On the one hand, as $B \subset B-K$, it follows

$$\mathbb{D}^{is}(A,B) = \inf_{b\in B} h^s(A-b) \geq \inf_{y\in B-K} h^s(A-y) = \mathbb{D}^{is}(A,B-K).$$

On the other hand, as $b-q \leq b$ for all $q \in K$, $b \in Y$, and $h^s(A-y)$ is K-decreasing in y by Proposition 3.1.17(i), we deduce that $h^s(A-b) \leq h^s(A-b+q)$, for all $q \in K$, $b \in Y$. Thus, $h^s(A-b) \leq \inf_{q\in K} h^s(A-b+q)$, for all $b \in Y$, and then

$$\mathbb{D}^{is}(A,B) = \inf_{b\in B} h^s(A-b) \leq \inf_{b\in B}\inf_{q\in K} h^s(A-b+q)) = \mathbb{D}^{is}(A,B-K).$$

Therefore, $\mathbb{D}^{is}(A,B-K) = \mathbb{D}^{is}(A,B)$.

(iii) This is Proposition 2.2.30.

(ii) By applying successively Lemma 3.1.5(ii) and part (iv) of the present theorem, we have

$$\widehat{\mathbb{D}}^{is}_K(A,B) = \mathbb{D}^{is}_{-K}(B,A) = \mathbb{D}^{is}_{-K}(B+K,A) = \widehat{\mathbb{D}}^{is}_K(A,B+K),$$
$$\widehat{\mathbb{D}}^{is}_K(A,B) = \mathbb{D}^{is}_{-K}(B,A) = \mathbb{D}^{is}_{-K}(B,A+K) = \widehat{\mathbb{D}}^{is}_K(A+K,B).$$

(v) This is Theorem 2.2.39(i). It is proved as (ii).

(vi) It is proved as (i) using Lemma 3.1.15(i) and the fact that $\mathbb{D}^{ii}(A,B) = h^i(A-B)$. □

In the following theorem, we establish the monotonicity of the functions $\bar{\mathbb{D}} \in \mathcal{D}$, but previously an additional result, whose proof is omitted because it is immediate, is stated.

Lemma 3.1.21. *Let $A, B \in \mathcal{P}_0(Y)$. If $A \subset B$, then $A \preccurlyeq^{\forall\exists} B$ and $B \curlyeqprec^{\forall\exists} A$.*

Theorem 3.1.22. *Let $A, B \in \mathcal{P}_0(Y)$.*

(i-a) $\mathbb{D}^{ss}(A, \cdot)$ is increasing w.r.t. $\subset$ and decreasing w.r.t. $\preccurlyeq^{\forall\forall}$, $\preccurlyeq^{\exists\forall}$ and $\curlyeqprec^{\forall\exists}$.

(i-b) $\mathbb{D}^{ss}(\cdot, B)$ is increasing w.r.t. $\subset$, $\preccurlyeq^{\forall\forall}$, $\curlyeqprec^{\exists\forall}$ and $\preccurlyeq^{\forall\exists}$.

(ii-a) $\widehat{\mathbb{D}}^{is}(A, \cdot)$ is increasing w.r.t. $\subset$ and decreasing w.r.t. $\preccurlyeq^{\forall\forall}$, $\preccurlyeq^{\exists\forall}$ and $\curlyeqprec^{\forall\exists}$.

(ii-b) $\widehat{\mathbb{D}}^{is}(\cdot, B)$ is decreasing w.r.t. $\subset$ and increasing w.r.t. $\preccurlyeq^{\forall\forall}$, $\preccurlyeq^{\exists\forall}$ and $\curlyeqprec^{\forall\exists}$.

(iii-a) $\mathbb{D}^{si}(A, \cdot)$ is increasing w.r.t. $\subset$ and decreasing w.r.t. $\preccurlyeq^{\forall\forall}$, $\preccurlyeq^{\exists\forall}$ and $\curlyeqprec^{\forall\exists}$.

(iii-b) $\mathbb{D}^{si}(\cdot, B)$ is decreasing w.r.t. $\subset$ and increasing w.r.t. $\preccurlyeq^{\forall\forall}$, $\preccurlyeq^{\exists\forall}$ and $\curlyeqprec^{\forall\exists}$.

(iv-a) $\mathbb{D}^{is}(A, \cdot)$ is decreasing w.r.t. $\subset$, $\preccurlyeq^{\forall\forall}$, $\curlyeqprec^{\exists\forall}$ and $\preccurlyeq^{\forall\exists}$.

(iv)-b) $\mathbb{D}^{is}(\cdot, B)$ is increasing w.r.t. $\subset$, $\preccurlyeq^{\forall\forall}$, $\curlyeqprec^{\exists\forall}$ and $\preccurlyeq^{\forall\exists}$.

(v-a) $\widehat{\mathbb{D}}^{si}(A, \cdot)$ is decreasing w.r.t. $\subset$, $\preccurlyeq^{\forall\forall}$, $\curlyeqprec^{\exists\forall}$ and $\preccurlyeq^{\forall\exists}$.

(v-b) $\widehat{\mathbb{D}}^{si}(\cdot, B)$ is increasing w.r.t. $\subset$, $\preccurlyeq^{\forall\forall}$, $\curlyeqprec^{\exists\forall}$ and $\preccurlyeq^{\forall\exists}$.

(vi-a) $\mathbb{D}^{ii}(A, \cdot)$ is decreasing w.r.t. $\subset$, $\preccurlyeq^{\forall\forall}$, $\curlyeqprec^{\exists\forall}$ and $\preccurlyeq^{\forall\exists}$.

(vi-b) $\mathbb{D}^{ii}(\cdot, B)$ is decreasing w.r.t. $\subset$ and increasing w.r.t. $\preccurlyeq^{\forall\forall}$, $\preccurlyeq^{\exists\forall}$ and $\curlyeqprec^{\forall\exists}$.

Proof. (i-a) and (i-b). 1. If $A_1 \preccurlyeq^{\forall\exists} A_2$, then $A_1 - b \preccurlyeq^{\forall\exists} A_2 - b$, for all $b \in Y$ by Lemma 1.3.8(i). In view of Lemma 3.1.19(ii), we obtain $h^s(A_1 - b) \leq h^s(A_2 - b)$. Taking $\sup_{b\in B}$, it follows that

$$\mathbb{D}^{ss}(A_1, B) = \sup_{b\in B} h^s(A_1 - b) \leq \sup_{b\in B} h^s(A_2 - b) = \mathbb{D}^{ss}(A_2, B),$$

i.e., $\mathbb{D}^{ss}(\cdot, B)$ is $\preccurlyeq^{\forall\exists}$-increasing.

2. If $B_1 \curlyeqprec_K^{\forall\exists} B_2$, then by Lemma 1.3.9(i), one has $B_2 \preccurlyeq_{-K}^{\forall\exists} B_1$. According to the previous point 1, $\mathbb{D}_{-K}^{ss}(\cdot, A)$ is $\preccurlyeq_{-K}^{\forall\exists}$-increasing, so $\mathbb{D}_{-K}^{ss}(B_2, A) \leq \mathbb{D}_{-K}^{ss}(B_1, A)$. Therefore $\mathbb{D}_K^{ss}(A, B_2) \leq \mathbb{D}_K^{ss}(A, B_1)$ since $\mathbb{D}_K^{ss}(A, B) = \mathbb{D}_{-K}^{ss}(B, A)$ by Lemma 3.1.5(iii). In consequence, $\mathbb{D}_K^{ss}(A, \cdot)$ is $\curlyeqprec^{\forall\exists}$-decreasing.

3. By Proposition 1.3.7(ii), $\preccurlyeq^{\forall\forall} \Rightarrow \curlyeqprec^{\exists\forall} \Rightarrow \preccurlyeq^{\forall\exists}$, and by Lemma 1.3.13 taking into account the previous point 1, we derive that $\mathbb{D}^{ss}(\cdot, B)$ is increasing w.r.t. $\preccurlyeq^{\forall\forall}$ and $\curlyeqprec^{\exists\forall}$.

4. Similarly, by Proposition 1.3.7(i), $\preccurlyeq^{\forall\forall} \Rightarrow \preccurlyeq^{\exists\forall} \Rightarrow \preccurlyeq^{\forall\exists}$, and by Lemma 1.3.13 taking into account the previous point 2, we deduce that $\mathbb{D}^{ss}(A,\cdot)$ is decreasing w.r.t. $\preccurlyeq^{\forall\forall}$ and $\preccurlyeq^{\exists\forall}$.

5. If $A_1 \subset A_2$, then $A_1 \preccurlyeq^{\forall\exists} A_2$ by Lemma 3.1.21, and using the previous point 1, we obtain that $\mathbb{D}^{ss}(A_1, B) \leq \mathbb{D}^{ss}(A_2, B)$.

6. If $B_1 \subset B_2$, then $B_2 \preccurlyeq^{\forall\exists} B_1$ by Lemma 3.1.21, and using the previous point 2, we derive that $\mathbb{D}^{ss}(A, B_1) \leq \mathbb{D}^{ss}(A, B_2)$.

In the rest of the parts from (ii) to (vi), we will not prove the points 3, 4, 5 and 6 because they are very similar to the proof of the same points in part (i).

(iv-a) and (iv-b). 1. If $B_1 \preccurlyeq^{\forall\exists} B_2$, that is, $B_1 \subset B_2 - K$, then $\inf_{z\in B_2-K} h^s(A - z) \leq \inf_{b_1\in B_1} h^s(A - b_1)$, that is, $\mathbb{D}^{is}(A, B_2 - K) \leq \mathbb{D}^{is}(A, B_1)$. Hence, by Proposition 3.1.20(iv) we have that $\mathbb{D}^{is}(A, B_2) \leq \mathbb{D}^{is}(A, B_1)$, which implies that $\mathbb{D}^{is}(A,\cdot)$ is $\preccurlyeq^{\forall\exists}$-decreasing.

2. If $A_1 \preccurlyeq^{\forall\exists} A_2$, then $A_1 - b \preccurlyeq^{\forall\exists} A_2 - b$, for all $b \in Y$ by Lemma 1.3.8(i). In view of Lemma 3.1.19(ii), we obtain $h^s(A_1 - b) \leq h^s(A_2 - b)$. Taking $\inf_{b\in B}$, it follows that $\mathbb{D}^{is}(A_1, B) \leq \mathbb{D}^{is}(A_2, B)$, that is, $\mathbb{D}^{is}(\cdot, B)$ is $\preccurlyeq^{\forall\exists}$-increasing.

(ii-a) and (ii-b). 1. If $A_1 \preccurlyeq_K^{\forall\exists} A_2$, by Lemma 1.3.9(i) we have that $A_2 \preccurlyeq_{-K}^{\forall\exists} A_1$. As $\mathbb{D}^{is}_{-K}(B,\cdot)$ is $\preccurlyeq_{-K}^{\forall\exists}$-decreasing by part (iv)-(a), it follows that $\mathbb{D}^{is}_{-K}(B, A_1) \leq \mathbb{D}^{is}_{-K}(B, A_2)$. Consequently, $\widehat{\mathbb{D}}^{is}(A_1, B) \leq \widehat{\mathbb{D}}^{is}(A_2, B)$ by Lemma 3.1.5(ii), i.e., $\widehat{\mathbb{D}}^{is}(\cdot, B)$ is $\preccurlyeq^{\forall\exists}$-increasing.

2. If $B_1 \preccurlyeq_K^{\forall\exists} B_2$, by Lemma 1.3.9(i) we have that $B_2 \preccurlyeq_{-K}^{\forall\exists} B_1$. As $\mathbb{D}^{is}_{-K}(\cdot, A)$ is $\preccurlyeq_{-K}^{\forall\exists}$-increasing by part (iv-b), point 2, we have $\mathbb{D}^{is}_{-K}(B_2, A) \leq \mathbb{D}^{is}_{-K}(B_1, A)$. Consequently, $\widehat{\mathbb{D}}^{is}(A, B_2) \leq \widehat{\mathbb{D}}^{is}(A, B_1)$ by Lemma 3.1.5(ii), i.e., $\widehat{\mathbb{D}}^{is}(A,\cdot)$ is $\preccurlyeq^{\forall\exists}$-decreasing.

Parts (iii) and (vi) are demonstrated similarly to (iv) and (i), resp., and part (v) in a similar way to (ii). □

Remark 3.1.23. In the empty cells of Table 3.1, the corresponding function $\bar{\mathbb{D}}(A,\cdot)$ or $\bar{\mathbb{D}}(\cdot, B)$ is not monotone.

Indeed, $\mathbb{D}^{ss}(A,\cdot)$ is not $\preccurlyeq^{\exists\exists}$-increasing. By contradiction, assume that $\mathbb{D}^{ss}(A,\cdot)$ is $\preccurlyeq^{\exists\exists}$-increasing. Then by Proposition 1.3.7 we have $\preccurlyeq^{\forall\forall} \Rightarrow \preccurlyeq^{\exists\exists}$ and from Lemma 1.3.13 it follows that $\mathbb{D}^{ss}(A,\cdot)$ is $\preccurlyeq^{\forall\forall}$-increasing, which is a contradiction.

Similarly, it is proved that $\mathbb{D}^{ss}(A,\cdot)$ is not increasing w.r.t. $\curlyeqprec^{\exists\forall}$ or $\preccurlyeq^{\forall\exists}$.

$\mathbb{D}^{ss}(A,\cdot)$ is not decreasing w.r.t. $\curlyeqprec^{\exists\forall}$ or $\preccurlyeq^{\forall\exists}$ or $\preccurlyeq^{\exists\exists}$ as the following data show: $Y=\mathbb{R}$, $K=\mathbb{R}_+$, $B_1=\{1\}$, $B_2=\{0,\,2\}$, $A=\{3\}$, one has $B_1 \curlyeqprec^{\exists\forall} B_2$ (and so $B_1 \preccurlyeq^{\forall\exists} B_2$ and $B_1 \preccurlyeq^{\exists\exists} B_2$) and however $\mathbb{D}^{ss}(A,B_1)=2<\mathbb{D}^{ss}(A,B_2)=3$.

For the rest of functions it is argued in a similar way or simple counter-examples can be found.

We collect the results of Theorem 3.1.22 in Table 3.1.

	$\subset$	$\preccurlyeq^{\forall\forall}$	$\preccurlyeq^{\exists\forall}$	$\curlyeqprec^{\forall\exists}$	$\curlyeqprec^{\exists\forall}$	$\preccurlyeq^{\forall\exists}$	$\preccurlyeq^{\exists\exists}$
$\mathbb{D}^{ss}(A,\cdot)$	Incr	Decr	Decr	Decr			
$\widehat{\mathbb{D}}^{is}(A,\cdot)$	Incr	Decr	Decr	Decr			
$\mathbb{D}^{si}(A,\cdot)$	Incr	Decr	Decr	Decr			
$\mathbb{D}^{is}(A,\cdot)$	Decr	Decr			Decr	Decr	
$\widehat{\mathbb{D}}^{si}(A,\cdot)$	Decr	Decr			Decr	Decr	
$\mathbb{D}^{ii}(A,\cdot)$	Decr	Decr			Decr	Decr	
$\mathbb{D}^{ss}(\cdot,B)$	Incr	Incr			Incr	Incr	
$\widehat{\mathbb{D}}^{is}(\cdot,B)$	Decr	Incr	Incr	Incr			
$\mathbb{D}^{si}(\cdot,B)$	Decr	Incr	Incr	Incr			
$\mathbb{D}^{is}(\cdot,B)$	Incr	Incr			Incr	Incr	
$\widehat{\mathbb{D}}^{si}(\cdot,B)$	Incr	Incr			Incr	Incr	
$\mathbb{D}^{ii}(\cdot,B)$	Decr	Incr	Incr	Incr			

Table 3.1. Theorem 3.1.22. Monotonicity of $\bar{\mathbb{D}}$ w.r.t. the set relations ("Decr" means decreasing and "Incr", increasing).

Next, we are going to prove a necessary result for what follows.

Proposition 3.1.24. *Let $A\in\mathcal{P}_0(Y)$. If $g:Y\to\mathbb{R}$ is continuous, then*

$$\inf_{a\in \operatorname{cl} A} g(a)=\inf_{a\in A} g(a) \quad \textit{and} \quad \sup_{a\in \operatorname{cl} A} g(a)=\sup_{a\in A} g(a).$$

Proof. First, we are going to show that $\inf_{a\in\operatorname{cl}A} g(a)=\inf_{a\in A} g(a)$. As $A\subset \operatorname{cl} A$, then $\alpha:=\inf_{z\in\operatorname{cl}A} g(z)\le\beta:=\inf_{a\in A} g(a)$. Assume that $\alpha<\beta$. Then, by definition of infimum we have at least a $z_0\in\operatorname{cl}A$ such that $\alpha\le g(z_0)<\beta$. As

$z_0 \in \operatorname{cl} A$, there exist a sequence $(a_n) \subset A$ such that $a_n \to z_0$ and, by continuity of the function g, we derive that $g(a_n) \to g(z_0)$. So, $g(a_n) < \beta$ for n large enough, which contradicts that $\beta = \inf_{a\in A} g(a)$.

Secondly, in a similar way, we demonstrate that $\sup_{a\in \operatorname{cl} A} g(a) = \sup_{a\in A} g(a)$. □

Now, we show closure properties for the functions $\bar{h}$ and $\bar{\mathbb{D}}$. The following result is an immediate consequence of Proposition 3.1.24 and Lemma 1.3.19(i).

Proposition 3.1.25. *Let $A \in \mathcal{P}_0(Y)$. Then, it holds that $\bar{h}(\operatorname{cl} A) = \bar{h}(A)$, for all $\bar{h} \in \mathcal{H}$.*

Remark 3.1.26. We deduce from the previous proposition and from Lemma 3.1.15 that $h^i(A) = h^i(\operatorname{cl} A) = h^i(\operatorname{cl}(A+K))$ and $h^s(A) = h^s(\operatorname{cl} A) = h^s(\operatorname{cl}(A-K))$.

Proposition 3.1.27. *Let $A, B \in \mathcal{P}_0(Y)$. Then, it is verified that*

$$\bar{\mathbb{D}}(\operatorname{cl} A, B) = \bar{\mathbb{D}}(A, \operatorname{cl} B) = \bar{\mathbb{D}}(A, B), \quad \forall \bar{\mathbb{D}} \in \mathcal{D}.$$

Proof. They are a consequence of Lemma 3.1.10 and Propositions 3.1.24 and 3.1.25. In the proof, we follow the order (i)-(vi) of Definition 3.1.1.

(i) For $\mathbb{D}^{ss}$. We are going to distinguish two cases.

Case 1. A is $(-K)$-bounded. Then $h^s(A - \cdot)$ is continuous by Proposition 3.1.17(i) and so we can apply Proposition 3.1.24. Therefore, we have

$$\mathbb{D}^{ss}(A, B) = \sup_{b\in B} h^s(A-b) = \sup_{b\in \operatorname{cl} B} h^s(A-b) = \mathbb{D}^{ss}(A, \operatorname{cl} B).$$

Moreover, from Proposition 3.1.25 it follows

$$\begin{aligned}\mathbb{D}^{ss}(\operatorname{cl} A, B) &= \sup_{b\in B} h^s(\operatorname{cl} A - b) = \sup_{b\in B} h^s(\operatorname{cl}(A-b)) = \sup_{b\in B} h^s(A-b)\\ &= \mathbb{D}^{ss}(A, B).\end{aligned}$$

Case 2. A is not $(-K)$-bounded. Then $h^s(A-y) = +\infty$ for all

$y \in Y$ by Proposition 3.1.12(ii), and so $\mathbb{D}^{ss}(A, B) = +\infty$ for all $B \in \mathcal{P}_0(Y)$. Consequently, $\mathbb{D}^{ss}(A, \operatorname{cl} B) = \mathbb{D}^{ss}(A, B)$ and $\mathbb{D}^{ss}(\operatorname{cl} A, B) = \mathbb{D}^{ss}(A, B)$ because $\operatorname{cl} A$ is not $(-K)$-closed either.

(ii)-(vi) It is similar for the rest of the functions. We have to apply Propositions 3.1.25, 3.1.12 and 3.1.17, but it is necessary to distinguish two cases, according to the following scheme: for $\widehat{\mathbb{D}}^{is}(A,B)$, B is or is not K-bounded; for $\mathbb{D}^{si}(A,B)$ and for $\mathbb{D}^{ii}(A,B)$, A is or is not K-proper; for $\mathbb{D}^{is}(A,B)$, A is or is not $(-K)$-bounded and for $\widehat{\mathbb{D}}^{si}(A,B)$, B is or is not $(-K)$-proper. □

We may give a remark for each function $\bar{\mathbb{D}} \in \mathcal{D}$ as Remark 3.1.26 taking into account Theorem 3.1.20.

3.2 Characterization by scalarization

In this section, we are going to derive new characterizations of the six set relations of Kuroiwa given in Definition 1.3.2 by using the six set scalarizations $\mathbb{D}^{\alpha}_{K}(A,B)$ and $\widehat{\mathbb{D}}^{\alpha}_{K}(A,B)$ given in Definition 3.1.1, which are introduced in the former section. Furthermore, some examples to illustrate the results obtained are provided but especially for emphasize that the assumptions required cannot be removed. The importance of these results lies in the fact that they could be applied in Section 4.1 to derive necessary and sufficient minimality conditions for a set optimization problem with the set criterion of solution.

Next, in the following theorem we propose a characterization to the set relations given in Definition 1.3.2 by using the set scalarization functions given in Definition 3.1.1.

Theorem 3.2.1. *Let $A, B \in \mathcal{P}_0(Y)$ and let K be closed. Then*

(i) $A \preccurlyeq^{\forall\forall} B$ if and only if $\mathbb{D}^{ss}(A,B) \leq 0$.

(ii) If $A \preccurlyeq^{\exists\forall} B$, then $\widehat{\mathbb{D}}^{is}(A,B) \leq 0$. The converse is true if A is K-compact and B is K-bounded.

(iii) If $A \preccurlyeq^{\forall\exists} B$, then $\mathbb{D}^{si}(A,B) \leq 0$. The converse is true if A is K-closed.

(iv) If $A \preccurlyeq^{\exists\forall} B$, then $\mathbb{D}^{is}(A,B) \leq 0$. The converse is true if A is $(-K)$-bounded and B is $(-K)$-compact.

(v) If $A \preccurlyeq^{\forall\exists} B$, then $\widehat{\mathbb{D}}^{si}(A,B) \leq 0$. The converse is true if B is $(-K)$-closed.

(vi) If $A \preccurlyeq^{\exists\exists} B$*, then* $\mathbb{D}^{ii}(A, B) \leq 0$*. The converse is true if* A *is* K*-compact and* B *is* $(-K)$*-compact.*

Proof. (i) ($\Rightarrow$) By hypothesis, for all $a \in A$ and all $b \in B$, we have that $a - b \in -K$. Hence, by Lemma 1.3.19(v) it follows that $D(a - b, -K) \leq 0$, for all $a \in A$ and all $b \in B$ and, consequently,

$$\mathbb{D}^{ss}(A, B) = \sup_{b \in B} \sup_{a \in A} D(a - b, -K) \leq 0. \tag{3.4}$$

($\Leftarrow$) Since (3.4) is satisfied, it results that $D(a - b, -K) \leq 0$, for all $a \in A$ and all $b \in B$. As K is closed, by Lemma 1.3.19(v) we deduce that $a - b \in -K$, for all $a \in A$ and all $b \in B$, that is, $A \preccurlyeq^{\forall\forall} B$.

(ii) ($\Rightarrow$) By hypothesis, there exists $a_0 \in A$ such that for all $b \in B$, we have that $a_0 - b \in -K$. Hence, by Lemma 1.3.19(v) it follows that $D(a_0 - b, -K) \leq 0$ for all $b \in B$ and, consequently, $\sup_{b \in B} D(a_0 - b, -K) \leq 0$. Therefore, it results that $\widehat{\mathbb{D}}^{is}(A, B) = \inf_{a \in A} \sup_{b \in B} D(a - b, -K) \leq 0$.

($\Leftarrow$) By contradiction, suppose that $A \not\preccurlyeq^{\exists\forall} B$. Then, for all $a \in A$, there exists $\widetilde{b} \in B$ (that depends on a) such that $a - \widetilde{b} \notin -K$. So, by Lemma 1.3.19(v) we derive that $D(a - \widetilde{b}, -K) > 0$ since K is closed and, consequently,

$$\widehat{\mathbb{D}}^{is}(a, B) = \widehat{h}^s(B - a) = \sup_{b \in B} D(a - b, -K) \geq D(a - \widetilde{b}, -K) > 0, \ \forall a \in A. \tag{3.5}$$

Since B is K-bounded and A is K-compact, by Corollary 3.1.18(ii) we have $\widehat{\mathbb{D}}^{is}(A, B) = \min_{a \in A} \widehat{h}^s(B - a)$, and so there exists $a_0 \in A$ such that

$$\widehat{\mathbb{D}}^{is}(A, B) = \widehat{\mathbb{D}}^{is}(a_0, B) > 0$$

in view of (3.5), which contradicts the hypothesis.

(iii) It is Theorem 2.3.1.

(iv) ($\Rightarrow$) By Lemma 1.3.9(ii), we have that $B \preccurlyeq^{\exists\forall}_{-K} A$, and applying part (ii) it follows that $\widehat{\mathbb{D}}^{is}_{-K}(B, A) \leq 0$. Using Lemma 3.1.5(ii), we conclude that $\mathbb{D}^{is}_K(A, B) \leq 0$.

($\Leftarrow$) By Lemma 3.1.5(ii), it holds that $\mathbb{D}^{is}_K(A, B) = \widehat{\mathbb{D}}^{is}_{-K}(B, A)$. By hypothesis, we have that $\widehat{\mathbb{D}}^{is}_{-K}(B, A) \leq 0$, B is $(-K)$-compact and A is $(-K)$-bounded, and using part (ii) it follows that $B \preccurlyeq^{\exists\forall}_{-K} A$. We conclude that $A \preccurlyeq^{\exists\forall} B$ by Lemma 1.3.9(ii).

(v) It is Theorem 2.3.17(i).

(vi) ($\Rightarrow$) By hypothesis, there exist $a_0 \in A$ and $b_0 \in B$ such that $a_0 - b_0 \in -K$. Therefore, by Lemma 1.3.19(v) we deduce that $D(a_0 - b_0, -K) \leq 0$ since K is closed and, consequently, $\mathbb{D}^{ii}(A, B) = \inf_{b\in B} \inf_{a\in A} D(a - b, -K) \leq 0$.

($\Leftarrow$) By hypothesis, $\mathbb{D}^{ii}(A, B) \leq 0$. Since A is K-compact and B is $(-K)$-compact, from Corollary 3.1.18(vi) it follows that

$$\mathbb{D}^{ii}(A, B) = \min_{b\in B} \min_{a\in A} D(a - b, -K) = D(a_0 - b_0, -K)$$

for some $a_0 \in A$ and $b_0 \in B$. Consequently, we have that

$$\mathbb{D}^{ii}(A, B) = D(a_0 - b_0, -K) \leq 0$$

and this implies that $a_0 - b_0 \in -K$ by Lemma 1.3.19(v) because K is closed, so we deduce that $A \preccurlyeq^{\exists\exists} B$. □

In [81, 83] for the Gerstewitz's function $g_{e,K}(y) := \inf\{t \in \mathbb{R} : y \in te - K\}$, we find close results to parts (iii), (v) and (vi) of Theorem 3.2.1 but assuming that it is attained $\inf_{a\in A} g_{e,K}(a - b)$ for all $b \in B$, $\inf_{b\in B} g_{e,K}(a - b)$ for all $a \in A$ and $\inf_{(a,b)\in A\times B} g_{e,K}(a - b)$, respectively. Moreover, Theorem 3.13 in [83] for the Gerstewitz's function $g_{e,K}$ (with an arbitrary closed set K) is similar to Theorem 3.2.1(i). Furthermore, results as parts (iii), (v) and (vi) of Theorem 3.2.1 are Propositions 3.8, 3.3 and 3.14 in [11] for the oriented distance but assuming that it is attained $\inf_{a\in A} D(a-b, -K)$, $\inf_{b\in B} D(a-b, -K)$ and $\inf_{(a,b)\in A\times B} D(a-b, -K)$; finally, part (i) of Theorem 3.2.1 is Proposition 3.12 in [11]. Conditions that allow to ensure that such infima are attained are not given.

To illustrate Theorem 3.2.1, we provide an example.

Example 3.2.2. With the data of Example 1.3.4, we can obtain the six scalarizations:

$$\begin{aligned}
&\mathbb{D}^{ss}(A, B) = a_s - b_i,\\
&\widehat{\mathbb{D}}^{is}(A, B) = \mathbb{D}^{si}(A, B) = a_i - b_i,\\
&\mathbb{D}^{is}(A, B) = \widehat{\mathbb{D}}^{si}(A, B) = a_s - b_s,\\
&\mathbb{D}^{ii}(A, B) = a_i - b_s.
\end{aligned}$$

Since A and B are K-compact and $(-K)$-compact, we can apply Theorem 3.2.1 and it results:

$$\begin{aligned}
A \preccurlyeq^{\forall\forall} B \;&\Leftrightarrow\; \mathbb{D}^{ss}(A,B) = a_s - b_i \leq 0 \;\Leftrightarrow a_s \leq b_i,\\
A \preccurlyeq^{\exists\forall} B \;&\Leftrightarrow\; \widehat{\mathbb{D}}^{is}(A,B) = a_i - b_i \leq 0 \;\Leftrightarrow\; a_i \leq b_i,\\
A \preceq^{\forall\exists} B \;&\Leftrightarrow\; \mathbb{D}^{si}(A,B) = a_i - b_i \leq 0 \;\Leftrightarrow\; a_i \leq b_i,\\
A \preceq^{\exists\forall} B \;&\Leftrightarrow\; \mathbb{D}^{is}(A,B) = a_s - b_s \leq 0 \;\Leftrightarrow\; a_s \leq b_s,\\
A \preccurlyeq^{\forall\exists} B \;&\Leftrightarrow\; \widehat{\mathbb{D}}^{si}(A,B) = a_s - b_s \leq 0 \;\Leftrightarrow\; a_s \leq b_s,\\
A \preccurlyeq^{\exists\exists} B \;&\Leftrightarrow\; \mathbb{D}^{ii}(A,B) = a_i - b_s \leq 0 \;\Leftrightarrow\; a_i \leq b_s.
\end{aligned}$$

which corroborates the results obtained in Example 1.3.4.

In the following examples, we can see that if the restrictions required in parts (ii)-(vi) of Theorem 3.2.1 are removed, then the reciprocal ones are not true.

Example 3.2.3. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = \{(-n, 1/n) : n \in \mathbb{N}\}$ and $B = \{(0,0)\}$. We have that

$$\mathbb{D}^{ii}(A,B) = \inf_{b\in B}\inf_{a\in A} D(a-b,-K) = \inf_{n\in\mathbb{N}} \frac{1}{n} = 0$$

and, however, $A \not\preccurlyeq^{\exists\exists} B$, that is, the reciprocal part of Theorem 3.2.1(vi) is not fulfilled because A is not K-compact. Moreover, $\widehat{\mathbb{D}}^{is}(A,B) = \inf_{a\in A}\sup_{b\in B} D(a-b,-K) = 0$ and, however, $A \not\preccurlyeq^{\exists\forall} B$, that is, the reciprocal part of Theorem 3.2.1(ii) is not satisfied because A is not K-compact. Finally, $\mathbb{D}^{si}(A,B) = 0$ and, however, $A \not\preceq^{\forall\exists} B$, that is, the reciprocal part of Theorem 3.2.1(iii) is not satisfied because A is not K-closed.

Example 3.2.4. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $B = \{(n, -1/n) : n \in \mathbb{N}\}$ and $A = \{(0,0)\}$. We have that $\mathbb{D}^{is}(A,B) = \inf_{b\in B}\sup_{a\in A} D(a-b,-K) = \inf_{n\in\mathbb{N}} 1/n = 0$ and, however, $A \not\preceq^{\exists\forall} B$, that is, the reciprocal part of Theorem 3.2.1(iv) is not true because B is not $(-K)$-compact. We also have that

$$\widehat{\mathbb{D}}^{si}(A,B) = \sup_{a\in A}\inf_{b\in B} D(a-b,-K) = 0$$

and, however, $A \not\preccurlyeq^{\forall\exists} B$, that is, the reciprocal part of Theorem 3.2.1(v) is not satisfied because B is not $(-K)$-closed.

3.3 Strict monotonicity

In this section, by considering a solid convex cone K and under suitable assumptions, strict monotonicity for the six set scalarizations $\mathbb{D}^{\alpha}_{K}(A,B)$ and $\widehat{\mathbb{D}}^{\alpha}_{K}(A,B)$ by using the six strict set relations of Kuroiwa, is investigated. To this purpose, some new important results wich deal with inequalities for the functions $\mathbb{D}^{\alpha}_{K}(A,B)$ and $\widehat{\mathbb{D}}^{\alpha}_{K}(A,B)$ when one of the variables is a sum of two sets are presented; moreover, it should be noted that theses results does not exist in the literature for the Gerstewitz's function. The results about strict monotonicity which are above mentioned could be applied in Section 4.2 to derive minimality conditions for a set optimization problem with the set criterion of solution. In the literature, there are very few authors that have researched strict monotonicity (see [5, 41, 51, 94, 96, 107]) and, in all these cases, Gerstewitz's function have been used. The results obtained represent an improvement since they require weakest assumptions.

We start with some needed results to achieve the objectives. We are going to show a kind of generalized subaditive property for the functions h^{α}.

Lemma 3.3.1. *Let $A, B \in \mathcal{P}_0(Y)$. The following inequalities hold:*

(i) $h^i(A) - \widehat{h}^s(B) \leq h^i(A+B) \leq h^i(A) + h^i(B)$.

(ii) $h^s(A) - \widehat{h}^i(B) \leq h^s(A+B) \leq h^s(A) + h^s(B)$.

Proof. (i) On the one hand, by Lemma 1.3.19(ix) we have

$$D(a+b,-K) \leq D(a,-K) + D(b,-K), \ \forall a \in A, \ \forall b \in B. \tag{3.6}$$

Taking infima in $a \in A$, $b \in B$, we deduce that $\inf_{a\in A, b\in B} D(a+b,-K) \leq \inf_{a\in A} D(a,-K) + \inf_{b\in B} D(b,-K)$, that is, we obtain $h^i(A+B) \leq h^i(A) + h^i(B)$ by Definition 3.1.8(i), which is the second inequality of (i).

On the other hand, by Lemma 1.3.19(ix) it follows

$$D(a,-K) = D(a+b-b,-K) \leq D(a+b,-K) + D(-b,-K),$$

that is,

$$D(a,-K) - D(b,K) \leq D(a+b,-K) \tag{3.7}$$

since $D(-b,-K) = D(b,K)$ by Lemma 1.3.19(xiii). Taking infima in $a \in A$, $b \in B$, we deduce

$$\inf_{a\in A} D(a,-K) + \inf_{b\in B}[-D(b,K)] \leq \inf_{a\in A, b\in B} D(a+b,-K) = h^i(A+B).$$

Hence, $h^i(A) - \sup_{b\in B} D(b,K) \leq h^i(A+B)$, which is the first inequality of (i) since $\widehat{h}^s(B) = \sup_{b\in B} D(b,K)$ by Definition 3.1.8(iv).

(ii) Taking suprema in $a \in A$, $b \in B$ in equation (3.6), we deduce

$$\sup_{a\in A, b\in B} D(a+b,-K) \leq \sup_{a\in A} D(a,-K) + \sup_{b\in B} D(b,-K),$$

that is, we obtain $h^s(A+B) \leq h^s(A) + h^s(B)$ by Definition 3.1.8(ii).

Taking now suprema in $a \in A$, $b \in B$ in (3.7), we derive

$$\sup_{a\in A} D(a,-K) + \sup_{b\in B}[-D(b,K)] \leq \sup_{a\in A, b\in B} D(a+b,-K) = h^s(A+B).$$

Hence, $h^s(A) - \inf_{b\in B} D(b,K) \leq h^s(A+B)$, which is the first inequality of (ii) since $\widehat{h}^i(B) = \inf_{b\in B} D(b,K)$ by Definition 3.1.8(iii). □

Now, let us see the strict monotonicity of the functions h^α.

Proposition 3.3.2. *Let $A, B \in \mathcal{P}_0(Y)$ and let K be solid.*

(i) If B is K-compact and $A \preccurlyeq_s^{\forall\exists} B$, then $h^i(A) < h^i(B)$.

(ii) If A is $(-K)$-compact and $A \preccurlyeq_s^{\forall\exists} B$, then $h^s(A) < h^s(B)$.

Proof. (i) As B is K-compact, by Definition 3.1.8(i) and Lemma 3.1.16, there exists $b_0 \in B$ such that $h^i(B) = \inf_{b\in B} D(b,-K) = D(b_0,-K)$. As $A \preccurlyeq_s^{\forall\exists} B$, then for all $b \in B$ there exists $a \in A$ such that $a \leq_{\operatorname{int} K} b$ and, in particular, for b_0 there exists $a_0 \in A$ such that $a_0 \leq_{\operatorname{int} K} b_0$. By Lemma 1.3.19(viii), we have $D(a_0,-K) < D(b_0,-K) = h^i(B)$, and consequently, $h^i(A) = \inf_{a\in A} D(a,-K) \leq D(a_0,-K) < h^i(B)$.

(ii) The proof of this part is analogous and is omitted. □

An immediate consequence of the previous proposition is the next result.

Corollary 3.3.3. *Let K be solid. Then the function h^i (resp., h^s) is strictly $\preccurlyeq_s^{\forall\exists}$ (resp., $\preccurlyeq_s^{\forall\exists}$) -increasing on K (resp., $(-K)$) -compact sets.*

Corollary 3.3.3, for $\preccurlyeq_s^{\forall\exists}$, is Proposition 3.3(ii) in [129].

Next, we provide inequalities for the functions $\bar{\mathbb{D}} \in \mathcal{D}$ when one of the variables is a sum of two sets.

Lemma 3.3.4. *Let $A, B, N \in \mathcal{P}_0(Y)$. Then, the following inequalities hold:*

(i-a) $\mathbb{D}^{ss}(A,B) - \widehat{h}^i(N) \leq \mathbb{D}^{ss}(A+N,B) \leq \mathbb{D}^{ss}(A,B) + h^s(N)$.

(i-b) $\mathbb{D}^{ss}(A,B) - h^i(N) \leq \mathbb{D}^{ss}(A,B+N) \leq \mathbb{D}^{ss}(A,B) + \widehat{h}^s(N)$.

(ii-a) $\widehat{\mathbb{D}}^{is}(A,B) - \widehat{h}^s(N) \leq \widehat{\mathbb{D}}^{is}(A+N,B) \leq \widehat{\mathbb{D}}^{is}(A,B) + h^i(N)$.

(ii-b) $\widehat{\mathbb{D}}^{is}(A,B) - h^i(N) \leq \widehat{\mathbb{D}}^{is}(A,B+N) \leq \widehat{\mathbb{D}}^{is}(A,B) + \widehat{h}^s(N)$.

(iii-a) $\mathbb{D}^{si}(A,B) - \widehat{h}^s(N) \leq \mathbb{D}^{si}(A+N,B) \leq \mathbb{D}^{si}(A,B) + h^i(N)$.

(iii-b) $\mathbb{D}^{si}(A,B) - h^i(N) \leq \mathbb{D}^{si}(A,B+N) \leq \mathbb{D}^{si}(A,B) + \widehat{h}^s(N)$.

(iv-a) $\mathbb{D}^{is}(A,B) - \widehat{h}^i(N) \leq \mathbb{D}^{is}(A+N,B) \leq \mathbb{D}^{is}(A,B) + h^s(N)$.

(iv-b) $\mathbb{D}^{is}(A,B) - h^s(N) \leq \mathbb{D}^{is}(A,B+N) \leq \mathbb{D}^{is}(A,B) + \widehat{h}^i(N)$.

(v-a) $\widehat{\mathbb{D}}^{si}(A,B) - \widehat{h}^i(N) \leq \widehat{\mathbb{D}}^{si}(A+N,B) \leq \widehat{\mathbb{D}}^{si}(A,B) + h^s(N)$.

(v-b) $\widehat{\mathbb{D}}^{si}(A,B) - h^s(N) \leq \widehat{\mathbb{D}}^{si}(A,B+N) \leq \widehat{\mathbb{D}}^{si}(A,B) + \widehat{h}^i(N)$.

(vi-a) $\mathbb{D}^{ii}(A,B) - \widehat{h}^s(N) \leq \mathbb{D}^{ii}(A+N,B) \leq \mathbb{D}^{ii}(A,B) + h^i(N)$.

(vi-b) $\mathbb{D}^{ii}(A,B) - h^s(N) \leq \mathbb{D}^{ii}(A,B+N) \leq \mathbb{D}^{ii}(A,B) + \widehat{h}^i(N)$.

Proof. We are going to prove the inequalities in the order (i), (iv), (vi), (iii), (ii) and (v).

(i-a) By Lemma 3.3.1(ii), we have

$$h^s(A-b) - \widehat{h}^i(N) \leq h^s(A+N-b) \leq h^s(A-b) + h^s(N). \tag{3.8}$$

By taking suprema with $b \in B$, we derive

$$\sup_{b\in B} h^s(A-b) - \widehat{h}^i(N) \leq \sup_{b\in B} h^s(A+N-b) \leq \sup_{b\in B} h^s(A-b) + h^s(N),$$

which is exactly the inequality (i-a) taking into account Lemma 3.1.10(i).

(i-b) By Lemma 3.1.10(i), we know that $\mathbb{D}^{ss}(A,B+N) = \sup_{b\in B, n\in N} h^s(A-b-n)$. By Lemma 3.3.1(ii), we deduce

$$h^s(A-b) - D(n,-K) \leq h^s(A-b-n) \leq h^s(A-b) + D(-n,-K). \tag{3.9}$$

By taking suprema with $b \in B$ and $n \in N$, we have

$$\sup_{b\in B} h^s(A-b) - \inf_{n\in N} D(n,-K) \leq \sup_{b\in B, n\in N} h^s(A-b-n)$$
$$\leq \sup_{b\in B} h^s(A-b) + \sup_{n\in N} D(-n,-K).$$

Consequently, by applying Lemma 3.1.10(i) and Definition 3.1.8 parts (i) and (iv), we obtain precisely part (i-b).

(iv-a) By taking infima with $b \in B$ in the inequality (3.8), we derive

$$\inf_{b\in B} h^s(A-b) - \widehat{h}^i(N) \leq \inf_{b\in B} h^s(A+N-b) \leq \inf_{b\in B} h^s(A-b) + h^s(N).$$

Therefore, by applying Lemma 3.1.10(iv), it results just part (iv-a).

(iv-b) By Lemma 3.1.10(iv), we know that $\mathbb{D}^{is}(A, B+N) = \inf_{b\in B, n\in N} h^s(A-b-n)$. Then, by taking infima with $b \in B$ and $n \in N$ in the inequality (3.9), we have

$$\inf_{b\in B} h^s(A-b) - \sup_{n\in N} D(n,-K) \leq \inf_{b\in B, n\in N} h^s(A-b-n)$$
$$\leq \inf_{b\in B} h^s(A-b) + \inf_{n\in N} D(-n,-K).$$

Consequently, by applying Lemma 3.1.10(iv) and Definition 3.1.8 parts (ii) and (iii), we obtain precisely part (iv-b).

Parts (vi) and (iii) are proved as parts (i) and (iv), by using Lemma 3.3.1(i) instead of Lemma 3.3.1(ii).

(ii-a) and (ii-b) are obtained from parts (iv-b) and (iv-a), respectively, for the particular case that we take $-K$ instead of K by Lemmas 3.1.5(ii) and 3.1.9.

(v-a) and (v-b) are obtained from parts (iii-b) and (iii-a), respectively, for the particular case that we take $-K$ instead of K by Lemmas 3.1.5(i) and 3.1.9. □

Remark 3.3.5. In the proof of Theorem 3.3.8, we will use Lemma 3.3.4 with $N = rU_0$, $r > 0$, and we will need the sign of $\bar{h}(rU_0)$ for all $\bar{h} \in \mathcal{H}$. If K is solid, then

$$\widehat{h}^i(rU_0) = h^i(rU_0) < 0 \quad \text{and} \quad \widehat{h}^s(rU_0) = h^s(rU_0) > 0.$$

Indeed, the equalities follow from Lemma 3.1.9 since $-rU_0 = rU_0$. The first inequality is true since

$$h^i(rU_0) = \inf_{u\in rU_0} D(u,-K) = \inf_{u\in rU_0\cap(-\operatorname{int} K)} D(u,-K) < 0$$

by Definition 1.3.18 and because $rU_0 \cap (-\operatorname{int} K) \neq \emptyset$. The inequality $h^s(rU_0) > 0$ is true since

$$h^s(rU_0) = \sup_{u \in rU_0} D(u, -K) = \sup_{u \in rU_0 \setminus (-K)} D(u, -K) > 0$$

by Definition 1.3.18 and because $rU_0 \setminus (-K) \neq \emptyset$.

In the following lemma, we show a property that will be useful in the sequel.

Lemma 3.3.6. *Let $A, B \in \mathcal{P}_0(Y)$ and let K be solid. If $A \preccurlyeq_s^{\forall\exists} B$ and B is K-compact, then there exits $r > 0$ such that $A \preccurlyeq_s^{\forall\exists} B + rU_0$, that is,*

$$B \subset A + \operatorname{int} K \Rightarrow \exists r > 0 : B + rU_0 \subset A + \operatorname{int} K.$$

Proof. If A is not K-proper, the result is evident. Thus, suppose that A is K-proper.

Since $a + K \subset A + K$ for all $a \in A$, then by Lemma 1.3.19(vii), (xii) and (xiii) we have $D(y, A + K) \leq D(y, a + K) = D(a - y, -K)$ for all $a \in A$ and all $y \in Y$ and, consequently, by Definition 3.1.1(iii), we have

$$D(y, A + K) \leq \inf_{a \in A} D(a - y, -K) = \mathbb{D}^{si}(A, y), \quad \forall y \in Y. \tag{3.10}$$

By hypothesis, $B \subset A + \operatorname{int} K$ and so $b \in A + \operatorname{int} K$ for all $b \in B$. Hence, by using Lemma 1.3.19(xii) and Definition 1.3.18, we have

$$D(b, A + K) = D(b, A + \operatorname{int} K) = -d(b, Y \setminus (A + \operatorname{int} K)), \quad \forall b \in B.$$

Therefore, by taking suprema with $b \in B$, we obtain

$$\begin{aligned} \sup_{b \in B} D(b, A + K) &= \sup_{b \in B} (-d(b, Y \setminus (A + \operatorname{int} K))) \\ &= -\inf_{b \in B} d(b, Y \setminus (A + \operatorname{int} K)) = -d(B, Y \setminus (A + \operatorname{int} K)). \end{aligned} \tag{3.11}$$

Let $r = d(B, Y \setminus (A + \operatorname{int} K))$. In view of (3.10) and (3.11), it follows

$$\sup_{b \in B} \mathbb{D}^{si}(A, b) \geq \sup_{b \in B} D(b, A + K) = -d(B, Y \setminus (A + \operatorname{int} K)) = -r. \tag{3.12}$$

As A is K-proper and B is K-compact, by Corollary 3.1.18(iii) there exists $b_0 \in B$ such that

$$\sup_{b \in B} \mathbb{D}^{si}(A, b) = \mathbb{D}^{si}(A, b_0). \tag{3.13}$$

Since $b_0 \in A + \operatorname{int} K$, there exists $a_0 \in A$ such that $a_0 - b_0 \in -\operatorname{int} K$, and using Definition 3.1.1(iii) and Lemma 1.3.19(iii) we get

$$\mathbb{D}^{si}(A, b_0) = \inf_{a \in A} D(a - b_0, -K) \leq D(a_0 - b_0, -K) < 0.$$

Taking into account (3.12) and (3.13), we derive that $0 > \mathbb{D}^{si}(A, b_0) \geq -r$, that is, $r > 0$. Therefore, since $\inf_{b \in B} d(b, Y \backslash (A + \operatorname{int} K)) = r$, it follows that $d(b, Y \backslash (A + \operatorname{int} K)) \geq r$, for all $b \in B$, and so $(b + rU_0) \cap (Y \backslash (A + \operatorname{int} K)) = \emptyset$, for all $b \in B$. This is equivalent to $b + rU_0 \subset A + \operatorname{int} K$ for all $b \in B$, that is, $B + rU_0 \subset A + \operatorname{int} K$. □

We are going to provide an identical result to Lemma 3.3.6 but in terms of the relation $\preccurlyeq_s^{\forall\exists}$.

Lemma 3.3.7. *Let $A, B \in \mathcal{P}_0(Y)$ and let K be solid. If $A \preccurlyeq_s^{\forall\exists} B$ and A is $(-K)$-compact, then there exits $r > 0$ such that $A + rU_0 \preccurlyeq_s^{\forall\exists} B$, that is,*

$$A \subset B - \operatorname{int} K \Rightarrow \exists r > 0 : A + rU_0 \subset B - \operatorname{int} K.$$

Next, under suitable assumptions, we establish the strict monotonicity for our six set scalarization functions with respect to each variable.

Theorem 3.3.8. *Let $A, B, C \in \mathcal{P}_0(Y)$ and let K be solid.*

(i-a) If C is $(-K)$-bounded, B is K-compact and $A \preccurlyeq_s^{\forall\exists} B$, then $\mathbb{D}^{ss}(C, A) > \mathbb{D}^{ss}(C, B)$.

(i-b) If C is K-bounded, A is $(-K)$-compact and $A \preccurlyeq_s^{\forall\exists} B$, then $\mathbb{D}^{ss}(A, C) < \mathbb{D}^{ss}(B, C)$.

(ii-a) If C is K-proper, B is K-compact and $A \preccurlyeq_s^{\forall\exists} B$, then $\widehat{\mathbb{D}}^{is}(C, A) > \widehat{\mathbb{D}}^{is}(C, B)$.

(ii-b) If C is K-bounded, B is K-compact and $A \preccurlyeq_s^{\forall\exists} B$, then $\widehat{\mathbb{D}}^{is}(A, C) < \widehat{\mathbb{D}}^{is}(B, C)$.

(iii-a) If C is K-proper, B is K-compact and $A \preccurlyeq_s^{\forall\exists} B$, then $\mathbb{D}^{si}(C, A) > \mathbb{D}^{si}(C, B)$.

(iii-b) If C is K-bounded, B is K-compact and $A \preccurlyeq_s^{\forall\exists} B$, then $\mathbb{D}^{si}(A, C) < \mathbb{D}^{si}(B, C)$.

(iv-a) If C is $(-K)$-bounded, A is $(-K)$-compact and $A \preccurlyeq_s^{\forall\exists} B$, then $\mathbb{D}^{is}(C, A) > \mathbb{D}^{is}(C, B)$.

(iv-b) If C is $(-K)$-proper, A is $(-K)$-compact and $A \preccurlyeq_s^{\forall\exists} B$, then $\mathbb{D}^{is}(A, C) < \mathbb{D}^{is}(B, C)$.

(v-a) If C is $(-K)$-bounded, A is $(-K)$-compact and $A \preccurlyeq_s^{\forall\exists} B$, then $\widehat{\mathbb{D}}^{si}(C, A) > \widehat{\mathbb{D}}^{si}(C, B)$.

(v-b) If C is $(-K)$-proper, A is $(-K)$-compact and $A \preccurlyeq_s^{\forall\exists} B$, then $\widehat{\mathbb{D}}^{si}(A, C) < \widehat{\mathbb{D}}^{si}(B, C)$.

(vi-a) If C is K-proper, A is $(-K)$-compact and $A \preccurlyeq_s^{\forall\exists} B$, then $\mathbb{D}^{ii}(C, A) > \mathbb{D}^{ii}(C, B)$.

(vi-b) If C is $(-K)$-proper, B is K-compact and $A \curlyeqprec_s^{\forall\exists} B$, then $\mathbb{D}^{ii}(A, C) < \mathbb{D}^{ii}(B, C)$.

(vii-a) Parts (i-a), (ii-a), (iii-a), (ii-b), (iii-b) and (vi-b) are also true if we assume $A \preccurlyeq_s^{\forall\forall} B$ or $A \preccurlyeq_s^{\exists\forall} B$ instead of $A \curlyeqprec_s^{\forall\exists} B$.

(vii-b) Parts (iv-a), (v-a), (vi-a), (i-b), (iv-b) and (v-b) are also true if we assume $A \preccurlyeq_s^{\forall\forall} B$ or $A \curlyeqprec_s^{\exists\forall} B$ instead of $A \preccurlyeq_s^{\forall\exists} B$.

Proof. (i-a) As B is K-compact, by Lemma 3.3.6, there exists $r > 0$ such that $A \curlyeqprec_s^{\forall\exists} B + rU_0$, which implies $A \curlyeqprec^{\forall\exists} B + rU_0$ by Proposition 1.3.7(iv). As the function $\mathbb{D}^{ss}(C, \cdot)$ is $\curlyeqprec^{\forall\exists}$-decreasing by Theorem 3.1.22(i), by applying Lemma 3.3.4(i-b), we have

$$\mathbb{D}^{ss}(C, A) \geq \mathbb{D}^{ss}(C, B + rU_0) \geq \mathbb{D}^{ss}(C, B) - h^i(rU_0) > \mathbb{D}^{ss}(C, B).$$

The last inequality is true because $h^i(rU_0) < 0$ by Remark 3.3.5, and $\mathbb{D}^{ss}(C, B) \in \mathbb{R}$ by Theorem 3.1.13(i) since C is $(-K)$-bounded and B is K-bounded as it is K-compact.

(i-b) As A is $(-K)$-compact, and $A \preccurlyeq_s^{\forall\exists} B$, then by Lemma 3.3.7 there exists $r > 0$ such that $A + rU_0 \preccurlyeq_s^{\forall\exists} B$, which implies $A + rU_0 \preccurlyeq^{\forall\exists} B$ by Proposition 1.3.7(iv). As the function $\mathbb{D}^{ss}(\cdot, C)$ is $\preccurlyeq^{\forall\exists}$-increasing by Theorem 3.1.22(iii), by applying Lemma 3.3.4(i-a) we have

$$\mathbb{D}^{ss}(B, C) \geq \mathbb{D}^{ss}(A + rU_0, C) \geq \mathbb{D}^{ss}(A, C) - \widehat{h}^i(rU_0) > \mathbb{D}^{ss}(A, C).$$

The last inequality is true because $\widehat{h}^i(rU_0) < 0$ by Remark 3.3.5, and $\mathbb{D}^{ss}(A,C) \in \mathbb{R}$ by Theorem 3.1.13(i) since C is K-bounded and A is $(-K)$-bounded as it is $(-K)$-compact.

Parts (ii)-(vi) are proved in a similar way.

(vii) By Proposition 1.3.7 parts (i) and (ii), we know that $\preccurlyeq^{\forall\forall} \Rightarrow \preccurlyeq^{\exists\forall} \Rightarrow \curlyeqprec^{\forall\exists}$ and $\preccurlyeq^{\forall\forall} \Rightarrow \curlyeqprec^{\exists\forall} \Rightarrow \preccurlyeq^{\forall\exists}$. Then, by parts (i)-(vi) and Lemma 1.3.13, we obtain the conclusion. □

We summarize the results of Theorem 3.3.8 in Table 3.2.

	$\precsim_s$						Assumptions		
T_C	$\preccurlyeq_s^{\forall\forall}$	$\preccurlyeq_s^{\exists\forall}$	$\curlyeqprec_s^{\forall\exists}$	$\curlyeqprec_s^{\exists\forall}$	$\preccurlyeq_s^{\forall\exists}$	$\preccurlyeq_s^{\exists\exists}$	C is	A is	B is
$\mathbb{D}^{ss}(C,\cdot)$	s.D.	s.D.	s.D.				$(-K)$-b.		K-c.
$\widehat{\mathbb{D}}^{is}(C,\cdot)$	s.D.	s.D.	s.D.				K-p.		K-c.
$\mathbb{D}^{si}(C,\cdot)$	s.D.	s.D.	s.D.				K-p.		K-c.
$\mathbb{D}^{is}(C,\cdot)$	s.D.			s.D.	s.D.		$(-K)$-b.	$(-K)$-c.	
$\widehat{\mathbb{D}}^{si}(C,\cdot)$	s.D.			s.D.	s.D.		$(-K)$-b.	$(-K)$-c.	
$\mathbb{D}^{ii}(C,\cdot)$	s.D.			s.D.	s.D.		K-p.	$(-K)$-c.	
$\mathbb{D}^{ss}(\cdot,C)$	s.I.			s.I.	s.I.		K-b.	$(-K)$-c.	
$\widehat{\mathbb{D}}^{is}(\cdot,C)$	s.I.	s.I.	s.I.				K-b.		K-c.
$\mathbb{D}^{si}(\cdot,C)$	s.I.	s.I.	s.I.				K-b.		K-c.
$\mathbb{D}^{is}(\cdot,C)$	s.I.			s.I.	s.I.		$(-K)$-p.	$(-K)$-c.	
$\widehat{\mathbb{D}}^{si}(\cdot,C)$	s.I.			s.I.	s.I.		$(-K)$-p.	$(-K)$-c.	
$\mathbb{D}^{ii}(\cdot,C)$	s.I.	s.I.	s.I.				$(-K)$-p.		K-c.

Table 3.2. Theorem 3.3.8, Strict monotonicity. $A \precsim_s B \Rightarrow T_C(A) < T_C(B)$ (s.I.) or $T_C(A) > T_C(B)$ (s.D.), where $T_C = \bar{\mathbb{D}}(C,\cdot)$ or $T_C = \bar{\mathbb{D}}(\cdot,C)$. ("s.D." means strictly decreasing; "s.I.", strictly increasing; "K-b.", K-bounded, "K-p.", K-proper and "K-c.", K-compact).

Parts (iii) and (v) of this theorem improve Proposition 2.3.15 and Theorem 2.3.17(vi), where it is additionally assumed that C is K-compact and $(-K)$-compact, respectively.

Results about strict monotonicity for extensions of Gerstewitz's function and for $\curlyeqprec_s^{\forall\exists}$ or $\preccurlyeq_s^{\forall\exists}$ have been considered in Maeda [107, Theorem 3.3] (with $Y = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$), Kuwano [91, Proposition 3.4], Kuwano et al. [94, Proposition 3.6] (they consider the five first relations of $\mathcal{R}$), Hernández and Rodríguez-Marín [51, Theorem 3.9], Araya [5, Theorem 3.2(xii) and (xiii)], Gutiérrez et al. [41, Theorem 3.5(g)], etc., most of them require that A, B and C are compact or cone-compact.

Remark 3.3.9. If the assumption on cone-compactness in Theorem 3.3.8 is not satisfied, the conclusion can be false. In this way, in Example 3.4.2(a), we have $A \curlyeqprec_s^{\forall\exists} B$ and however (we choose $C = B$) $\mathbb{D}^{si}(A, B) = \mathbb{D}^{si}(B, B) = 0$, note that B is not K-compact. In Example 3.4.2(b), we have $A \preccurlyeq_s^{\forall\exists} B$ and however (we choose $C = A$) $\widehat{\mathbb{D}}^{si}(A, A) = \widehat{\mathbb{D}}^{si}(A, B) = 0$, observe that A is not $(-K)$-compact.

3.4 Characterization by scalarization of the strict set relations

In this section, by considering a solid convex cone K, new characterizations of six strict set relations of Kuroiwa are derived by using the six set scalarizations $\mathbb{D}^{\alpha}_K(A, B)$ and $\widehat{\mathbb{D}}^{\alpha}_K(A, B)$. Moreover, some examples to illustrate the results obtained are provided with the aim to emphasize that the assumptions required cannot be removed. These results will be used in Section 4.2 to derive weak minimality conditions for a set optimization problem with the set criterion of solution.

In the following theorem, by using the six set oriented distances presented in Definition 3.1.1, we characterize the strict set relations which have been introduced in Definition 1.3.3 considering a solid convex cone K, that is, $\operatorname{int} K \neq \emptyset$. This result plays a crucial role in the following chapter when we want to investigate weak optimality in set optimization problems.

Next, in the following theorem we present a characterization for the six strict set relations by means of the six set oriented distance scalarization functions that will be used later on.

Theorem 3.4.1. *Let $A, B \in \mathcal{P}_0(Y)$ and K be solid. Then*

(i) If $\mathbb{D}^{ss}(A,B) < 0$, then $A \preccurlyeq_s^{\forall\forall} B$. The reciprocal implication is true if A is $(-K)$-compact and B is K-compact.

(ii) If $\widehat{\mathbb{D}}^{is}(A,B) < 0$, then $A \preccurlyeq_s^{\exists\forall} B$. The reciprocal implication is true if B is K-compact.

(iii) If $\mathbb{D}^{si}(A,B) < 0$, then $A \preccurlyeq_s^{\forall\exists} B$. The reciprocal implication is true if B is K-compact.

(iv) If $\mathbb{D}^{is}(A,B) < 0$, then $A \preccurlyeq_s^{\exists\forall} B$. The reciprocal implication is true if A is $(-K)$-compact.

(v) If $\widehat{\mathbb{D}}^{si}(A,B) < 0$, then $A \preccurlyeq_s^{\forall\exists} B$. The reciprocal implication is true if A is $(-K)$-compact.

(vi) $\mathbb{D}^{ii}(A,B) < 0$ if and only if $A \preccurlyeq_s^{\exists\exists} B$.

Proof. We are going to prove the theorem in the order (i), (iv), (ii), (iii), (v) and (vi).

(i) ($\Rightarrow$) Since $\mathbb{D}^{ss}(A,B) = \sup_{b\in B}\sup_{a\in A} D(a-b,-K) < 0$, it follows that $D(a-b,-K) < 0$ for all $a \in A$ and all $b \in B$. Then, by applying Lemma 1.3.19(iii) we deduce $a - b \in -\operatorname{int} K$ for all $a \in A$ and all $b \in B$, that is, $A \preccurlyeq_s^{\forall\forall} B$.

($\Leftarrow$) By hypothesis, for all $a \in A$ and all $b \in B$ we have $a - b \in -\operatorname{int} K$ and, therefore, $D(a-b,-K) < 0$ by Lemma 1.3.19(iii). As A is $(-K)$-compact and B is K-compact, then we obtain $\mathbb{D}^{ss}(A,B) = \max_{b\in B}\max_{a\in A} D(a-b,-K)$ by Corollary 3.1.18(i), and so there exist $a_0 \in A$ and $b_0 \in B$ such that $\mathbb{D}^{ss}(A,B) = D(a_0 - b_0, -K)$. Therefore, $\mathbb{D}^{ss}(A,B) < 0$.

(iv) ($\Rightarrow$) By contradiction, suppose that $A \not\preccurlyeq_s^{\exists\forall} B$. Then, for all $b \in B$, there exists $\widetilde{a} \in A$ such that $\widetilde{a} - b \notin -\operatorname{int} K$. Then, by Lemma 1.3.19(iii), we have $D(\widetilde{a} - b, -K) \geq 0$ and, consequently, by Definition 3.1.8(ii) we have $h^s(A-b) = \sup_{a\in A} D(a-b,-K) \geq D(\widetilde{a}-b,-K) \geq 0$, for all $b \in B$. Hence, by Lemma 3.1.10(iv), we have $\mathbb{D}^{is}(A,B) = \inf_{b\in B} h^s(A-b) \geq 0$, which is a contradiction.

($\Leftarrow$) By hypothesis, there exists $b_0 \in B$ such that for all $a \in A$ we have $a - b_0 \in -\operatorname{int} K$, that is, $D(a-b_0,-K) < 0$ by Lemma 1.3.19(iii). As A is $(-K)$-compact, then $A - b_0$ is $(-K)$-compact too, and by Lemma 3.1.16(ii) there exists $a_0 \in A$ such that $h^s(A-b_0) = \sup_{a\in A} D(a-b_0,-K) = D(a_0-b_0,-K)$. Therefore,

$h^s(A-b_0)<0$ and, consequently, $\mathbb{D}^{is}(A,B)=\inf_{b\in B} h^s(A-b)\le h^s(A-b_0)<0$.

(ii) ($\Rightarrow$) By Lemma 3.1.5(ii), it is verified that $\widehat{\mathbb{D}}^{is}(A,B)=\mathbb{D}^{is}_{-K}(B,A)$. By using the hypothesis, we have $\mathbb{D}^{is}_{-K}(B,A)<0$, and applying part (iv) it follows $B\preccurlyeq^{\exists\forall}_{s,-K} A$. Therefore, we conclude $A\preccurlyeq^{\exists\forall}_{s} B$ by Lemma 1.3.9(iv).

($\Leftarrow$) By hypothesis $A\preccurlyeq^{\exists\forall}_{s} B$, and by Lemma 1.3.9(iv), we have $B\preccurlyeq^{\exists\forall}_{s,-K} A$. As A is $(-K)$-compact, applying part (iv) it follows $\mathbb{D}^{is}_{-K}(B,A)<0$. By using Lemma 3.1.5(ii), we conclude $\widehat{\mathbb{D}}^{is}(A,B)<0$.

(iii) Although it is Theorem 2.3.11, we give a different proof here. The "if" part is as the previous ones. For the reciprocal implication, assume that $A\preccurlyeq^{\forall\exists}_{s} B$. As B is K-compact (and so it is also K-bounded), we can apply Theorem 3.3.8(iii-b) to the function $\mathbb{D}^{si}(\cdot,B)$, and it results $\mathbb{D}^{si}(A,B)<\mathbb{D}^{si}(B,B)=0$. The last equality is true by Proposition 2.2.34 since B is K-proper.

(v) It is Theorem 2.3.17(v), but it can also be proved following the same ideas as the ones used in part (iii).

(vi) ($\Rightarrow$) By hypothesis, we have $\inf_{b\in B}\inf_{a\in A} D(a-b,-K)<0$. Then, there exists $b_0\in B$ such that $\inf_{a\in A} D(a-b_0,-K)<0$ and, therefore, for that $b_0\in B$ there exists $a_0\in A$ such that $D(a_0-b_0,-K)<0$. So, $a_0-b_0\in -\operatorname{int} K$ by Lemma 1.3.19(iii), that is, $A\preccurlyeq^{\exists\exists}_{s} B$.

($\Leftarrow$) By hypothesis, there exists $a_0\in A$ and exists $b_0\in B$ such that $a_0-b_0\in -\operatorname{int} K$. Then, $D(a_0-b_0,-K)<0$ by Lemma 1.3.19(iii) and, hence, $\inf_{a\in A} D(a-b_0,-K)<0$. Consequently, $\mathbb{D}^{ii}(A,B)=\inf_{b\in B}\inf_{a\in A} D(a-b,-K)<0$. □

Similar results to Theorem 3.4.1 for extensions of Gerstewitz's function and for $\preccurlyeq^{\forall\exists}_{s}$ or $\preccurlyeq^{\forall\exists}_{s}$ have been considered in Maeda [107, Theorem 3.4] (with $Y=\mathbb{R}^n$ and $K=\mathbb{R}^n_+$ and assuming that A and B are compact sets), Hernández and Rodríguez-Marín [51, Corollary 3.11] and Araya [5, Theorem 3.2(xi)], where the two last ones require that A and B are cone-compact.

We are going to consider some examples in which we can see that without the conditions required in (i)-(v) of Theorem 3.4.1, the corresponding results are not true.

Example 3.4.2. Assume $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$.

(a) Let $A = \{(0,0)\}$ and $B = \{(n, \frac{1}{n}) : n = 1,2,3,\dots\}$. Clearly, we have that $A \preccurlyeq_s^{\forall\forall} B$, $A \preccurlyeq_s^{\exists\forall} B$ and $A \preccurlyeq_s^{\forall\exists} B$ and, however, $\mathbb{D}^{ss}(A,B) = \widehat{\mathbb{D}}^{is}(A,B) = \mathbb{D}^{si}(A,B) = \sup_n D((-n, \frac{-1}{n}), -K) = 0$. Hence, in Theorem 3.4.1, the reciprocal implication in parts (i), (ii) and (iii) is not verified because B is not K-compact.

(b) Let $A = \{(-n, \frac{-1}{n}) : n = 1,2,3,\dots\}$ and $B = \{(0,0)\}$. We can check that $A \preccurlyeq_s^{\exists\forall} B$ and $A \preccurlyeq_s^{\forall\exists} B$, but $\mathbb{D}^{is}(A,B) = \widehat{\mathbb{D}}^{si}(A,B) = \sup_n D((-n, \frac{-1}{n}), -K) = 0$. Therefore, in Theorem 3.4.1, the reciprocal implication in parts (iv) and (v) is not verified because A is not $(-K)$-compact.

We illustrate Theorem 3.4.1 in the next example.

Example 3.4.3. With the data of Example 1.3.5, first of all we are going to prove the following expressions for the six scalarizations:

$$\left.\begin{array}{ll} \mathbb{D}^{ss}(A,B) = D(a_s - b_i, -K), & \mathbb{D}^{is}(A,B) = D(a_s - b_s, -K), \\ \widehat{\mathbb{D}}^{is}(A,B) = D(a_i - b_i, -K), & \widehat{\mathbb{D}}^{si}(A,B) = D(a_s - b_s, -K), \\ \mathbb{D}^{si}(A,B) = D(a_i - b_i, -K), & \mathbb{D}^{ii}(A,B) = D(a_i - b_s, -K). \end{array}\right\} \tag{3.14}$$

Indeed, the following relations are clear for $[u,v]_K$ with $u \leq_K v$:

$$[u,v]_K + K = u + K \quad \text{and} \quad [u,v]_K - K = v - K.$$

Now, if we apply the above relations and Theorem 3.1.20, we obtain:

$$\begin{aligned} \mathbb{D}^{ss}(A,B) &= \mathbb{D}^{ss}(A - K, B + K) = \mathbb{D}^{ss}(a_s - K, b_i + K) = \mathbb{D}^{ss}(a_s, b_i) \\ &= D(a_s - b_i, -K). \end{aligned}$$

The rest of formulas in (3.14) are deduced by using the same ideas.

Secondly, since A and B are K-compact and $(-K)$-compact, which is checked via the definition, we can apply Theorem 3.4.1 and Lemma 1.3.19(iii) and we obtain:

$$\begin{aligned} A \preccurlyeq_s^{\forall\forall} B &\Leftrightarrow \mathbb{D}^{ss}(A,B) = D(a_s - b_i, -K) < 0 \Leftrightarrow a_s \leq_{\operatorname{int} K} b_i, \\ A \preccurlyeq_s^{\exists\forall} B &\Leftrightarrow \widehat{\mathbb{D}}^{is}(A,B) = D(a_i - b_i, -K) < 0 \Leftrightarrow a_i \leq_{\operatorname{int} K} b_i, \\ A \preccurlyeq_s^{\forall\exists} B &\Leftrightarrow \mathbb{D}^{si}(A,B) = D(a_i - b_i, -K) < 0 \Leftrightarrow a_i \leq_{\operatorname{int} K} b_i, \\ A \preccurlyeq_s^{\exists\forall} B &\Leftrightarrow \mathbb{D}^{is}(A,B) = D(a_s - b_s, -K) < 0 \Leftrightarrow a_s \leq_{\operatorname{int} K} b_s, \\ A \preccurlyeq_s^{\forall\exists} B &\Leftrightarrow \widehat{\mathbb{D}}^{si}(A,B) = D(a_s - b_s, -K) < 0 \Leftrightarrow a_s \leq_{\operatorname{int} K} b_s, \\ A \preccurlyeq_s^{\exists\exists} B &\Leftrightarrow \mathbb{D}^{ii}(A,B) = D(a_i - b_s, -K) < 0 \Leftrightarrow a_i \leq_{\operatorname{int} K} b_s. \end{aligned}$$

which corroborates the results obtained in Example 1.3.5.

Next, we state other characterization for $\prec_s^{\forall\exists}$ and $\preccurlyeq_s^{\forall\exists}$.

Proposition 3.4.4. *Let $A, B \in \mathcal{P}_0(Y)$ and let K be solid.*

(i) If B is K-compact, then

$$A \prec_s^{\forall\exists} B \Leftrightarrow \mathbb{D}^{si}(A, y) < \mathbb{D}^{si}(B, y), \ \forall y \in Y.$$

(ii) If A is $(-K)$-compact, then

$$A \preccurlyeq_s^{\forall\exists} B \Leftrightarrow \widehat{\mathbb{D}}^{si}(y, A) > \widehat{\mathbb{D}}^{si}(y, B), \ \forall y \in Y.$$

Proof. (i) ($\Rightarrow$) It is a consequence of Theorem 3.3.8(iii-b).

($\Leftarrow$) By contradiction, let us suppose $A \not\prec_s^{\forall\exists} B$. Then, there exits $b_0 \in B$ such that $a - b_0 \notin -\operatorname{int} K$ for all $a \in A$, and so, by Lemma 1.3.19(iii), we obtain that $D(a - b_0, -K) \geq 0$. In consequence, $\mathbb{D}^{si}(A, b_0) = \inf_{a \in A} D(a - b_0, -K) \geq 0$ by Definition 3.1.1(iii). Moreover, $\mathbb{D}^{si}(B, b_0) = \inf_{b \in B} D(b - b_0, -K) \leq 0$ since $b_0 \in B$, and so $\mathbb{D}^{si}(A, b_0) \geq \mathbb{D}^{si}(B, b_0)$, which contradicts the hypothesis with $y = b_0$.

(ii) It follows from part (i) taking $-K$ instead of K, and by applying Lemmas 1.3.9(iii) and 3.1.5(i). □

Chapter 4

Application to set optimization problems

In this chapter, by considering a set relation $\precsim$ (see [65, 87–89, 94]), in the sequel some types of $\precsim$-optimal solutions are defined to a set optimization problem ($\precsim$-SOP) with the set criterion of solution. Recall that Y is a real normed space ordered by a convex cone $K \subset Y$. Let S be the decision space (an arbitrary non-empty set) and $F : S \rightrightarrows Y$ be a set-valued map with $F(x) \neq \emptyset$, for all $x \in S$. We are going to study the following set optimization problem ($\precsim$-SOP), for the minimun case, with the set criterion of solution by considering the set relation $\precsim$:

$$(\precsim\text{-SOP}) \text{ minimize } F(x) \quad \text{subject to } x \in S.$$

As application of the results obtained in the previous sections, we aim to characterize by scalarization several types of solution to a set optimization problem.

In Bao and Mordukhovich [6], Hamel and Heyde [46], Heyde et al. [53] and Khan et al. [71], we can find some practical problems which are modeled by set optimization problems.

The results stated in this chapter are collected in [68, Section 5] and [69, Section 5].

4.1 Characterization by scalarization of minimal solutions

In this section, the monotonicity of the six set oriented distances (see Theorem 3.1.22) and the characterization of the set relations of Kuroiwa (see Theorem 3.2.1) are applied to derive several characterizations of minimal solution to a set optimization problem where the images of the set-valued ctive map are compared with one of the set relations belonging to $\mathcal{R} = \{\curlyeqprec^{\forall\forall},\ \curlyeqprec^{\exists\forall},\ \preccurlyeq^{\forall\exists},\ \preccurlyeq^{\exists\forall},\ \curlyeqprec^{\forall\exists},\ \curlyeqprec^{\exists\exists}\}$ (we use the so-called set criterion of solution). Moreover, $K \subset Y$ is a proper closed convex cone not necessarily solid (we do not assume that K is pointed).

First of all, we are going to introduce some needed definitions which will be used along the section. We star with the definition of $\precsim$-minimal solution. In the literature, this concept of solution have been considered in several works (see, for example, Jahn and Ha [65], Kuroiwa [86], Kuroiwa [88], Kuwano et al. [95]) with respect to the preorders $\preccurlyeq^{\forall\exists}$ and $\curlyeqprec^{\forall\exists}$.

Definition 4.1.1. Let $x_0 \in S$. It is said that x_0 is a $\precsim$-minimal solution to ($\precsim$-SOP) if $F(x) \precsim F(x_0)$ for some $x \in S$, implies $F(x_0) \precsim F(x)$.

Definition 4.1.2. Let $g : S \to \mathbb{R} \cup \{+\infty\}$. (a) We say that $M \subset S$ is the strict solution set of the scalar problem $\min\{g(x) : x \in S\}$ if

(i) $g(x) > 0$ for all $x \in S \setminus M$, and

(ii) $g(x) = 0$ for all $x \in M$.

(b) We say that $x_0 \in S$ (with $g(x_0) \in \mathbb{R}$) is a solution of the scalar problem $\min\{g(x) : x \in S\}$ if $g(x) \geq g(x_0)$ for all $x \in S$.

Given $A, B \in \mathcal{P}_0(Y)$ and $x_0 \in S$, we denote:

$A \sim^{\precsim} B$ if and only if $A \precsim B$ *and* $B \precsim A$,

$\mathrm{E}(x_0, \precsim) = \{x \in S : F(x) \sim^{\precsim} F(x_0)\}$,

$\mathrm{Lev}(x_0, \precsim) = \{x \in S : F(x) \precsim F(x_0)\}$.

Finally, we denote by $\mathcal{F}$ the family of all sets $F(x)$ with $x \in S$. It is said that a map $T : \mathcal{F} \to \mathbb{R} \cup \{+\infty\}$ is $\precsim$-increasing until x_0 if $x \in S$ and $F(x) \precsim F(x_0)$

implies $T(F(x)) \leq T(F(x_0))$. Whenever "N" denotes some property of sets in Y, we say that F is "N"-valued on X if $F(x)$ has the property "N" for every $x \in S$.

It is obvious that $\mathrm{E}(x_0, \precsim) \subset \mathrm{Lev}(x_0, \precsim)$. The inverse inclusion is also true for a $\precsim$-minimal solution. The proof is immediate.

Lemma 4.1.3. *The point $x_0 \in S$ is a $\precsim$-minimal solution of ($\precsim$-SOP) if and only if* $\mathrm{E}(x_0, \precsim) = \mathrm{Lev}(x_0, \precsim)$.

The following result is also immediate and, consequently, its proof is omitted.

Lemma 4.1.4. *If $\precsim$ is transitive and $x_0 \in S$ is a $\precsim$-minimal solution of ($\precsim$-SOP), then each element of* $\mathrm{E}(x_0, \precsim)$ *is a $\precsim$-minimal solution of ($\precsim$-SOP).*

The next lemma shows us when the scalarizations are not $-\infty$.

Lemma 4.1.5. *(i) One has $\mathbb{D}^{ss}(A, B) \neq -\infty$ for all $A, B \in \mathcal{P}_0(Y)$.*

(ii) If A is K-proper, then $\mathbb{D}^{si}(A, B) \neq -\infty$ for all $B \in \mathcal{P}_0(Y)$.

(iii) If B is $(-K)$-proper, then $\widehat{\mathbb{D}}^{si}(A, B) \neq -\infty$ for all $A \in \mathcal{P}_0(Y)$.

Proof. (i) It is Remark 3.1.7.

(ii) It is Corollary 2.2.6

(iii) It is a consequence of Lemma 3.1.5(i) and Proposition 2.2.5. □

Next, we are going to give a result which deal with $\precsim$-equivalent sets.

Lemma 4.1.6. *(Equivalence Lemma) Let $A, B \in \mathcal{P}_0(Y)$.*

(i) $A \sim^{\preccurlyeq^{\forall\forall}} B$ if and only if $\mathbb{D}^{ss}(A, B) = 0$ and $\mathbb{D}^{ss}(B, A) = 0$.

(ii) If $A \sim^{\preccurlyeq^{\exists\forall}} B$, then $\widehat{\mathbb{D}}^{is}(A, B) = 0$ and $\widehat{\mathbb{D}}^{is}(B, A) = 0$. The reciprocal implication is true if A and B are K-compact.

(iii) Let B be K-proper. If $A \sim^{\preccurlyeq^{\forall\exists}} B$, then $\mathbb{D}^{si}(A, B) = 0$ and $\mathbb{D}^{si}(B, A) = 0$. The reciprocal implication is true if A and B are K-closed.

(iv) If $A \sim^{\preccurlyeq^{\exists\forall}} B$, then $\mathbb{D}^{is}(A, B) = 0$ and $\mathbb{D}^{is}(B, A) = 0$. The reciprocal implication is true if A and B are $(-K)$-compact.

(v) Let B be $(-K)$-proper. If $A \sim^{\preccurlyeq^{\forall\exists}} B$, then $\widehat{\mathbb{D}}^{si}(A, B) = 0$ and $\widehat{\mathbb{D}}^{si}(B, A) =$ 0. The reciprocal implication is true if A and B are $(-K)$-closed.

Proof. (iii) and (v) are, respectively, Proposition 2.2.35 and Theorem 2.2.39(iv).

(ii) $A \sim^{\preccurlyeq^{\exists\forall}} B$ if and only if

$$A \preccurlyeq^{\exists\forall} B \quad \text{and} \quad B \preccurlyeq^{\exists\forall} A. \tag{4.1}$$

These conditions mean that there exist $a_0 \in A$ and $b_0 \in B$ such that $B \subset a_0 + K$ and $A \subset b_0 + K$, which implies that A and B are K-bounded, and so they are K-proper. In view of (4.1), by Proposition 1.3.7(i) we have that $A \preccurlyeq^{\forall\exists} B$ and $B \preccurlyeq^{\forall\exists} A$, that is, $A \sim^{\preccurlyeq^{\forall\exists}} B$. Hence, by part (iii), we obtain that $\mathbb{D}^{si}(A,B) = 0$ and $\mathbb{D}^{si}(B,A) = 0$. By applying Lemma 3.1.4(i) we derive that $0 = \mathbb{D}^{si}(A,B) \leq \widehat{\mathbb{D}}^{is}(A,B)$ and $0 = \mathbb{D}^{si}(B,A) \leq \widehat{\mathbb{D}}^{is}(B,A)$. Now, if we consider Theorem 3.2.1(ii), in view of (4.1) we deduce that $\widehat{\mathbb{D}}^{is}(A,B) \leq 0$ and $\widehat{\mathbb{D}}^{is}(B,A) \leq 0$. Therefore, we conclude that $\widehat{\mathbb{D}}^{is}(A,B) = 0$ and $\widehat{\mathbb{D}}^{is}(B,A) = 0$.

The reciprocal implication is an immediate consequence of Theorem 3.2.1(ii).

(iv) $A \sim^{\preccurlyeq^{\exists\forall}} B$ if and only if $A \preccurlyeq_K^{\exists\forall} B$ and $B \preccurlyeq_K^{\exists\forall} A$. Using Lemma 1.3.9(ii), we deduce that $B \preccurlyeq_{-K}^{\exists\forall} A$ and $A \preccurlyeq_{-K}^{\exists\forall} B$. By part (ii) we get $\mathbb{D}_{-K}^{is}(B,A) = \mathbb{D}_{-K}^{is}(A,B) = 0$. Finally, using Lemma 3.1.5(ii), we obtain the conclusion.

The reciprocal implication is an immediate consequence of Theorem 3.2.1(iv).

(i) $A \sim^{\preccurlyeq^{\forall\forall}} B$ if and only if $A \preccurlyeq^{\forall\forall} B$ and $B \preccurlyeq^{\forall\forall} A$. By Theorem 3.2.1(i), we deduce $\mathbb{D}^{ss}(A,B) \leq 0$ and $\mathbb{D}^{ss}(B,A) \leq 0$. On the other hand, by Proposition 1.3.7(ii) we have $A \preccurlyeq^{\exists\forall} B$ and $B \preccurlyeq^{\exists\forall} A$, that is, $A \sim^{\preccurlyeq^{\exists\forall}} B$. Hence, by part (ii), we obtain $\widehat{\mathbb{D}}^{is}(A,B) = 0$ and $\widehat{\mathbb{D}}^{is}(B,A) = 0$. By applying Lemma 3.1.4(i), we deduce $0 = \widehat{\mathbb{D}}^{is}(A,B) \leq \mathbb{D}^{ss}(A,B)$ and $0 = \widehat{\mathbb{D}}^{is}(B,A) \leq \mathbb{D}^{ss}(A,B)$. Therefore, we conclude $\mathbb{D}^{ss}(A,B) = 0$ and $\mathbb{D}^{ss}(B,A) = 0$.

The "if" part is an immediate consequence of Theorem 3.2.1(i). □

The following theorem is the main result in this section, and it establishes necessary and sufficient conditions of minimality.

Theorem 4.1.7. *For each relation $\precsim \in \{\preccurlyeq^{\forall\forall}, \preccurlyeq^{\exists\forall}, \preccurlyeq^{\forall\exists}, \preccurlyeq^{\exists\forall}, \preccurlyeq^{\forall\exists}\}$, let us consider problem ($\precsim$-SOP) with $x_0 \in S$ and suppose that F satisfies assumption $A(\precsim)$ of Table 4.1.*

Assumption	*Requirements*
$A(\preccurlyeq^{\forall\forall})$	*There is not special requirements*
$A(\preccurlyeq^{\exists\forall})$	*F is K-compact valued*
$A(\curlyeqprec^{\forall\exists})$	*F is K-proper valued and K-closed valued*
$A(\curlyeqprec^{\exists\forall})$	*F is $(-K)$-compact valued*
$A(\preccurlyeq^{\forall\exists})$	*$F(x_0)$ is $(-K)$-proper and F is $(-K)$-closed valued*

Table 4.1. Assumptions in Theorem 4.1.7.

Then for each pair

$$(\precsim,\bar{\mathbb{D}})\in\left\{\left(\preccurlyeq^{\forall\forall},\mathbb{D}^{ss}\right),\ \left(\preccurlyeq^{\exists\forall},\widehat{\mathbb{D}}^{is}\right),\ \left(\curlyeqprec^{\forall\exists},\mathbb{D}^{si}\right),\ \left(\curlyeqprec^{\exists\forall},\mathbb{D}^{is}\right),\ \left(\preccurlyeq^{\forall\exists},\widehat{\mathbb{D}}^{si}\right)\right\},$$

the following statements are equivalent:

(a) x_0 is a $\precsim$-minimal solution of ($\precsim$-SOP).

(b) $\bar{\mathbb{D}}(F(x),F(x_0))>0$ for all $x\in S\setminus \mathrm{E}(x_0,\precsim)$.

(c) For all $x\in S$ one has $\bar{\mathbb{D}}(F(x),F(x_0))>0$ or $\bar{\mathbb{D}}(F(x_0),F(x))\le 0$.

(d) There exists a map $T:\mathcal{F}\to\mathbb{R}\cup\{+\infty\}$ which is $\precsim$-increasing until x_0 and such that

(d_1) $T(F(x))>0$ for all $x\in S\setminus \mathrm{E}(x_0,\precsim)$,

(d_2) if $x\in S$ and $F(x)\precsim F(x_0)$, then $T(F(x))=0$.

Proof. We prove the case $(\precsim,\bar{\mathbb{D}})=\left(\preccurlyeq^{\exists\forall},\widehat{\mathbb{D}}^{is}\right)$, the rest of the cases are similar, by applying the results of Table 4.1.

Case $(\precsim,\bar{\mathbb{D}})=\left(\preccurlyeq^{\exists\forall},\widehat{\mathbb{D}}^{is}\right)$.

(a)⇒(b). By Lemma 4.1.3, we have that $\mathrm{E}(x_0,\preccurlyeq^{\exists\forall})=\mathrm{Lev}(x_0,\preccurlyeq^{\exists\forall})$. So, if $x\in S\setminus\mathrm{E}(x_0,\preccurlyeq^{\exists\forall})$, then $x\notin\mathrm{Lev}(x_0,\preccurlyeq^{\exists\forall})$, i.e., $F(x)\not\preccurlyeq^{\exists\forall}F(x_0)$, and by Theorem 3.2.1(ii) we have that $\widehat{\mathbb{D}}^{is}(F(x),F(x_0))>0$ because $F(x_0)$ is K-bounded and $F(x)$ is K-compact since F is K-compact valued.

(b)⇒(c). If $x\in S\setminus\mathrm{E}(x_0,\preccurlyeq^{\exists\forall})$, then by hypothesis, $\widehat{\mathbb{D}}^{is}(F(x),F(x_0))>0$. If $x\in\mathrm{E}(x_0,\preccurlyeq^{\exists\forall})$, by Lemma 4.1.6(ii) it follows that $\widehat{\mathbb{D}}^{is}(F(x_0),F(x))=0$.

(c)⇒(d). Choose $T:\mathcal{F}\to\mathbb{R}$ given by $T(F(x))=\widehat{\mathbb{D}}^{is}(F(x),F(x_0))$. We have that T is $\preccurlyeq^{\exists\forall}$-increasing until x_0 by Theorem 3.1.22(ii-b), and $T(F(x))\in\mathbb{R}$ by Theorem 3.1.13(ii) because $F(x)$ is K-proper and $F(x_0)$ is K-bounded since F is K-compact valued.

Let us see that $\mathrm{E}(x_0, \preccurlyeq^{\exists\forall}) = \mathrm{Lev}(x_0, \preccurlyeq^{\exists\forall})$. By contradiction, suppose that there exists $x \in \mathrm{Lev}(x_0, \preccurlyeq^{\exists\forall})\backslash \mathrm{E}(x_0, \preccurlyeq^{\exists\forall})$. Then $F(x) \preccurlyeq^{\exists\forall} F(x_0)$ and $F(x_0) \not\preccurlyeq^{\exists\forall} F(x)$. By Theorem 3.2.1(ii) applied to both inequalities we obtain $\widehat{\mathbb{D}}^{is}(F(x), F(x_0)) \leq 0$ and $\widehat{\mathbb{D}}^{is}(F(x_0), F(x)) > 0$ because F is K-compact valued, which contradicts the hypothesis (c). So, $\mathrm{E}(x_0, \preccurlyeq^{\exists\forall}) = \mathrm{Lev}(x_0, \preccurlyeq^{\exists\forall})$.

Therefore, if $x \in S$ and $F(x) \preccurlyeq^{\exists\forall} F(x_0)$, then $x \in \mathrm{E}(x_0, \preccurlyeq^{\exists\forall})$, and by Lemma 4.1.6(ii) it follows that $T(F(x)) = \widehat{\mathbb{D}}^{is}(F(x), F(x_0)) = 0$. Thus, (d_2) is proved.

If $x \in S\backslash \mathrm{E}(x_0, \preccurlyeq^{\exists\forall})$, then $F(x) \not\preccurlyeq^{\exists\forall} F(x_0)$, and by Theorem 3.2.1(ii) it follows that $T(F(x)) = \widehat{\mathbb{D}}^{is}(F(x), F(x_0)) > 0$ since F is K-compact valued. Thus, (d_1) holds.

(d)⇒(a). By contradiction, assume that x_0 is not a $\preccurlyeq^{\exists\forall}$-minimal solution. Then, there exists $x_1 \in S$ such that $F(x_1) \preccurlyeq^{\exists\forall} F(x_0)$ and $F(x_0) \not\preccurlyeq^{\exists\forall} F(x_1)$. From (d_1), it follows that $T(F(x_1)) > 0$ since $x_1 \notin \mathrm{E}(x_0, \preccurlyeq^{\exists\forall})$, and by (d_2), we have $T(F(x_1)) = 0$, which is a contradiction. The proof is finished. □

Rel $\precsim$	Func $\bar{\mathbb{D}}$	Char $A \precsim B$	Equival.	Finite	Increase
$\preccurlyeq^{\forall\forall}$	$\mathbb{D}^{ss}$	Th. 3.2.1(i)	Lem. 4.1.6(i)	Lem. 4.1.5(i)	Th. 3.1.22(i-b)
$\preccurlyeq^{\exists\forall}$	$\widehat{\mathbb{D}}^{is}$	Th. 3.2.1(ii)	Lem. 4.1.6(ii)	Th. 3.1.13(ii)	Th. 3.1.22(ii-b)
$\preccurlyeq^{\forall\exists}$	$\mathbb{D}^{si}$	Th. 3.2.1(iii)	Lem. 4.1.6(iii)	Lem. 4.1.5(ii)	Th. 3.1.22(iii-b)
$\preccurlyeq^{\exists\forall}$	$\mathbb{D}^{is}$	Th. 3.2.1(iv)	Lem. 4.1.6(iv)	Th. 3.1.13(iv)	Th. 3.1.22(iv-b)
$\preccurlyeq^{\forall\exists}$	$\widehat{\mathbb{D}}^{si}$	Th. 3.2.1(v)	Lem. 4.1.6(v)	Lem. 4.1.5(iii)	Th. 3.1.22(v-b)

Table 4.2. Results that are applied in the proof of Theorem 4.1.7.

Remark 4.1.8. (i) Let us observe that the monotonicity of T is not used to prove (d)⇒(a). This is useful if we apply (d) as a sufficient condition. However, in order to have other equivalent expressions to (d) it is necessary that T be $\precsim$-increasing until x_0 (see Remark 4.1.10).

(ii) Under the assumptions of Theorem 4.1.7, the following statement is also equivalent to (a), (b), (c) and (d):

$(\bar{d})$ There exists a map $T : \mathcal{A} \to \mathfrak{R}$ which is $\precsim$-increasing and such that (d_1) and (d_2) holds, where $\mathcal{A}$ and $\mathfrak{R}$ are given in Table 4.3.

	$\preccurlyeq^{\forall\forall}$	$\preccurlyeq^{\exists\forall}$	$\preccurlyeq^{\forall\exists}$	$\preccurlyeq^{\exists\forall}$	$\preccurlyeq^{\forall\exists}$
$\mathcal{A}$	$\mathcal{P}_0(Y)$	$\mathcal{C}_K(Y)$	$\mathcal{P}_{0,K}(Y)$	$\mathcal{C}_{-K}(Y)$	$\mathcal{P}_{0,-K}(Y)$
$\mathfrak{R}$	$\mathbb{R}\cup\{+\infty\}$	$\mathbb{R}$	$\mathbb{R}\cup\{+\infty\}$	$\mathbb{R}$	$\mathbb{R}\cup\{+\infty\}$

Table 4.3. Values of $\mathcal{A}$ and $\mathfrak{R}$ in statement $(\bar{d})$
($\mathcal{C}_K(Y)$ is the family of all K-compact subsets of Y).

Indeed, it is clear that $(\bar{d}) \Rightarrow (d)$. The proof of $(c) \Rightarrow (\bar{d})$ is the same that the one of $(c) \Rightarrow (d)$ in Theorem 4.1.7, but now we define $T(A) = \bar{\mathbb{D}}(A, F(x_0))$ for all $A \in \mathcal{A}$.

(iii) In the statement (d) of Theorem 4.1.7 and in Remark 4.1.10, the map T actually takes its values in $\mathfrak{R}$ according Table 4.3.

In the following remarks, we provide other equivalent expressions for statements (b), (c) and (d) of Theorem 4.1.7.

Remark 4.1.9. (i) The following statements are equivalent to (c):

(c') It does not exist $x \in S$ such that

$$\bar{\mathbb{D}}(F(x), F(x_0)) \leq 0 \quad \text{and} \quad \bar{\mathbb{D}}(F(x_0), F(x)) > 0.$$

(c'') For each $x \in S$ one has $\bar{\mathbb{D}}(F(x), F(x_0)) > 0$ or $\bar{\mathbb{D}}(F(x_0), F(x)) = 0$.

(c''') For each $x \in S$, exactly one of the following statements is true:

(c_1''') $\bar{\mathbb{D}}(F(x), F(x_0)) > 0$,

(c_2''') $\bar{\mathbb{D}}(F(x), F(x_0)) = 0$ and $\bar{\mathbb{D}}(F(x_0), F(x)) = 0$.

(ii) The following statements are equivalent to (c_2'''):

(e_1) $\bar{\mathbb{D}}(F(x), F(x_0)) \leq 0$ and $\bar{\mathbb{D}}(F(x_0), F(x)) \leq 0$.

(e_2) $\bar{\mathbb{D}}(F(x), F(x_0)) \leq 0$ and $\bar{\mathbb{D}}(F(x_0), F(x)) = 0$.

(e_3) $\bar{\mathbb{D}}(F(x), F(x_0)) = 0$ and $\bar{\mathbb{D}}(F(x_0), F(x)) \leq 0$.

So, (c_2''') can be replaced for any of them.

Proof. (i) $(c) \Leftrightarrow (c')$. It is clear since *not* (c') is just *not* (c).

$(c'') \Rightarrow (c)$. It is also clear, and $(c) \Rightarrow (c'')$ because $(c) \Rightarrow (b)$ and $(b) \Rightarrow (c'')$ (see the proof of $(b) \Rightarrow (c)$ in Theorem 4.1.7 where Lemma 4.1.6 is applied). The same reasoning serves to prove that $(c''') \Leftrightarrow (c)$.

(ii) It is clear that $(c_2''') \Rightarrow (e_3) \Rightarrow (e_1)$ and $(c_2''') \Rightarrow (e_2) \Rightarrow (e_1)$. By Theorem 3.2.1, under the assumptions of Table 4.3, one has $(e_1) \Rightarrow F(x) \sim^{\precsim} F(x_0)$, and from Lemma 4.1.6 it follows that $F(x) \sim^{\precsim} F(x_0) \Rightarrow (c_2''')$.

Property (c) (or (c') or (c'') or (c''')) is interesting because it is not necessary to determine $\mathrm{E}(x_0, \precsim)$ as in (b) or $\mathrm{Lev}(x_0 \precsim)$ as in (d). If we have a procedure to calculate $\bar{\mathbb{D}}(F(x), F(y))$ for all $x, y \in S$, property (c) could be useful to obtain all $\precsim$-minimal solutions. In this situation, it would be easy to construct an algorithm if S is finite.

Remark 4.1.10. Let $T : \mathcal{F} \to \mathbb{R} \cup \{+\infty\}$ and consider the following statements:

(d') (d_1') $T(F(x)) > 0$ for all $x \in S \setminus \mathrm{E}(x_0, \precsim)$,

(d_2') if $x \in \mathrm{E}(x_0, \precsim)$, then $T(F(x)) = 0$.

(d'') (d_1'') $T(F(x)) \geq 0$ for all $x \in S$,

(d_2'') if $x \in S$ then, $x \in \mathrm{E}(x_0, \precsim) \Leftrightarrow T(F(x)) = 0$.

(d''') (d_1''') $T(F(x)) > 0$ for all $x \in S \setminus \mathrm{E}(x_0, \precsim)$,

(d_2''') if $x \in S$ then, $x \in \mathrm{E}(x_0, \precsim) \Leftrightarrow T(F(x)) = 0$.

Then

(i) $(d) \Rightarrow (d') \Leftrightarrow (d'') \Leftrightarrow (d''')$.

(ii) If T is $\precsim$-increasing until x_0 and

$$x_0 \in \mathrm{E}(x_0, \precsim) \tag{4.2}$$

then $(d'') \Rightarrow (d)$, and so the four statements (d), (d'), (d'') and (d''') are equivalent.

Proof. (i) $(d) \Rightarrow (d')$. In Theorem 4.1.7, without using any assumption, it has been proved that $(d) \Rightarrow (a)$, and so by Lemma 4.1.3 we have $\mathrm{E}(x_0, \precsim) = \mathrm{Lev}(x_0, \precsim)$. Now it is clear that $(d) \Rightarrow (d')$.

$(d') \Leftrightarrow (d'')$. We only have to prove the '$\Leftarrow$' part of (d_2'') since the rest is clear. Let $T(F(x)) = 0$. If $x \notin \mathrm{E}(x_0, \precsim)$, then by (d_1'), $T(F(x)) > 0$, a contradiction, and so $x \in \mathrm{E}(x_0, \precsim)$.

$(d'') \Leftrightarrow (d''')$. It is immediate.

(ii) We only have to prove (d_2) since $(d'') \Leftrightarrow (d')$ and $(d_1) \equiv (d_1')$. Suppose that $F(x) \precsim F(x_0)$. As (4.2) holds, by using (d_2'') we derive $T(F(x_0)) = 0$. Now, as T is $\precsim$-increasing until x_0, we deduce that $T(F(x)) \leq T(F(x_0)) = 0$. By (d_1''), we obtain $T(F(x)) \geq 0$, and so we conclude that $T(F(x)) = 0$.

Observe that statement (d') is equivalent to say that $\mathrm{E}(x_0, \precsim)$ is the strict solution set of the scalar problem $\min\{T(F(x)) : x \in S\}$ according to Definition 4.1.2. Note that the condition (4.2) is satisfied for all $x_0 \in S$ for the reflexive relations $\preccurlyeq^{\forall\exists}$ and $\precsim^{\forall\exists}$.

We point out that, if (4.2) holds, then statement (d''_1) is equivalent to say that x_0 is a solution of the scalar problem $\min\{T(F(x)) : x \in S\}$ since $T(F(x_0)) = 0$ by (d''_2).

Remark 4.1.11. (i) The following statements are equivalent to (b):

(b') (b'_1) $\bar{\mathbb{D}}(F(x), F(x_0)) > 0$ for all $x \in S \setminus \mathrm{E}(x_0, \precsim)$,

(b'_2) if $x \in \mathrm{E}(x_0, \precsim)$, then $\bar{\mathbb{D}}(F(x), F(x_0)) = 0$.

(b'') (b''_1) $\bar{\mathbb{D}}(F(x), F(x_0)) \geq 0$ for all $x \in S$,

(b''_2) $x \in S$, $x \in \mathrm{E}(x_0, \precsim) \Leftrightarrow \bar{\mathbb{D}}(F(x), F(x_0)) = 0$.

(b''') (b'''_1) $\bar{\mathbb{D}}(F(x), F(x_0)) > 0$ for all $x \in S \setminus \mathrm{E}(x_0, \precsim)$,

(b'''_2) $x \in S$, $x \in \mathrm{E}(x_0, \precsim) \Leftrightarrow \bar{\mathbb{D}}(F(x), F(x_0)) = 0$.

(b^{iv}) (b^{iv}_1) $\bar{\mathbb{D}}(F(x), F(x_0)) \geq 0$ for all $x \in S$,

(b^{iv}_2) if $x \in S$ and $\bar{\mathbb{D}}(F(x), F(x_0)) = 0$, then $x \in \mathrm{E}(x_0, \precsim)$.

The proof of equivalences $(b') \Leftrightarrow (b'') \Leftrightarrow (b''')$ is identical to the one given in Remark 4.1.10 only by changing $T(F(x))$ for $\bar{\mathbb{D}}(F(x), F(x_0))$. The proof of $(b) \Leftrightarrow (b')$ is clear since $(b) \equiv (b'_1)$ and (b'_2) is true by Lemma 4.1.6, parts (i) to (v) as appropriate according to $\precsim$ and Table 4.3. The implication $(b'') \Rightarrow (b^{iv})$ is obvious. Finally, to prove $(b^{iv}) \Rightarrow (b)$ we take $x \in S \setminus \mathrm{E}(x_0, \precsim)$. Then, by (b^{iv}_2) we deduce that $\bar{\mathbb{D}}(F(x), F(x_0)) \neq 0$, and taking into account (b^{iv}_1), we conclude that $\bar{\mathbb{D}}(F(x), F(x_0)) > 0$.

(ii) Statement (b') is equivalent to say that $\mathrm{E}(x_0, \precsim)$ is the strict solution set of the scalar problem

$$\min\{\bar{\mathbb{D}}(F(x), F(x_0)) : x \in S\} \tag{4.3}$$

according to Definition 4.1.2.

(iii) We point out that, if $x_0 \in \mathrm{E}(x_0, \precsim)$, then statement (b''_1) is equivalent to say that x_0 is a solution of the scalar problem (4.3) since $\bar{\mathbb{D}}(F(x_0), F(x_0)) = 0$ by (b''_2).

The following example shows a set optimization problem with a minimal solution which is not solution of the scalarized problem (4.3).

Example 4.1.12. Consider $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = \{(1,1)\}$, $B = \{(2,0),(0,2)\}$, $S = \{0,1\}$, $F(0) = B$ and $F(1) = A$. We have that 0 is a $\preccurlyeq^{\exists\forall}$-minimal solution. However, 0 is not a solution of the scalar problem $\min\{\widehat{\mathbb{D}}^{is}(F(x),F(0)) : x \in S\}$ since $\widehat{\mathbb{D}}^{is}(F(1),F(0)) = 1$ and $\widehat{\mathbb{D}}^{is}(F(0),F(0)) = 2$.

Remark 4.1.13. Results that have some similarity with Theorem 4.1.7 are the following:

1) Hernández and Rodríguez-Marín [51], Theorems 4.1 and 4.7 and Corollary 4.8 for $\preccurlyeq^{\forall\exists}$, in the form $(a) \Leftrightarrow (d')$, $(a) \Leftrightarrow (d'')$ and $(a) \Leftrightarrow (b'')$, by using an extension of the Gerstewitz's function and requiring F is K-bounded valued.

2) Kuwano et al. [94], Corollary 4.1 for $\preccurlyeq^{\forall\exists}$ in the form $(a) \Leftrightarrow (b'')$, Corollary 4.2 for $\preccurlyeq^{\forall\forall}$ and $\preccurlyeq^{\exists\forall}$ in the form $(b'') \Rightarrow (a)$, and Corollary 4.3 for $\preccurlyeq^{\exists\forall}$ in the form $(b'') \Rightarrow (a)$, by using several extensions of the Gerstewitz's function. Results for $\preccurlyeq^{\exists\exists}$ are not given as our in Theorem 4.1.15.

3) Gutiérrez et al. [41], Corollary 4.4(b) for $\preccurlyeq^{\forall\exists}$, in the form $(a) \Leftrightarrow (b''')$, using a scalarization based on the oriented distance.

4) Xu and Li [129], Theorems 4.1 and 4.7 for $\preccurlyeq^{\forall\exists}$, in the form $(a) \Leftrightarrow (d')$ and $(a) \Leftrightarrow (b'')$, using a scalarization based on the oriented distance.

5) Khoshkhabar-amiranloo et al. [73], Theorems 3.1(i) and 4.1(i) for $\preccurlyeq^{\forall\exists}$ and $\preccurlyeq^{\forall\exists}$ resp., both in the form $(a) \Leftrightarrow (d'')$, using a nonegative modification of an extension of the Gerstewitz's function.

6) Jiménez et al. [67], Theorems 5.4 and 5.12 for $\preccurlyeq^{\forall\exists}$ and $\preccurlyeq^{\forall\exists}$, resp., both in the form $(a) \Leftrightarrow (b^{iv})$, using the functions $\mathbb{D}^{si}$ and $\widehat{\mathbb{D}}^{si}$.

The case $(\precsim, \bar{\mathbb{D}}) = \left(\preccurlyeq^{\exists\exists}, \mathbb{D}^{ii}\right)$ has not been considered in Theorem 4.1.7 because it presents some peculiarities and we prefer to deal with it as a separate case.

First, we examine the equivalence. As a consequence of Theorem 3.2.1(vi) we have the following result.

Lemma 4.1.14. *Let $A, B \in \mathcal{P}_0(Y)$. If $A \sim^{\preccurlyeq^{\exists\exists}} B$, then $\mathbb{D}^{ii}(A, B) \leq 0$ and $\mathbb{D}^{ii}(B, A) \leq 0$. The reciprocal implication is true if A is K-compact and B is $(-K)$-compact.*

Note that, unlike Lemma 4.1.6, now we do not obtain the equality to zero.

Theorem 4.1.15. *Consider the problem ($\preccurlyeq^{\exists\exists}$-SOP) and a point $x_0 \in S$. Assume that*

$$F \text{ is } K\text{-compact valued and } (-K)\text{-compact valued.} \tag{4.4}$$

Then, the following statements are equivalent:

(a) x_0 is a $\preccurlyeq^{\exists\exists}$-minimal solution of ($\preccurlyeq^{\exists\exists}$-SOP).

(b) $\mathbb{D}^{ii}(F(x), F(x_0)) > 0$ for all $x \in S \setminus \mathrm{E}(x_0, \preccurlyeq^{\exists\exists})$.

(c) For all $x \in S$ one has $\mathbb{D}^{ii}(F(x), F(x_0)) > 0$ or $\mathbb{D}^{ii}(F(x_0), F(x)) \leq 0$.

(d) There exists a map $T : \mathcal{F} \to \mathbb{R}$ such that

(d_1) $T(F(x)) > 0$ for all $x \in S \setminus \mathrm{E}(x_0, \preccurlyeq^{\exists\exists})$,

(d_2) if $x \in S$ and $F(x) \preccurlyeq^{\exists\exists} F(x_0)$, then $T(F(x)) \leq 0$.

Proof. (a)⇒(b). By Lemma 4.1.3, we have $\mathrm{E}(x_0, \preccurlyeq^{\exists\exists}) = \mathrm{Lev}(x_0, \preccurlyeq^{\exists\exists})$. So, if $x \in S \setminus \mathrm{E}(x_0, \preccurlyeq^{\exists\exists})$, then $x \notin \mathrm{Lev}(x_0, \preccurlyeq^{\exists\exists})$, i.e., $F(x) \not\preccurlyeq^{\exists\exists} F(x_0)$, and by Theorem 3.2.1(vi) we have that $\mathbb{D}^{ii}(F(x), F(x_0)) > 0$ because $F(x)$ is K-compact and $F(x_0)$ is $(-K)$-compact since (4.4) holds.

(b)⇒(c). If $x \in S \setminus \mathrm{E}(x_0, \preccurlyeq^{\exists\exists})$, by hypothesis, $\mathbb{D}^{ii}(F(x), F(x_0)) > 0$. If $x \in \mathrm{E}(x_0, \preccurlyeq^{\exists\exists})$, by Lemma 4.1.14 it follows that $\mathbb{D}^{ii}(F(x_0), F(x)) \leq 0$.

(c)⇒(d). Choose $T : \mathcal{F} \to \mathbb{R}$ given by $T(F(x)) = \mathbb{D}^{ii}(F(x), F(x_0))$. We have that $T(F(x)) \in \mathbb{R}$ by Theorem 3.1.13(vi) because $F(x)$ is K-proper and $F(x_0)$ is $(-K)$-bounded since (4.4) holds.

Let us see that $\mathrm{E}(x_0, \preccurlyeq^{\exists\exists}) = \mathrm{Lev}(x_0, \preccurlyeq^{\exists\exists})$. By contradiction, suppose that there exists $x \in \mathrm{Lev}(x_0, \preccurlyeq^{\exists\exists}) \setminus \mathrm{E}(x_0, \preccurlyeq^{\exists\exists})$. Then $F(x) \preccurlyeq^{\exists\exists} F(x_0)$ and $F(x_0) \not\preccurlyeq^{\exists\exists} F(x)$. By Theorem 3.2.1(vi) applied to both inequalities since (4.4) holds, we obtain $\mathbb{D}^{ii}(F(x), F(x_0)) \leq 0$ and $\mathbb{D}^{ii}(F(x_0), F(x)) > 0$, which contradicts the hypothesis (c). So, $\mathrm{E}(x_0, \preccurlyeq^{\exists\exists}) = \mathrm{Lev}(x_0, \preccurlyeq^{\exists\exists})$.

Therefore, if $x \in S$ and $F(x) \preccurlyeq^{\exists\exists} F(x_0)$, then $x \in \mathrm{E}(x_0, \preccurlyeq^{\exists\exists})$, and by Lemma 4.1.14 it follows that $T(F(x)) = \mathbb{D}^{ii}(F(x), F(x_0)) \leq 0$. Thus, ($d_2$) is proved.

If $x \in S \backslash \mathrm{E}(x_0, \preccurlyeq^{\exists\exists})$, then $F(x) \not\preccurlyeq^{\exists\exists} F(x_0)$, and by Theorem 3.2.1(vi) it follows that $T(F(x)) = \mathbb{D}^{ii}(F(x), F(x_0)) > 0$ since (4.4) holds.

(d)$\Rightarrow$(a). By contradiction, assume that x_0 is not a $\preccurlyeq^{\exists\exists}$-minimal solution of ($\preccurlyeq^{\exists\exists}$-SOP). Then there exists $x_1 \in S$ such that $F(x_1) \preccurlyeq^{\exists\exists} F(x_0)$ and $F(x_0) \not\preccurlyeq^{\exists\exists} F(x_1)$. From (d_1), it follows that $T(F(x_1)) > 0$ since $x_1 \notin \mathrm{E}(x_0, \preccurlyeq^{\exists\exists})$, and by (d_2), we have $T(F(x_1)) \leq 0$, which is a contradiction. The proof is finished. □

Note that the only differences w.r.t. Theorem 4.1.7 are that it is not required that T is increasing until x_0 and in (d_2) we conclude $T(F(x)) \leq 0$. Other differences appear in Remarks 4.1.16, 4.1.17 and 4.1.18.

Note also that T may be defined on $\mathcal{P}_{0,K}(Y)$ and is finite.

In the following remarks, we provide other equivalent expressions for statements (b), (c) and (d) in Theorem 4.1.15.

Remark 4.1.16. The following statements are equivalent to (c):

(c') It does not exist $x \in S$ such that

$$\mathbb{D}^{ii}(F(x), F(x_0)) \leq 0 \quad \text{and} \quad \mathbb{D}^{ii}(F(x_0), F(x)) > 0.$$

(c'') For each $x \in S$, exactly one of the following statements is true:

(c''_1) $\mathbb{D}^{ii}(F(x), F(x_0)) > 0$,

(c''_2) $\mathbb{D}^{ii}(F(x), F(x_0)) \leq 0$ and $\mathbb{D}^{ii}(F(x_0), F(x)) \leq 0$.

Proof. $(c) \Leftrightarrow (c')$. It is clear since *not* (c') is just *not* (c).

$(c'') \Rightarrow (c)$. It is also clear, and $(c) \Rightarrow (c'')$ because $(c) \Rightarrow (b)$ and $(b) \Rightarrow (c'')$ (see the proof of $(b) \Rightarrow (c)$ in Theorem 4.1.15 where Lemma 4.1.14 is applied).

Remark 4.1.17. Let $T : \mathcal{F} \to \mathbb{R}$ and consider the following statements:

(d') (d'_1) $T(F(x)) > 0$ for all $x \in S \setminus \mathrm{E}(x_0, \preccurlyeq^{\exists\exists})$,

(d'_2) if $x \in \mathrm{E}(x_0, \preccurlyeq^{\exists\exists})$, then $T(F(x)) \leq 0$.

(d'') If $x \in S$ then, $x \in \mathrm{E}(x_0, \preccurlyeq^{\exists\exists}) \Leftrightarrow T(F(x)) \leq 0$.

Then

(i) $(d) \Rightarrow (d') \Leftrightarrow (d'')$.

(ii) If T is $\preccurlyeq^{\exists\exists}$-increasing until x_0, then $(d') \Rightarrow (d)$, and so the three statements (d), (d') and (d'') are equivalent.

Proof. (i) $(d) \Rightarrow (d')$. In Theorem 4.1.15, without using any assumption, it has been proved that $(d) \Rightarrow (a)$, and so by Lemma 4.1.3 we have $\mathrm{E}(x_0, \preccurlyeq^{\exists\exists}) = \mathrm{Lev}(x_0, \preccurlyeq^{\exists\exists})$. Now it is clear that $(d) \Rightarrow (d')$.

$(d') \Leftrightarrow (d'')$. We only have to prove the '$\Leftarrow$' part of (d'') since the other implications are clear. Let $T(F(x)) \leq 0$. If $x \notin \mathrm{E}(x_0, \preccurlyeq^{\exists\exists})$, then by (d'_1), $T(F(x)) > 0$, a contradiction, and consequently $x \in \mathrm{E}(x_0, \preccurlyeq^{\exists\exists})$.

(ii) $(d'') \Rightarrow (d)$. We only have to prove (d_2) since $(d'') \Leftrightarrow (d')$ and $(d_1) \equiv (d'_1)$. Suppose that $F(x) \preccurlyeq^{\exists\exists} F(x_0)$. As T is $\preccurlyeq^{\exists\exists}$-increasing until x_0, we deduce that $T(F(x)) \leq T(F(x_0))$. Now, as $x_0 \in \mathrm{E}(x_0, \preccurlyeq^{\exists\exists})$ since $\preccurlyeq^{\exists\exists}$ is reflexive, by (d'') we derive that $T(F(x_0)) \leq 0$, and therefore, $T(F(x)) \leq T(F(x_0)) \leq 0$.

We point out that statement (d'') is equivalent to say that $\mathrm{E}(x_0, \preccurlyeq^{\exists\exists})$ is exactly the 0-sublevel set of $T \circ F$.

Remark 4.1.18. The following statements are equivalent to (b):

(b') (b'_1) $\mathbb{D}^{ii}(F(x), F(x_0)) > 0$ for all $x \in S \setminus \mathrm{E}(x_0, \preccurlyeq^{\exists\exists})$,

(b'_2) if $x \in \mathrm{E}(x_0, \preccurlyeq^{\exists\exists})$, then $\mathbb{D}^{ii}(F(x), F(x_0)) \leq 0$.

(b'') If $x \in S$ then, $x \in \mathrm{E}(x_0, \preccurlyeq^{\exists\exists}) \Leftrightarrow \mathbb{D}^{ii}(F(x), F(x_0)) \leq 0$.

Its proof is identical to the one of Remark 4.1.17 only by changing $T(F(x))$ for $\mathbb{D}^{ii}(F(x), F(x_0))$. But to prove the implication $(b) \Rightarrow (b')$ we use Lemma 4.1.14. In this case, $(b') \Rightarrow (b)$ is obvious.

We also point out that statement (b'') is equivalent to say that $\mathrm{E}(x_0, \preccurlyeq^{\exists\exists})$ is exactly the 0-sublevel set of $\mathbb{D}^{ii}(F(\cdot), F(x_0))$.

To illustrate our results we provide an example.

Example 4.1.19. Consider $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, and $F : \mathbb{N} \rightrightarrows \mathbb{R}^2$ given by

$$F(x) = [(x, -x), (x + 1, -x - 1/2)],$$

where $[u, v]$ denotes the interval of $\mathbb{R}^2$ of extremes u and v. We suppose that $S = \mathbb{N}$. After some calculations we arrive to the following results:

1. $$\mathbb{D}^{ss}(F(a), F(b)) = \begin{cases} b - a + 1/2 & \text{if } a < b \\ a - b + 1 & \text{if } a > b \\ 1 & \text{if } a = b \end{cases}$$

2. $\mathbb{D}^{is}(F(a), F(b)) = \widehat{\mathbb{D}}^{is}(F(a), F(b)) = \begin{cases} |a-b| & \text{if } a \neq b \\ 1/3 & \text{if } a = b \end{cases}$

3. $\mathbb{D}^{si}(F(a), F(b)) = \widehat{\mathbb{D}}^{si}(F(a), F(b)) = |a-b|$.

4. $\mathbb{D}^{ii}(F(a), F(b)) = \begin{cases} b-a-1/2 & \text{if } a < b \\ a-b-1 & \text{if } a > b \\ 0 & \text{if } a = b \end{cases}$

Let us note that F is K-compact valued and $(-K)$-compact valued. Now it is easy to determine all minimal solutions for the six problems SOP.

Discussion:

1. ($\preccurlyeq^{\forall\forall}$-SOP). For each $b \in S$, we have $\mathbb{D}^{ss}(F(a), F(b)) > 0$ for all $a \in S$. So, statement (c) of Theorem 4.1.7 is fulfilled, and therefore, every $b \in S$ is a $\preccurlyeq^{\forall\forall}$-minimal solution.

2. ($\preccurlyeq^{\exists\forall}$-SOP) and ($\precsim^{\exists\forall}$-SOP). These problems are exactly as the previous case using, respectively, $\mathbb{D}^{is}$ and $\widehat{\mathbb{D}}^{is}$.

3. ($\preccurlyeq^{\forall\exists}$-SOP). For each $b \in S$, we have $\mathbb{D}^{si}(F(a), F(b)) > 0$ for all $a \in S \setminus \{b\}$ and for $a = b$, $\mathbb{D}^{si}(F(b), F(a)) = 0$. So, (c) holds and, by applying Theorem 4.1.7, we conclude that every $b \in S$ is a $\preccurlyeq^{\forall\exists}$-minimal solution.

4. ($\precsim^{\forall\exists}$-SOP). This problem is as the previous case.

5. ($\preccurlyeq^{\exists\exists}$-SOP). For each $b \in \mathbb{N}$, $\mathrm{E}(b, \preccurlyeq^{\exists\exists}) = \{b\}$ and we have $\mathbb{D}^{ii}(F(a), F(b)) > 0$ for all $a \neq b$ except if $a - b - 1 = 0$ with $a > b$, which is true if and only if $a = b + 1$. Therefore, statement (b) of Theorem 4.1.15 is not satisfied and, accordingly, each $b \in S$ is not a $\preccurlyeq^{\exists\exists}$-minimal solution. Instead, if $S = \{x \in \mathbb{N} :\ x \text{ is even}\}$, then every point of S is a $\preccurlyeq^{\exists\exists}$-minimal solution.

4.2 Characterization by scalarization of weak minimal solutions

In this section, the strict monotonicity of the six set oriented distances (see Theorem 3.3.8) and the characterization of strict set relations of Kuroiwa (see Theorem 3.4.1) are applied to derive several characterizations of weak minimal solution to a set optimization problem with a set-valued map as objective where the images are compared with one of the set relations belonging to $\mathcal{R}_s = \{\preccurlyeq_s^{\forall\forall}, \preccurlyeq_s^{\exists\forall}, \curlyeqprec_s^{\forall\exists}, \curlyeqprec_s^{\exists\forall}, \preccurlyeq_s^{\forall\exists}, \preccurlyeq_s^{\exists\exists}\}$ (we use the set criterion of solution). Moreover, $K \subset Y$ is a proper closed convex solid cone (we do not assume that K is pointed).

First of all, we are going to introduce some needed definitions wich will be used along the section. We star with the definition of weak $\precsim$-minimal solution.

Definition 4.2.1. Let $x_0 \in S$.

(a) It is said that x_0 is a $\precsim$-minimal (resp., weak $\precsim$-minimal) solution to ($\precsim$-SOP) if $F(x) \precsim F(x_0)$ (resp., $F(x) \precsim_s F(x_0)$) for some $x \in S$, implies $F(x_0) \precsim F(x)$ (resp., $F(x_0) \precsim_s F(x)$).

(b) It is said that x_0 is a strict weak $\precsim$-minimal solution to ($\precsim$-SOP) if $F(x) \not\precsim F(x_0)$ for all $x \in S$.

The notion of weak $\precsim$-minimal solution is very common and the notion of strict weak $\precsim$-minimal solution has been considered, for example, in [91, Definition 3.1(b)].

It is clear that each strict weak $\precsim$-minimal solution is also a weak $\precsim$-minimal solution. The relationship between $\precsim$-minimal solution and weak $\precsim$-minimal solution is given in the next proposition.

Proposition 4.2.2. *Consider problem ($\precsim$-SOP) and let $x_0 \in S$.*

(i) For $\precsim \in \{\preccurlyeq^{\forall\forall}, \preccurlyeq^{\exists\forall}, \curlyeqprec^{\exists\forall}\}$, one has that if x_0 is a $\precsim$-minimal solution to ($\precsim$-SOP), then x_0 is a strict weak $\precsim$-minimal solution to ($\precsim$-SOP).

(ii) If x_0 is a $\curlyeqprec^{\forall\exists}$-minimal (resp., $\preccurlyeq^{\forall\exists}$-minimal) solution to ($\precsim$-SOP), then x_0 is a weak $\curlyeqprec^{\forall\exists}$-minimal (resp., $\preccurlyeq^{\forall\exists}$-minimal) solution to ($\precsim$-SOP).

Proof. (i) 1. For $\preccurlyeq^{\exists\forall}$. Suppose, to the contrary, that x_0 is not a strict weak $\preccurlyeq^{\exists\forall}$-minimal solution to ($\preccurlyeq^{\exists\forall}$-SOP). So, there exists $x_1 \in S$ such that $F(x_1) \preccurlyeq_s^{\exists\forall} F(x_0)$. So, by Definition 1.3.3 there exists $y_1 \in F(x_1)$ satisfying

$$y_1 - F(x_0) \subset -\operatorname{int} K. \tag{4.5}$$

Moreover, by Proposition 1.3.7(iv), we have $F(x_1) \preccurlyeq^{\exists\forall} F(x_0)$, and as by hypothesis x_0 is a $\preccurlyeq^{\exists\forall}$-minimal solution to ($\preccurlyeq^{\exists\forall}$-SOP), it follows that $F(x_0) \preccurlyeq^{\exists\forall} F(x_1)$. So, by Definition 1.3.2 there exists $y_0 \in F(x_0)$ such that $y_0 - F(x_1) \subset -K$. From this last inclusion and (4.5) we deduce $y_1 - y_0 \in (-\operatorname{int} K) \cap K$. But this leads to $0 \in \operatorname{int} K$ and, as K is a cone, we derive that $K = Y$, which is a contradiction since K is proper.

2. For $\preccurlyeq^{\exists\forall}$ and $\preccurlyeq^{\forall\forall}$. The proof is similar, and it is omitted.

(ii) It is Proposition 2.7(i) in [51]. □

Remark 4.2.3. Proposition 4.2.2 is not true for $\preccurlyeq^{\exists\exists}$ as the following data show: $S = \{0, 1\}$, $Y = \mathbb{R}$, $K = \mathbb{R}_+$, $F(0) = [0, 1]$ and $F(1) = \{1\}$. Then, one has that $x_0 = 1$ is a $\preccurlyeq^{\exists\exists}$-minimal solution but is not a weak $\preccurlyeq^{\exists\exists}$-minimal solution.

In the former section, we have provided several characterizations for $\precsim$-minimal solutions of ($\precsim$-SOP). Now, we focus on weak $\precsim$-minimal solutions.

Given $A \in \mathcal{P}_0(Y)$, $x_0 \in S$ and $\precsim \in \mathcal{R} \cup \mathcal{R}_s$, we denote:

$[A, \precsim] = \{B \in \mathcal{P}_0(Y) : A \precsim B \text{ and } B \precsim A\}$,

$\mathrm{E}(x_0, \precsim) = \{x \in S : F(x) \in [F(x_0), \precsim]\}$.

It is obvious that $\mathrm{E}(x_0, \precsim_s) \subset \mathrm{Lev}(x_0, \precsim_s)$. The inverse inclusion is also true for a weak $\precsim$-minimal solution. The proof is immediate.

Lemma 4.2.4. *Let $\precsim \in \mathcal{R}$. The point $x_0 \in S$ is a weak $\precsim$-minimal solution of ($\precsim$-SOP) if and only if* $\mathrm{E}(x_0, \precsim_s) = \mathrm{Lev}(x_0, \precsim_s)$.

In the following lemmas we state some basics properties of the sets $[B, \precsim]$.

Lemma 4.2.5. *Let $B \in \mathcal{P}_0(Y)$. Then, $[B, \precsim_s] \subset [B, \precsim]$ for all $\precsim \in \mathcal{R}$.*

Proof. It is obvious because $A \precsim_s B$ implies $A \precsim B$, for all $\precsim \in \mathcal{R}$ by Proposition 1.3.7(iv). □

Next, we are going to recall the Lemma 4.1.6 about $\precsim$-equivalent sets but by using the new notation.

Lemma 4.2.6. *(Equivalence Lemma) Let $A, B \in \mathcal{P}_0(Y)$.*

(i) $A \in [B, \preccurlyeq^{\forall\forall}]$ if and only if $\mathbb{D}^{ss}(A, B) = 0$ and $\mathbb{D}^{ss}(B, A) = 0$.

(ii) If $A \in [B, \preccurlyeq^{\exists\forall}]$, then $\widehat{\mathbb{D}}^{is}(A, B) = 0$ and $\widehat{\mathbb{D}}^{is}(B, A) = 0$.

(iii) Let B be K-proper. If $A \in [B, \curlyeqprec^{\forall\exists}]$, then $\mathbb{D}^{si}(A, B) = 0$ and $\mathbb{D}^{si}(B, A) =$ 0.

(iv) If $A \in [B, \curlyeqprec^{\exists\forall}]$, then $\mathbb{D}^{is}(A, B) = 0$ and $\mathbb{D}^{is}(B, A) = 0$.

(v) Let B be $(-K)$-proper. If $A \in [B, \preccurlyeq^{\forall\exists}]$, then $\widehat{\mathbb{D}}^{si}(A, B) = 0$ and $\widehat{\mathbb{D}}^{si}(B, A) =$ 0.

Next, we are going to calculate $[A, \precsim_s]$ for the first five strict set order relations.

Lemma 4.2.7. *Let $B \in \mathcal{P}_0(Y)$. It holds that:*

(i) $[B, \preccurlyeq_s^{\forall\forall}] = \emptyset$.

(ii) $[B, \preccurlyeq_s^{\exists\forall}] = \emptyset$.

(iii) If B is K-compact, then $[B, \curlyeqprec_s^{\forall\exists}] = \emptyset$.

(iv) $[B, \curlyeqprec_s^{\exists\forall}] = \emptyset$.

(v) If B is $(-K)$-compact, then $[B, \preccurlyeq_s^{\forall\exists}] = \emptyset$.

Proof. (ii) By contradiction, assume that there exists a nonempty set $A \in [B, \preccurlyeq_s^{\exists\forall}]$. Then, $A \preccurlyeq_s^{\exists\forall} B$ and $B \preccurlyeq_s^{\exists\forall} A$. By definition, there exist $a_0 \in A$ and $b_0 \in B$ such that $a_0 - B \subset -\operatorname{int} K$ and $b_0 - A \subset -\operatorname{int} K$. Therefore, $a_0 - b_0 \in (-\operatorname{int} K) \cap \operatorname{int} K$, which leads to $K = Y$, a contradiction since K is proper.

(i) It follows from part (ii) since $[B, \preccurlyeq_s^{\forall\forall}] \subset [B, \preccurlyeq_s^{\exists\forall}]$ by Proposition 1.3.7(iii).

(iii) Assume, to the contrary, that there exists a nonempty set $A \in [B, \curlyeqprec_s^{\forall\exists}]$. Then, we have that $A \curlyeqprec_s^{\forall\exists} B$ and by Theorem 3.4.1(iii) we have $\mathbb{D}^{si}(A, B) < 0$ since B is K-compact. Moreover, from Lemma 4.2.5 it follows that $[B, \curlyeqprec_s^{\forall\exists}] \subset [B, \curlyeqprec^{\forall\exists}]$, and so $A \in [B, \curlyeqprec^{\forall\exists}]$. By Lemma 4.1.6(iii), we derive that $\mathbb{D}^{si}(A, B) = 0$, which is a contradiction.

(iv) It follows from (ii) since $[B, \curlyeqprec_{s,K}^{\exists\forall}] = [B, \preccurlyeq_{s,-K}^{\exists\forall}]$ by Lemma 1.3.9(iv).

(v) It follows from (iii) since $[B, \preccurlyeq_{s,K}^{\forall\exists}] = [B, \curlyeqprec_{s,-K}^{\forall\exists}]$ by Lemma 1.3.9(iii). $\square$

In the next theorem we establish necessary and sufficient conditions of weak minimality in several forms for the five first relations of $\mathcal{R}$ and in Theorem 4.2.12 for the sixth one.

Theorem 4.2.8. *For each set relation* $\precsim\ \in \{\preccurlyeq^{\forall\forall}, \preccurlyeq^{\exists\forall}, \preccurlyeq^{\forall\exists}, \preccurlyeq^{\exists\forall}, \preccurlyeq^{\forall\exists}\}$, *consider problem* ($\precsim$*-SOP), a point* $x_0 \in S$ *and suppose that for each pair* $(\precsim, \bar{\mathbb{D}})$, F *satisfies the assumptions of Table 4.4.*

$\precsim$	$\bar{\mathbb{D}}$	*Assumptions*
$\preccurlyeq^{\forall\forall}$	$\mathbb{D}^{ss}$	F *is* $(-K)$*-compact valued and* $F(x_0)$ *is* K*-compact*
$\preccurlyeq^{\exists\forall}$	$\widehat{\mathbb{D}}^{is}$	F *is* K*-proper valued and* $F(x_0)$ *is* K*-compact*
$\preccurlyeq^{\forall\exists}$	$\mathbb{D}^{si}$	F *is* K*-proper valued and* $F(x_0)$ *is* K*-compact*
$\preccurlyeq^{\exists\forall}$	$\mathbb{D}^{is}$	F *is* $(-K)$*-compact valued*
$\preccurlyeq^{\forall\exists}$	$\widehat{\mathbb{D}}^{si}$	F *is* $(-K)$*-compact valued*

Table 4.4. Assumptions in Theorem 4.2.8.

Then, for each pair $(\precsim, \bar{\mathbb{D}})$ *in Table 4.4 the following statements are equivalent:*

(a) x_0 *is a weak* $\precsim$*-minimal solution of (*$\precsim$*-SOP).*

(b) x_0 *is a strict weak* $\precsim$*-minimal solution of (*$\precsim$*-SOP).*

(c) $\bar{\mathbb{D}}(F(x), F(x_0)) \geq 0$ *for all* $x \in S$.

(d) *There exists a map* $T : \mathcal{F} \to \mathbb{R}$, *which is* $\precsim$*-increasing on* $\mathcal{F}$, *strictly* $\precsim_s$*-increasing on* K_1*-compact sets and such that*

(d_1) $T(F(x)) \geq 0$ *for all* $x \in S$,

(d_2) *if* $x \in S$ *and* $F(x) \precsim_s F(x_0)$, *then* $T(F(x)) < 0$,

where $K_1 = -K$ *for* $\preccurlyeq^{\forall\forall}$, $\preccurlyeq^{\exists\forall}$ *and* $\preccurlyeq^{\forall\exists}$, *and* $K_1 = K$ *for* $\preccurlyeq^{\exists\forall}$ *and* $\preccurlyeq^{\forall\exists}$.

(e) *There exists a map* $T : \mathcal{F} \to \mathbb{R}$ *such that* (d_1) *and* (d_2) *are satisfied.*

Proof. We prove the case $(\precsim, \bar{\mathbb{D}}) = \left(\preccurlyeq^{\forall\forall}, \mathbb{D}^{ss}\right)$, since the remaining cases are similar, by applying the results of Table 4.4.

Case $(\precsim, \bar{\mathbb{D}}) = \left(\preccurlyeq^{\forall\forall}, \mathbb{D}^{ss}\right)$.

$(a) \Rightarrow (b)$. By contradiction, suppose that there exists $x_1 \in S$ such that $F(x_1) \preccurlyeq_s^{\forall\forall} F(x_0)$, that is, $x_1 \in \mathrm{Lev}(x_0, \preccurlyeq_s^{\forall\forall})$. As x_0 is a weak $\preccurlyeq^{\forall\forall}$-minimal solution, by applying Lemma 4.2.4 we have $\mathrm{Lev}(x_0, \preccurlyeq_s^{\forall\forall}) = \mathrm{E}(x_0, \preccurlyeq_s^{\forall\forall})$ and, therefore,

$x_1 \in \mathrm{E}(x_0, \preccurlyeq_s^{\forall\forall})$. However, this contradicts the fact that $\mathrm{E}(x_0, \preccurlyeq_s^{\forall\forall}) = \emptyset$, which is true since by Lemma 4.2.7(i) one has $[F(x_0), \preccurlyeq_s^{\forall\forall}] = \emptyset$.

$(b) \Rightarrow (c)$. The hypothesis (b) means that $F(x) \not\preccurlyeq_s^{\forall\forall} F(x_0)$ for all $x \in S$, and this implies that $\mathbb{D}^{ss}(F(x), F(x_0)) \geq 0$ for all $x \in S$ by Theorem 3.4.1(i), so (c) holds.

$(c) \Rightarrow (d)$. We define the map $T : \mathcal{F} \to \mathbb{R}$ given by $T(F(x)) = \mathbb{D}^{ss}(F(x), F(x_0))$. Firstly, $T(F(x)) \in \mathbb{R}$ by Theorem 3.1.13(i) because $F(x)$ is $(-K)$-bounded and $F(x_0)$ is K-bounded since F is $(-K)$-compact valued and $F(x_0)$ is K-compact by hypothesis (see Table 4.4). Moreover, T is $\preccurlyeq^{\forall\forall}$-increasing on $\mathcal{F}$ by Theorem 3.1.22(iii) and strictly $\preccurlyeq_s^{\forall\forall}$-increasing on $(-K)$-compact sets by Theorem 3.3.8(i-b). Secondly, (d_1) holds by assumption (c). And thirdly, let us prove (d_2). Suppose that $x \in S$ and $F(x) \preccurlyeq_s^{\forall\forall} F(x_0)$. We can apply Theorem 3.4.1(i) because $F(x)$ is $(-K)$-compact and $F(x_0)$ is K-compact by hypothesis. So, we conclude $T(F(x)) = \mathbb{D}^{ss}(F(x), F(x_0)) < 0$.

$(d) \Rightarrow (e)$. It is obvious.

$(e) \Rightarrow (a)$. By contradiction, assume that x_0 is not a weak $\preccurlyeq^{\forall\forall}$-minimal solution. Then, there exists $x_1 \in S$ such that $F(x_1) \preccurlyeq_s^{\forall\forall} F(x_0)$ and $F(x_0) \not\preccurlyeq_s^{\forall\forall} F(x_1)$. From (d_1), it follows that $T(F(x_1)) \geq 0$, and by (d_2), we have $T(F(x_1)) < 0$, which is a contradiction. The proof is finished. □

Next, we summarize the results needed in the proof of Theorem 4.2.8.

$\precsim$	$\bar{\mathbb{D}}$	Ch. $A \precsim_s B$	$\mathrm{E}(x_0, \precsim_s) = \emptyset$	Finite	$\precsim$-incr./str. $\precsim_s$-incr.
$\preccurlyeq^{\forall\forall}$	$\mathbb{D}^{ss}$	Th. 3.4.1(i)	Lem. 4.2.7(i)	Th. 3.1.13(i)	Ths. 3.1.22(iii)/3.3.8(i-b)
$\preccurlyeq^{\exists\forall}$	$\widehat{\mathbb{D}}^{is}$	Th. 3.4.1(ii)	Lem. 4.2.7(ii)	Th. 3.1.13(ii)	Ths. 3.1.22(iv)/3.3.8(ii-b)
$\preccurlyeq^{\forall\exists}$	$\mathbb{D}^{si}$	Th. 3.4.1(iii)	Lem. 4.2.7(iii)	Th. 3.1.13(iii)	Ths. 3.1.22(iv)/3.3.8(iii-b)
$\preccurlyeq^{\exists\forall}$	$\mathbb{D}^{is}$	Th. 3.4.1(iv)	Lem. 4.2.7(iv)	Th. 3.1.13(iv)	Ths. 3.1.22(iii)/3.3.8(iv-b)
$\preccurlyeq^{\forall\exists}$	$\widehat{\mathbb{D}}^{si}$	Th. 3.4.1(v)	Lem. 4.2.7(v)	Th. 3.1.13(v)	Ths. 3.1.22(iii)/3.3.8(v-b)

Table 4.5. Results that are applied in the proof of Theorem 4.2.8.

Let us observe that the map T in statement (d) has the same properties as $\bar{\mathbb{D}}(\cdot, F(x_0))$. To obtain necessary minimality conditions it is better to have a lot of properties; however, viewed (d) as a sufficient condition for (a) it is easier to find a map T satisfying statement (e).

Remark 4.2.9. Statement (b) is equivalent to

(b') $F(x) \not\precsim_s F(x_0)$ for all $x \in S\backslash\{x_0\}$.

Indeed, under the assumptions of Theorem 3.3.8 one has $F(x_0) \not\precsim_s F(x_0)$, since otherwise $F(x_0) \in [F(x_0), \precsim_s]$, which contradicts Lemma 4.2.7.

Remark 4.2.10. If it holds

$$F(x_0) \precsim F(x_0), \tag{4.6}$$

then statement (c) is equivalent to:

(c') x_0 is a solution of the scalar problem $\min\{\bar{\mathbb{D}}(F(x), F(x_0)) : x \in S\}$.

Indeed, $\bar{\mathbb{D}}(F(x_0), F(x_0)) = 0$ by Lemma 4.1.6(i)-(v) (let us observe that under the assumptions of Theorem 4.2.8, we can apply Lemma 4.1.6(iii) and (v) since $F(x_0)$ is K-proper and $(-K)$-proper, respectively). Note that condition (4.6) is satisfied for all $x_0 \in S$ for the reflexive relations $\preccurlyeq^{\forall\exists}$ and $\curlyeqprec^{\forall\exists}$.

Results about weak minimal solutions of a set optimization problem for $\preccurlyeq_s^{\forall\exists}$ or $\curlyeqprec_s^{\forall\exists}$, using extensions of Gerstewitz's function, have been provided in Hernández and Rodríguez-Marín [51, Theorem 4.2 and Corollary 4.11] (in the forms $(a) \Leftrightarrow (d)$ and $(a) \Leftrightarrow (c')$), Araya [5, Theorems 5.2 and 5.4] (in the form $(a) \Leftrightarrow (c')$), Gutiérrez et al. [41, Corollary 4.4(a)] (in the form $(a) \Leftrightarrow (c')$) and Khoshkhabar-amiranloo et al. [73, Theorems 3.2 and 4.2] (in the form $(a) \Leftrightarrow (c')$). By using a scalarization based on the oriented distance, Xu and Li [129, Theorems 4.6 and 4.8] (in the forms $(a) \Leftrightarrow (d)$ and $(a) \Leftrightarrow (c')$), and using the functions $\mathbb{D}^{si}$ and $\widehat{\mathbb{D}}^{si}$, Jiménez et al. [67, Theorems 5.7 and 5.15] (in the forms $(a) \Leftrightarrow (b) \Leftrightarrow (c')$). A result as Lemma 4.2.7 is not given in none of these papers.

The case $(\precsim, \bar{\mathbb{D}}) = \left(\curlyeqprec^{\exists\exists}, \mathbb{D}^{ii}\right)$ has not been approached in Theorem 4.2.8 because it presents some peculiarities as, for example, Lemma 4.2.7 is not true for $\curlyeqprec_s^{\exists\exists}$, so we prefer to deal with it as a separate case.

The following result is a direct consequence of Theorem 3.4.1(vi)

Lemma 4.2.11. *Let $A, B \in \mathcal{P}_0(Y)$. Then, $A \in [B, \curlyeqprec_s^{\exists\exists}]$ if and only if $\mathbb{D}^{ii}(A, B) < 0$ and $\mathbb{D}^{ii}(B, A) < 0$.*

Theorem 4.2.12. *Consider problem ($\preccurlyeq^{\exists\exists}$-SOP), a point $x_0 \in S$ and suppose that the following assumption holds:*

$$(A) \begin{cases} \text{Either } F \text{ is } K\text{-proper valued and } F(x_0) \text{ is } (-K)\text{-bounded, or} \\ F \text{ is } K\text{-bounded valued and } F(x_0) \text{ is } (-K)\text{-proper.} \end{cases}$$

Then, the following statements are equivalent:

(a) *x_0 is a weak $\preccurlyeq^{\exists\exists}$-minimal solution of ($\preccurlyeq^{\exists\exists}$-SOP).*

(b) *$\mathbb{D}^{ii}(F(x), F(x_0)) \geq 0$ for all $x \in S \setminus \mathrm{E}(x_0, \preccurlyeq_s^{\exists\exists})$.*

(c) *For all $x \in S$ one has $\mathbb{D}^{ii}(F(x), F(x_0)) \geq 0$ or $\mathbb{D}^{ii}(F(x_0), F(x)) < 0$.*

(d) *There exists a map $T : \mathcal{F} \to \mathbb{R}$ such that*

(d_1) *$T(F(x)) \geq 0$ for all $x \in S \setminus \mathrm{E}(x_0, \preccurlyeq_s^{\exists\exists})$,*

(d_2) *if $x \in S$ and $F(x) \preccurlyeq_s^{\exists\exists} F(x_0)$, then $T(F(x)) < 0$.*

Proof. (a) $\Rightarrow$ (b). By Lemma 4.2.4, we obtain $\mathrm{E}(x_0, \preccurlyeq_s^{\exists\exists}) = \mathrm{Lev}(x_0, \preccurlyeq_s^{\exists\exists})$. So, if $x \in S \setminus \mathrm{E}(x_0, \preccurlyeq_s^{\exists\exists})$, then $x \notin \mathrm{Lev}(x_0, \preccurlyeq_s^{\exists\exists})$, that is, $F(x) \not\preccurlyeq_s^{\exists\exists} F(x_0)$, and by Theorem 3.4.1(vi) we derive that $\mathbb{D}^{ii}(F(x), F(x_0)) \geq 0$.

(b) $\Rightarrow$ (c). For each $x \in S$ one has $x \notin \mathrm{E}(x_0, \preccurlyeq_s^{\exists\exists})$ or $x \in \mathrm{E}(x_0, \preccurlyeq_s^{\exists\exists})$. In the first case, by (b) we deduce $\mathbb{D}^{ii}(F(x), F(x_0)) \geq 0$, and in the second case, by Lemma 4.2.11 we get $\mathbb{D}^{ii}(F(x_0), F(x)) < 0$.

(c) $\Rightarrow$ (a). By contradiction, suppose that x_0 is not a weak $\preccurlyeq^{\exists\exists}$-minimal solution of ($\preccurlyeq^{\exists\exists}$-SOP). Then, there exists $x_1 \in S$ such that $F(x_1) \preccurlyeq_s^{\exists\exists} F(x_0)$ and $F(x_0) \not\preccurlyeq_s^{\exists\exists} F(x_1)$. By Theorem 3.4.1(vi) applied to both inequalities we obtain $\mathbb{D}^{ii}(F(x_1), F(x_0)) < 0$ and $\mathbb{D}^{ii}(F(x_0), F(x_1)) \geq 0$, which contradicts the hypothesis (c). Therefore, statements (a), (b) and (c) are equivalent.

(b) $\Rightarrow$ (d). We define the mapping $T : \mathcal{F} \to \mathbb{R}$ given by $T(F(x)) = \mathbb{D}^{ii}(F(x), F(x_0))$. We have $T(F(x)) \in \mathbb{R}$ by Theorem 3.1.13(vi) since assumption (A) holds.

If $x \in S$ and $F(x) \preccurlyeq_s^{\exists\exists} F(x_0)$, then by Theorem 3.4.1(vi) we deduce $T(F(x)) = \mathbb{D}^{ii}(F(x), F(x_0)) < 0$. Thus, ($d_2$) holds.

If $x \in S \setminus \mathrm{E}(x_0, \preccurlyeq_s^{\exists\exists})$, then $T(F(x)) = \mathbb{D}^{ii}(F(x), F(x_0)) \geq 0$ by hypothesis (b), and so, (d_1) is proved.

(d) $\Rightarrow$ (a). By contradiction, suppose that x_0 is not a weak $\preccurlyeq^{\exists\exists}$-minimal solution of ($\preccurlyeq^{\exists\exists}$-SOP). Then, there exists $x_1 \in S$ such that $F(x_1) \preccurlyeq_s^{\exists\exists} F(x_0)$ and

$F(x_0) \not\preccurlyeq_s^{\exists\exists} F(x_1)$. From (d_1), it follows that $T(F(x_1)) \geq 0$ since $x_1 \notin \mathrm{E}(x_0, \preccurlyeq_s^{\exists\exists})$, and by (d_2), we have $T(F(x_1)) < 0$, which is a contradiction. □

In the following remarks, we provide other equivalent expressions for statements (b), (c) and (d) in Theorem 4.2.12.

Remark 4.2.13. The following statement is equivalent to (c):

(c') It does not exist $x \in S$ such that

$$\mathbb{D}^{ii}(F(x), F(x_0)) < 0 \quad \text{and} \quad \mathbb{D}^{ii}(F(x_0), F(x)) \geq 0.$$

It is clear since *not* (c') is just *not* (c).

Remark 4.2.14. Let $T : \mathcal{F} \to \mathbb{R}$ and consider the following statements:

(d') (d_1') $T(F(x)) \geq 0$ for all $x \in S \setminus \mathrm{E}(x_0, \preccurlyeq_s^{\exists\exists})$,

(d_2') if $x \in \mathrm{E}(x_0, \preccurlyeq_s^{\exists\exists})$, then $T(F(x)) < 0$.

(d'') If $x \in S$ then, $x \in \mathrm{E}(x_0, \preccurlyeq^{\exists\exists}) \Leftrightarrow T(F(x)) < 0$.

Then

(i) $(d) \Rightarrow (d') \Leftrightarrow (d'')$.

(ii) If T is $\preccurlyeq^{\exists\exists}$-increasing until x_0 (that is, $F(x) \preccurlyeq^{\exists\exists} F(x_0)$ implies $T(F(x)) \leq T(F(x_0))$) and $x_0 \in \mathrm{E}(x_0, \preccurlyeq_s^{\exists\exists})$, then $(d') \Rightarrow (d)$, and so the three statements (d), (d') and (d'') are equivalent.

Proof. (i) $(d) \Rightarrow (d')$. It is obvious since $\mathrm{E}(x_0, \preccurlyeq_s^{\exists\exists}) \subset \mathrm{Lev}(x_0, \preccurlyeq_s^{\exists\exists})$.

$(d') \Leftrightarrow (d'')$. We only have to prove the '$\Leftarrow$' part of (d''). Let $x \in S$ such that $T(F(x)) < 0$. If $x \notin \mathrm{E}(x_0, \preccurlyeq_s^{\exists\exists})$, then by (d_1'), it follows $T(F(x)) \geq 0$, which is a contradiction and, consequently, $x \in \mathrm{E}(x_0, \preccurlyeq_s^{\exists\exists})$.

(ii) We only have to prove (d_2) since $(d_1) \equiv (d_1')$. If we take $x \in S$ such that $F(x) \preccurlyeq_s^{\exists\exists} F(x_0)$, then $F(x) \preccurlyeq^{\exists\exists} F(x_0)$ by Proposition 1.3.7(iv) and, as T is $\preccurlyeq^{\exists\exists}$-increasing until x_0, we deduce $T(F(x)) \leq T(F(x_0))$. Now, as $x_0 \in \mathrm{E}(x_0, \preccurlyeq_s^{\exists\exists})$ by hypothesis, using (d_2') we derive $T(F(x_0)) < 0$ and, therefore, $T(F(x)) < 0$.

Remark 4.2.15. The following statements are equivalent to (b):

(b') (b_1') $\mathbb{D}^{ii}(F(x), F(x_0)) \geq 0$ for all $x \in S \setminus \mathrm{E}(x_0, \preccurlyeq_s^{\exists\exists})$,

(b_2') if $x \in \mathrm{E}(x_0, \preccurlyeq_s^{\exists\exists})$, then $\mathbb{D}^{ii}(F(x), F(x_0)) < 0$.

(b'') If $x \in S$ then, $x \in \mathrm{E}(x_0, \preccurlyeq_s^{\exists\exists}) \Leftrightarrow \mathbb{D}^{ii}(F(x), F(x_0)) < 0$.

Its proof is identical to the one of Remark 4.2.14 only by changing $T(F(x))$ for $\mathbb{D}^{ii}(F(x), F(x_0))$. But to prove the implication $(b) \Rightarrow (b'_2)$ we use Lemma 4.2.11. In this case, $(b') \Rightarrow (b)$ is obvious.

We illustrate our results with two examples.

Example 4.2.16. Consider $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $S = \mathbb{R}_+$, and $F : \mathbb{R}_+ \rightrightarrows \mathbb{R}^2$ defined by

$$F(x) = [(x, x), (x + r, x + r)]_K,$$

where r is a fixed positive number. Using (3.14), it is easy to obtain the six scalarizations: for all $x, x_0 \in \mathbb{R}_+$, one has

$$\begin{aligned}
\mathbb{D}^{ss}(F(x), F(x_0)) &= (x - x_0 + r)\varphi(x - x_0 + r),\\
\widehat{\mathbb{D}}^{is}(F(x), F(x_0)) &= \mathbb{D}^{si}(F(x), F(x_0)) = \mathbb{D}^{is}(F(x), F(x_0)) = \widehat{\mathbb{D}}^{si}(F(x), F(x_0))\\
&= (x - x_0)\varphi(x - x_0),\\
\mathbb{D}^{ii}(F(x), F(x_0)) &= (x - x_0 - r)\varphi(x - x_0 - r),
\end{aligned}$$

where $\varphi : \mathbb{R} \to \mathbb{R}_+$ is given by $\varphi(x) = 1$ if $x < 0$ and $\varphi(x) = \sqrt{2}$ if $x \geq 0$.

Let us note that F is K-compact valued and $(-K)$-compact valued. Now it is easy to determine all minimal solutions for the six problems ($\precsim$-SOP).

1. $\preccurlyeq^{\forall\forall}$. We want to check statement (c) of Theorem 4.2.8. Let $x, x_0 \in \mathbb{R}_+$. We have $\mathbb{D}^{ss}(F(x), F(x_0)) \geq 0$ for all $x \in \mathbb{R}_+$ if and only if $x - x_0 + r \geq 0$ for all $x \in \mathbb{R}_+$, which is true just if $x_0 \leq r$. So, we obtain that $[0, r]$ is the set of weak $\preccurlyeq^{\forall\forall}$-minimal solutions.

2. $\preccurlyeq^{\exists\forall}$, $\preccurlyeq^{\forall\exists}$, $\preccurlyeq^{\exists\forall}$ and $\preccurlyeq^{\forall\exists}$. These four cases have the same scalarization associated: $\bar{\mathbb{D}}(F(x), F(x_0)) = (x - x_0)\varphi(x - x_0)$. Then, $\bar{\mathbb{D}}(F(x), F(x_0)) \geq 0$ for all $x \in \mathbb{R}_+$ if and only if $x_0 = 0$. Therefore, by Theorem 4.2.8 we conclude that the unique weak minimal solution is 0.

3. $\preccurlyeq^{\exists\exists}$. Now, we check statement (c) of Theorem 4.2.12. If $\mathbb{D}^{ii}(F(x), F(x_0)) \geq 0$, this assertion is equivalent to $x - x_0 - r \geq 0$, is true if $x \geq x_0 + r$. The second assertion of (c), $\mathbb{D}^{ii}(F(x_0), F(x)) < 0$ is fulfilled if $x_0 - x - r < 0$, that is, if $x > x_0 - r$. So, each $x \in \mathbb{R}_+$ satisfying this inequality satisfies also the first

one. Therefore, it must be $x_0 < x + r$ for all $x \in \mathbb{R}_+$, which is true just if $x_0 < r$. In consequence, $[0, r)$ is the set of weak $\preccurlyeq^{\exists\exists}$-minimal solutions.

Example 4.2.17. Suppose that $x_0 \in S$, F is K-proper valued and $F(x_0)$ is K-compact. Let us prove that x_0 is a weak $\preccurlyeq^{\forall\exists}$-minimal solution of ($\preccurlyeq^{\forall\exists}$-SOP) if and only if for all $x \in S$ there exists $\bar{y} \in F(x_0)$ such that

$$(F(x) - \bar{y}) \cap (-\operatorname{int} K) = \emptyset. \tag{4.7}$$

($\Rightarrow$) By Theorem 4.2.8 we have $\mathbb{D}^{si}(F(x), F(x_0) \geq 0$ for all $x \in S$. By Corollary 3.1.18(iii), there exists $\bar{y} \in F(x_0)$ such that $\mathbb{D}^{si}(F(x), F(x_0)) = h^i(F(x) - \bar{y})$, and so $h^i(F(x) - \bar{y}) = \inf_{y \in F(x)} D(y - \bar{y}, -K) \geq 0$. By Lemma 1.3.19(iii) it follows that $y - \bar{y} \notin -\operatorname{int} K$ for all $y \in F(x)$, i.e., (4.7) holds.

($\Leftarrow$) Condition (4.7) implies that $\mathbb{D}^{si}(F(x), \bar{y}) = \inf_{y \in F(x)} D(y - \bar{y}, -K) \geq 0$, and so $\mathbb{D}^{si}(F(x), F(x_0)) = \sup_{y \in F(x_0)} \mathbb{D}^{si}(F(x), y) \geq 0$ for all $x \in S$. Therefore, by Theorem 4.2.8, we conclude that x_0 is a weak $\preccurlyeq^{\forall\exists}$-minimal solution.

The result proved in this example is inspired in [1, Proposition 18], which is more general.

Chapter 5

Conclusions and future lines of development

In this last chapter, the most important results derived throughout this memory are summarized with the aim of providing a global vision of the work which has been carried out. Moreover, some future lines of development are proposed, which could be interesting directions of research that have appeared along the thesis.

5.1 Conclusions

The main goal of this thesis focuses on the study of set scalarization functions and their applications in set-valued optimization problems, in order to derive necessary and sufficient conditions of minimality and weak minimality with the set criterion of solution. In this criterion of solution, set relations on the power set of the objective space Y relying on the ordering structure given in Y play one of the most essential roles in set optimization problems since they act as preference relations which provide a natural way to compare the values of the set-valued objective map. Through these binary relations, given two sets, it is possible to decide if one set dominates another set in a certain sense. In this memory, the comparison between two sets needed in the set criterion of solution has been carried out by means of the set relations of Kuroiwa (see Definition 1.3.2 in Section 1.2) associated with a cone K.

First of all, with a solid convex cone K, by considering the norm $\|.\|_e$ generated by Minkowski's functional of an order interval (see [35, 38, 61, 62, 103, 109, 133]) with $e \in \operatorname{int} K$, we have established a relationship between the Khoshkhabar-amiranloo and Soleimani-damaneh function [73] and the excess of a set over the conic extension of another set (see Theorem 1.3.30).

In Chapter 2, we have gathered the main set scalarization functions between two sets existing in the literature. On the one hand, the extensions of Gerstewitz's function (see [48, 95, 96, 113, 131]) which are defined in a real topological linear space with a solid convex cone K and, on the other hand, the extensions of the oriented distance function of Hiriart-Urruty (see [18, 38, 45, 129]) which are defined in a normed space with a not necessarily solid convex cone K. It should be highlighted that the main advantage of the oriented distance in contrast with Gerstewitz's function is that the solidness of the convex cone K is not required.

Section 2.1 is concerned with the study of the relationships among the set scalarization functions which can be found in the different papers existing in the literature. To the best of our knowledge, so far some relations between Gerstewitz's function (see Definition 1.3.14 in Section 1.2) and the oriented distance function of Hiriart-Urruty (see Definition 1.3.18 in Section 1.2) can be found in [98]. However, as far as we know, there are no research where such comparisons have been analyzed for the corresponding set scalarization functions except in [38, Theorem 6.15] with a solid convex cone K, where Gutiérrez, Jiménez, Miglierina and Molho showed that their set extension of the oriented distance, denoted by $\Delta_B(A)$, coincides with Gerstewitz's function $h^l_{\inf}(A, B)$, by considering the norm $\|.\|_e$ generated by Minkowski's functional of an order interval with $e \in \operatorname{int} K$. For this reason, this section is very interesting since the results presented help to clarify the relationships among the different set scalarization functions for the first time in the literature.

Moreover, we study some new relationships among the existing set scalarization functions. To be precise, we have proved a characterization for the distance

$\mathcal{D}(A, B)$ of Crespi, Ginchev and Rocca [18] (see Theorem 2.1.2), a relation between the oriented distances of Crespi et al. and the set oriented distance $\mathfrak{D}_A(B)$ of Xu and Li [129] (see Theorem 2.1.7), an interesting result where the oriented distance $\Delta_B(A)$ of Gutiérrez et al. [38] is related to the distances of Crespi et al. and the one of Xu and Li (see Corollary 2.1.8) and, moreover, we have presented a characterization to the extension of Gutiérrez et al. by relating this function to the oriented distance function $D(\cdot, A)$ of Hiriart-Urruty (see Proposition 2.1.9).

Also, we have presented two set scalarization functions which are set extensions of type sup-inf of the oriented distance function, the first one introduced by Ha [45], and the second one, a new set extension introduced by us (see Definition 2.1.14), which are denoted by $\mathbb{D}^{si}$ and $\widehat{\mathbb{D}}^{si}$, respectively, and that can be called set oriented distances. Furthermore, some new relationships among the extensions of type sup-inf mentioned and the set scalarization functions existing in the literature are established (see Proposition 2.1.21, Theorem 2.1.24 and Theorem 2.1.25).

In Section 2.2, to achieve our aims in set optimization problems, new important properties for the set scalarization functions $\mathbb{D}^{si}$ and $\widehat{\mathbb{D}}^{si}$ are derived. More specifically, by using cone-properness, cone-boundedness and a new concept of cone-boundedness with respect to a set which have been introduced by us (see Definition 2.2.14), some results about the finitude of the function $\mathbb{D}^{si}$ are presented (see Proposition 2.2.5, Corollaries 2.2.6 and 2.2.11). Besides, some new good properties as convexity of the function $\mathbb{D}^{si}(\cdot, B)$ (see Proposition 2.2.25), positive homogeneity (see Proposition 2.2.26), Lipschitz continuity for the functions $\mathbb{D}^{si}(A, y)$ and $\mathbb{D}^{si}(y, B)$ (see Theorems 2.2.27 and 2.2.28), invariance respect to conic extensions (see Proposition 2.2.30 and Theorem 2.2.39(i)), invariance by using equivalents sets (see Propositions 2.2.31, 2.2.33 and 2.2.35, and Theorem 2.2.39(ii),(iv) and (v)), diagonal null (see Proposition 2.2.34 and Theorem 2.2.39(iii)), monotonicity (see Proposition 2.2.36 and Theorem 2.2.39(vi)), invariance with respect to closure (see Proposition 2.2.37 and Theorem 2.2.39(vii)), etc. are presented. As we have said in the corresponding section of the memory, we have found various results that represent an improvement with respect to the corresponding results existing in the literature for another set scalarizations since we ask weaker assumptions.

In Section 2.3, by using the useful properties which have been shown in the former section, new characterizations of the lower set less relation $\curlyeqprec^{\forall\exists}$ and the upper set less relation $\preccurlyeq^{\forall\exists}$ of Kuroiwa by means of the set scalarization functions $\mathbb{D}^{si}$ and $\widehat{\mathbb{D}}^{si}$ are provided (see Theorems 2.3.1 and 2.3.17(i)). Furthermore, if K is a solid convex cone, characterizations for the strict set relations $\curlyeqprec_s^{\forall\exists}$ and $\preccurlyeq_s^{\forall\exists}$ corresponding of the set relations of Kuroiwa (see Theorems 2.3.11 and 2.3.17(v)) by requiring assumptions of cone-compactness are derived. We also deal with strict monotonicity for the functions $\mathbb{D}^{si}$ and $\widehat{\mathbb{D}}^{si}$ (see Proposition 2.3.15 and Theorem 2.3.17(vi)) by using the strict lower set less relation $\curlyeqprec_s^{\forall\exists}$ and the strict upper set less relation $\preccurlyeq_s^{\forall\exists}$. As we have pointed out in the corresponding section of the memory, some of our results represent an improvement with respect to the results existing in the literature for another set scalarizations because we require weaker assumptions.

Chapter 3 is devoted to introducing new set scalarization functions which are extensions of the oriented distance function of Hiriart-Urruty. It should be pointed out that the results achieved have been derived by applying the new set oriented distances and, therefore, these results are new too.

In Section 3.1, six set scalarizations of type sup-inf and inf-sup, which are generalizations of the oriented distance function, denoted by $\mathbb{D}^{\alpha}$ and $\widehat{\mathbb{D}}^{\alpha}$, have been presented (see Definition 3.1.1), four of which are new. Relationships among them are presented (see Lemmas 3.1.4 and 3.1.5), characterizations to these six set scalarizations are given (see Lemma 3.1.10) and, moreover, some of their main properties have been studied as, for example, finitude under suitable assumptions of cone-properness and cone-boundedness (see Theorem 3.1.13), invariance by conic extensions (see Theorem 3.1.20), monotonicity (see Theorem 3.1.22) by considering the six set relations $\preccurlyeq^{\alpha\beta}$ and $\curlyeqprec^{\alpha\beta}$ of Kuroiwa introduced in Definition 1.3.2, closure property (see Proposition 3.1.27), etc.

In Section 3.2, new characterizations of the six set relations $\preccurlyeq^{\alpha\beta}$ and $\curlyeqprec^{\alpha\beta}$ of Kuroiwa (see Theorem 3.2.1) have been derived, by using the six set scalarizations

$\mathbb{D}^\alpha$ and $\widehat{\mathbb{D}}^\alpha$ which are introduced in the former section. Furthermore, some examples to illustrate the results obtained are provided, especially to emphasize that the assumptions required cannot be removed. The importance of these results lies in the fact that they could be applied in Section 4.1 to analyze minimality conditions for a set optimization problem with the set criterion of solution.

In Section 3.3, by considering a solid convex cone K and under suitable assumptions, strict monotonicity for the six set scalarizations $\mathbb{D}^\alpha$ and $\widehat{\mathbb{D}}^\alpha$ (see Theorem 3.3.8) by using the six set relations $\preccurlyeq^{\alpha\beta}$ and $\curlyeqprec^{\alpha\beta}$ of Kuroiwa has been investigated. For this purpose, some new important results which deal with inequalities for the functions $\mathbb{D}^\alpha$ and $\widehat{\mathbb{D}}^\alpha$ when one of their variables is a sum of two sets, are presented; moreover, it should be emphasized that these results do not exist in the literature for the set Gerstewitz's function. The results about strict monotonicity mentioned above are very important because they could be applied in Section 4.2 to derive weak minimality conditions for a set optimization problem with the set criterion of solution. It is worth mentioning that in the literature, there are very few authors who have researched strict monotonicity (see [5, 41, 51, 94, 96, 107]) and, in all these cases, set Gerstewitz's function has been used. As we have said in the corresponding section of the memory, the results proved represent an extension since they require weakest assumptions.

In Section 3.4, by considering a solid convex cone K, new characterizations of the six strict set relations $\preccurlyeq_s^{\alpha\beta}$ and $\curlyeqprec_s^{\alpha\beta}$ of Kuroiwa (see Theorem 3.4.1) have been derived by using the six set scalarizations $\mathbb{D}^\alpha$ and $\widehat{\mathbb{D}}^\alpha$. Moreover, some examples to illustrate the results obtained are provided with the aim to emphasize that the assumptions required cannot be removed. These results will be used in section 4.2 to deduce weak minimality conditions for a set optimization problem with the set criterion of solution.

In Chapter 4, applications to set optimization problems with set criterion of solution by means of set relations $\preccurlyeq^{\alpha\beta}$ and $\curlyeqprec^{\alpha\beta}$ of Kuroiwa are searched. To be precise, we characterize by scalarization several types of solution.

In Section 4.1, by considering some good properties of the six set scalarizations $\mathbb{D}^\alpha$ and $\widehat{\mathbb{D}}^\alpha$ which have been presented in Section 3.1 as, for example, finitude, monotonicity, performance with respect to equivalent sets, and by applying the characterizations of the set relations of Kuroiwa, which have been given in Section 3.2, several new characterizations by scalarization of minimal solutions for six set optimization problems with the set criterion have been derived (see Theorems 4.1.7 and 4.1.15). Also, to illustrate our results we have provided an example.

In Section 4.2, by considering some useful properties of the six set scalarizations $\mathbb{D}^\alpha$ and $\widehat{\mathbb{D}}^\alpha$ presented in Section 3.1 as, for example, finitude and monotonicity, as well as their strict monotonicity studied in Section 3.3 and by applying the characterizations of the strict set relations of Kuroiwa which are given in Section 3.4, several new characterizations by scalarization of weak minimal solutions for six set optimization problems (see Theorems 4.2.8 and 4.2.12) with the set criterion, are achieved. Also, to illustrate our results we have provided an example.

5.2 Future lines of development

In this section, the lines of further research we present are deduced as a result of the work carried out. In general, the proposed developments open new directions of research related to the accomplished study.

Line 1. In the literature different set order relations have been used to the comparison of sets in order to solve set optimization problems. In [50, 75, 82, 83] a more general set order relation were introduced, where the involved set D describing the domination structure does not need to be a convex cone K. To be precise, the generalized upper set less order relation (see [50, 82, 83]) and the generalized lower set less order relation [75] were introduced.

In [98] some properties to the oriented distance function when the associated set is neither a cone nor a convex set were analyzed. In [29] the properties of the oriented distance function with respect to a co-radiant set were investigated. Let us note that co-radiant sets are more general than a cone and are main tools in the study of approximate solutions in vector optimization problems. In [74] Gerstewitz's function with a nonempty proper subset of Y was considered and in [83] some generalizations of Gerstewitz's function $z^{D,k}$ to characterize generalized set order relations were treated where the involved set D does not need to be a convex cone K.

In [22, 108] set relations in analogy to set relations of Kuroiwa were defined by considering improvement sets (see [13, 40]) with the aim of defining new solution concept for set-valued problems with the set criterion of solution [22] and to investigate stability of the solution sets for set optimization problems and to study the upper semi-continuity and lower semi-continuity of solution mapping to parametric set optimization problems.

In this thesis we have presented new set extensions $\mathbb{D}^\alpha$ and $\widehat{\mathbb{D}}^\alpha$ of type sup-inf and inf-sup of the oriented distance function of Hiriart-Urruty which have been defined by using a convex cone not necessarily solid. We think that a line of research could be to introduce the six set oriented distances $\mathbb{D}^\alpha$ and $\widehat{\mathbb{D}}^\alpha$ by using a set D which does not have to be a convex cone K as, for instance, by using free disposal, improvement sets [13, 40], or co-radiant sets [29]. In general, to extend the results we have presented in this thesis in the framework mentioned.

Line 2. In set-valued optimization problems with the set criterion, a set-valued objective map is minimized (or maximized) with respect to a set order relation. The variable ordering structures [26] are introduced by a cone-valued map [25] that associates with each element of the linear space an ordering and, therefore, generalize the concept of ordering structures in vector and set optimization problems. In the literature, different types of set order relations with a variable order structure to compare sets in a linear topological space have been recently taking into account (see, for instance, [3,80,84]). By considering a variable order structure, in respect of scalarization methods to characterize set order relations and minimal elements of set optimization problems with the set criterion, in [84] a modified Gerstewitz's function $z^{D,k}$ was used but changing the cone D for a variable order structure. Furthermore, in [3] the authors have characterized set order relations defined with a variable order structure, through a version of the oriented distance function which is given with a variable order structure.

We considerer that a future research could be to define the six set oriented distances $\mathbb{D}^\alpha$ and $\widehat{\mathbb{D}}^\alpha$ presented in this thesis but taking into account a variable ordering structure instead of a convex cone K as, for example, by using $K = \cup_{a\in A}\mathcal{K}(a)$ where $A \subset Y$ is a nonempty set and $\mathcal{K} : Y \rightrightarrows Y$ is a cone-valued map. We have another approach by using a Bishop-Phelps cone-valued map to introduce the variable order structure. In this case, for these six set scalarization functions, it might be useful to investigate their relationships and their properties, to characterize generalize set order relations and different kinds of minimal solutions to set optimization problems and, in general, to extend the results which we have presented in this thesis to the mentioned framework.

Line 3. In the literature we can find many concepts which may involve the convergence of sequences of sets (see [22, 30, 74, 116]). The most usual concepts of set convergence in the power set of a partially ordered normed space Y supplied with a set order relation are the set convergence of Painlevé-Kuratowski or Pompeiu-Hausdorff.

We think it could be interesting to define some concepts of set convergence by using the set oriented distances $\mathbb{D}^\alpha$ presented in this thesis, which could be called upper and lower $\mathbb{D}^\alpha$-convergence, in the power set of a partially ordered normed space Y supplied with a set order relation. We could define $\mathbb{D}^\alpha$-convergence, as follows:

$A_n \rightharpoonup A \Leftrightarrow \mathbb{D}^\alpha(A_n, A) \to 0, \quad A_n \rightharpoondown A \Leftrightarrow \mathbb{D}^\alpha(A, A_n) \to 0,$

$A_n \to A \Leftrightarrow (A_n \rightharpoonup A \text{ and } A_n \rightharpoondown A) \Leftrightarrow (\mathbb{D}^\alpha(A_n, A) \to 0 \text{ and } \mathbb{D}^\alpha(A, A_n) \to 0).$

It might be interesting to compare the $\mathbb{D}^\alpha$-convergence with other concepts of set convergence existing in the literature as, for example, the set convergence of Painlevé-Kuratowski or Pompeiu-Hausdorff, to investigate their properties and compatibility with respect to the set relations considered, their behavior by passing to the limit, etc.

For example, for all n, by using $\mathbb{D}^{si}$-convergence, it could be interesting to prove:

(i) If $A_n \preccurlyeq^{\forall\exists} B$ and $\mathbb{D}^{si}(A, A_n) \to 0$, then $A \preccurlyeq^{\forall\exists} B$.

(ii) If $B \preccurlyeq^{\forall\exists} A_n$ and $\mathbb{D}^{si}(A_n, A) \to 0$, then $B \preccurlyeq^{\forall\exists} A$.

(iii) If $A \preccurlyeq^{\forall\exists} B$, $\mathbb{D}^{si}(A, A_n) \to 0$ and $\mathbb{D}^{si}(A_n, A) \to 0$, then $A_n \preccurlyeq^{\forall\exists} B$.

(iv) If $B \preccurlyeq^{\forall\exists} A$, $\mathbb{D}^{si}(A_n, A) \to 0$ and $\mathbb{D}^{si}(A, A_n) \to 0$, then $B \preccurlyeq^{\forall\exists} A_n$.

(v) If $\mathbb{D}^{si}(A, A_n) \to 0$, $\mathbb{D}^{si}(B, B_n) \to 0$ and $A_n \preccurlyeq^{\forall\exists} B_n$, then $A \preccurlyeq^{\forall\exists} B$.

(vi) If $\mathbb{D}^{si}(A_n, A) \to 0$, $\mathbb{D}^{si}(B_n, B) \to 0$ and $B_n \preccurlyeq^{\forall\exists} A_n$, then $B \preccurlyeq^{\forall\exists} A$.

(vii) If $A_n \to A$ and $B_n \to B$, then $A_n + B_n \to A + B$, that is, $\mathbb{D}^{si}(A+B, A_n + B_n) \to 0$ and $\mathbb{D}^{si}(A_n + B_n, A + B) \to 0$.

(viii) If $A_n \to A$ and $B_n \to B$, then $\lambda A_n + \mu B_n \to \lambda A + \mu B$, where $\lambda, \mu \in \mathbb{R}_+$, that is, $\mathbb{D}^{si}(\lambda A + \mu B, \lambda A_n + \mu B_n) \to 0$ and $\mathbb{D}^{si}(\lambda A_n + \mu B_n, \lambda A + \mu B) \to 0$.

Line 4. Many concepts in set-valued analysis such as stability, differentiation and approximation may involve the convergence of sequences of sets [30]. In the power set of a space supplied with an ordering, some usual notions of set convergence are Kuratowski-Painlevé or Pompeiu-Hausdorff convergences [22, 116]. In [30,74] a concept of set convergence compatible with the ordering on the power set of a partially ordered normed space was used. By using set convergence, stability of minimal sets and minimal solutions to set-valued optimization problems with the set criterion was investigated. Moreover, the upper and lower convergence of approximate minimal solution sets in Kuratowski-Painlevé sense was established [74], and the asymptotic behavior of sequences of lower bounded sets was studied [30].

We think that further directions of research could be to investigate stability results for minimal sets and minimal solutions to set-valued optimization problems with the set criterion, by considering some new concepts of set convergence defined by using the set oriented distances $\mathbb{D}^\alpha$ and $\widehat{\mathbb{D}}^\alpha$. Likewise, it might be fruitful to study the asymptotic behavior of sequences of both minimal sets and minimal solutions to set-valued optimization problems by using the set convergence above mentioned. More precisely, considering a sequence of K-bounded subsets A_n converging in the sense of the new set convergence to a K-bounded set A, it could be interesting to study the convergence of the minimal elements of A_n towards the minimal element of A. Afterwards, considering a sequence of set-valued optimization problems SOP_n, the data of which converging to the data of a set-valued optimization problem SOP, it might be useful to investigate the convergence of the minimal solutions to SOP_n towards the minimal solutions of SOP.

Line 5. In 1966, Tykhonov [125] introduced well-posedness for scalar optimization problems, which guarantees the convergence of minimizing sequences to the unique solution of the problem. More recently, well-posedness for scalar optimization problems was introduced in [104]. In [43, 134] a generalized version of Gerstewiz's function given in [51] was used to introduce the well-posedness property in the setting of set optimization problems with the set criterion and, therefore, the well-posedness defined are applicable only for a solid convex cone K. In [134] by using K-bounded sets, it is established the equivalent between three kinds of well-posedness for a set optimization problem with set criterion at a given minimal solution x_0 and well-posedness of the three kinds of scalar optimization problems. In [43] the notion of well-posedness due to Zhang et al. [134] and K-proper sets was used. So, in [43] an extension of the results of [134] is given since it is not required K-bounded objective sets; moreover, the equivalence between the well-posedness of the original set optimization problem at a given strict minimal solution x_0 [45] and the classical Tykhonov well-posedness of the scalarized problem was obtained under the assumption that the set $F(x_0)$ is a K-proper set.

In [20] a definition of global well-posedness for set optimization problems was given, and by using an embedding technique a well-posedness property of a class of generalized convex set-valued maps was proposed and a class of quasiconvex se-valued maps which guarantees well-posedness of the set optimization problem was introduced.

In [37] three types of well-posedness for a set optimization problem with the set criterion of solution were studied by using the u-preorder of Kuroiwa and a generalization of the oriented distance introduced by Xu and Li [129] and, therefore, the well-posedness defined are applicable for a not necessarily solid cone K; moreover, necessary conditions for the well-posedness by using the Hausdorff set-convergence were obtained. It is worth noting that the results obtained in [37] are more general than in [43] where the authors studied a notion of well-posedness for a set optimization problem under the assumption that the ordering cone K is solid.

We consider that it could be interesting to use the set oriented distances $\mathbb{D}^\alpha$ and $\mathbb{D}^\alpha$-convergence which have been defined with a convex cone K not necessarily solid, in order to introduce different classes of well-posedness for a set optimization problem with the set criterion of solution and K-bounded sets, by considering well-posedness of scalar optimization problems; moreover, to establish the equivalence between the well-posedness of ($\preceq$-SOP) and the well-posedness of scalar optimization problems, by using the functions $\mathbb{D}^\alpha(\cdot, F(x_0))$ where x_0 is a $\preceq$-minimal or weak $\preceq$-minimal solution to the set optimization problem considered.

Line 6. Via scalarization techniques, a vector optimization problem is replaced by an associated family of scalar optimization problems, which allows to relate qualitatively the solutions of both problems and to solve optimization problems through numerical methods applicable to scalar problems, that is, solutions of optimization problems can be characterized and computed as solutions of appropriate scalar optimization problems.

In [113, 131, 132] it was proved that the problem to calculate each value of the scalarizing functions can be computed in a finite dimensional Euclidean space for certain polytope sets with a convex polyhedral cone inducing the ordering. This problem can be decomposed into finite numbers of linear programming subproblems.

We observe that in the literature there exist a few papers dealing with concrete calculation process to compute values of scalarization functions. However, this is of great importance since usually many properties on set-valued maps and set optimization are described by scalarization. In all mentioned papers, Gerstewitz's function is the only scalarization that has been used. In the particular case of a finite dimensional Euclidean space by using the six set oriented distances $\mathbb{D}^\alpha$ and $\widehat{\mathbb{D}}^\alpha$ and polyhedral cones, we believe that it might be fruitful to formulate an algorithm which allows the comparison of two polytopes sets (that is, convex hull to a finite set of points), and evaluate whether both sets fulfill an inequality in a discrete numerical manner.

In [18, 29, 34], if A is a convex set we can find the following expression:

$$D(y, A) = \sup_{\|\xi\|=1} (\langle \xi, y\rangle - \sup_{a\in A}\langle \xi, a\rangle).$$

Moreover, in the case that $A = K$ is a convex cone it follows that

$$D(y, -K) = \sup_{\|\xi\|=1, \xi\in K^+} \langle \xi, y\rangle.$$

To be precise, to decide if a set relation is fulfilled by two sets evaluating one inequality it is necessary to compute numerically the inequality mentioned via scalarization functions. For this purpose, we need to propose a calculation process that allows us to compute values of scalarization functions mentioned. So, we think that considering the representation above introduced, it can be possible to reform the six set scalarization functions $\mathbb{D}^\alpha$ and $\widehat{\mathbb{D}}^\alpha$ with the aim of carrying out the comparison between two sets to evaluate one inequality. For instance, the function $\mathbb{D}_K^{si}(A,B)$ could be reformulated as follows:

$$\mathbb{D}_K^{si}(A,B) = \sup_{y\in B}\inf_{x\in A} D(x-y,-K) = \sup_{y\in B}\inf_{x\in A}\sup_{\|\xi\|=1,\xi\in K^+}\langle \xi, x-y\rangle.$$

Line 7. An interesting topic in optimization theory is to characterize cone-convexity and cone-quasiconvexity of vector-valued or set-valued objective functions in terms of usual convexity or quasiconvexity of certain real-valued functions, that is, taking into account some appropriate scalarization functions [97, 103]. Hence, it is important and useful to study what kind of scalarizing functions can be used so that, for example, a set-valued map will inherit its properties by means of a composition between the set-valued maps given and the scalarization functions which are considered.

For a set-valued map F, in [43, 72, 79, 92, 95, 96, 106, 111] inherit properties of different kinds of convexity and continuity are studied by composing the set-valued map F with scalarization functions. In [43, 72, 95, 96, 106] Gerstewitz's function was considered. In [92], compositions of a set-valued map F with extensions of Gerstewitz's functions were used to prove continuity for the set-valued map under some convexity assumptions. To be precise, four types of nonconvex scalarizing functions for set-valued maps based on the compositions $h^\alpha_{\inf}\circ F$ and $h^\alpha_{\inf}\circ F$ where $\alpha\in\{\preccurlyeq^{\forall\exists}, \curlyeqprec^{\forall\exists}\}$ were used to show continuity for the set-valued map F.

We think that further direction of research might be to investigate properties of the composition of a set-valued map F and the six set scalarization functions $\mathbb{D}^\alpha$ and $\widehat{\mathbb{D}}^\alpha$. So, it could be useful to study inherited properties of, for instance, $\mathbb{D}^\alpha \circ F$ according to some assumptions of F by defining $(\mathbb{D}^\alpha(\cdot, F(x_0))\circ F)(x) = \mathbb{D}^\alpha(F(x), F(x_0))$. The inverse results could be interesting too.

Line 8. In the literature, different generalizations $h^\alpha_{\inf}(A,B)$ and $h^\alpha_{\sup}(A,B)$, where $\alpha\in\{\forall\forall,\exists\forall,\forall\exists,\exists\exists\}$, of Gerstewitz's function have been introduced (see, for example, [79, 94, 95]) by using the six set relations of Kuroiwa. For these functions, several properties have been shown. However, some new interesting properties and results which have been proved in this thesis for the six set oriented distances $\mathbb{D}^\alpha$ and $\widehat{\mathbb{D}}^\alpha$ by using the six set relations of Kuroiwa, do not exist for the Gerstewitz's function.

We consider that it might be useful to investigate if it is possible to extend some properties and results presented in this thesis for the functions $\mathbb{D}^{\alpha}$ and $\widehat{\mathbb{D}}^{\alpha}$, for the set extensions of Gerstewitz's function as, for example, a property given by the inequalities in Lemma 3.3.4, where one of the variables is a sum of two sets. Moreover, by considering a solid convex cone K, it could be fruitful to investigate the relationships between the different set extensions of Gerstewitz's function and the six set oriented distances $\mathbb{D}^{\alpha}$ and $\widehat{\mathbb{D}}^{\alpha}$, for example, by using the norm induced by Minkowski's functional [62, 109, 133]. Also, it could be interesting to look for similar results of Corollary 3.1.18 for different generalizations of Gerstewitz's function (see, for instance, [79, 94, 95]) which have been introduced by using the six set relations of Kuroiwa.

Line 9. It is well known that Gerstewitz's function is continuous and convex. Moreover, the continuity and convexity of the extensions of Gerstewitz's function (see [71, 92]) play an important role in the study of the existence of solutions and the stability of the set of solutions for set optimization problems. These properties have been shown under some suitable conditions in [49] and were used to consider the upper semicontinuity and the lower semicontinuity of strongly approximate solution mappings to the parametric set optimization problems.

We consider that it could be interesting to investigate continuity and convexity properties for the functions $\mathbb{D}^{\alpha}$ and $\widehat{\mathbb{D}}^{\alpha}$ given in Definition 3.1.1.

Line 10. In Theorem 2.2.1, we have shown that $\mathbb{D}^{si}(A, B) \leq r$ if and only if $B \subset \mathrm{cl}(rU_0 + A + K)$, where $A, B \in \mathcal{P}_0(Y)$ and $r \geq 0$. We think it could be interesting to prove $\mathbb{D}^{si}(A, B) = \min\{r \geq 0 : B \subset \mathrm{cl}(rU_0 + A + K)\}$. That is, $\mathbb{D}^{si}(A, B)$ is the smallest number $r \geq 0$ such that $B \subset \mathrm{cl}(rU_0 + A + K)$. Moreover, we considerer it could be interesting to prove a result of this kind for $\mathbb{D}^{si}(A, B) < r$ as, for example, $\mathbb{D}^{si}(A, B) < r$ if and only if $B \subset \mathrm{int}(rU_0 + A + K)$, under suitable assumptions. Also, we think it might be useful to find results of type $\mathbb{D}^{si}(A, B) \geq r$ if and only if $A \subset \mathrm{cl}(rU_0 + B + K)$ and $\mathbb{D}^{si}(A, B) > r$ if and only if $A \subset \mathrm{int}(rU_0 + B + K)$.

Line 11. Minimax theory treats a class of extremum problems which involves, not simply minimization or maximization, but a combination of both [115]. It is well-known that saddle point assertions play an important role in scalar optimization due to their relations with other fundamental tools and theories such as Kuhn-Tucker optimality conditions, duality, minimax theory, etc.

Given $A, B \in \mathcal{P}_0(Y)$ and a function $\varphi : A \times B \to \mathbb{R} \cup \{\pm\infty\}$, it is possible to build the functions $\sup_{y \in B} \inf_{x \in A} \varphi(x, y)$ and $\sup_{x \in A} \inf_{y \in B} \varphi(x, y)$. In [107] we can find a result of this type but using Gerstewitz's function.

If we define $\varphi(x, y) = D(x - y, -K)$, we think that it might be useful to investigate when the expressions sup-inf and inf-sup above mentioned are equal. Therefore, it could be interesting to find under which conditions there exists a saddle point for the function $\varphi(x, y)$, that is, to furnish conditions for $\mathbb{D}^{si}(A, B) = \widehat{\mathbb{D}}^{is}(A, B)$ and $\widehat{\mathbb{D}}^{si}(A, B) = \mathbb{D}^{is}(A, B)$.

Line 12. In order to obtain minimal solutions to set optimization problems with the set criterion, one has to analyze whether one set dominates another set in a certain sense, that is, it is necessary to choose the best set in some sense, according to the set relation that has been considered. To compare two sets by means of a set order relation, it is usually necessary to characterize the set order relations by means of a set scalarization function. In this memory we have applied the six set order relations of Kuroiwa (Definition 1.3.2) and the six scalarization functions $\mathbb{D}^\alpha$ and $\widehat{\mathbb{D}}^\alpha$ (Definition 3.1.1).

In [4, 70, 76] new set order relations by using Minkowski difference of sets have been introduced to compare sets. We think that a line of future research is concerned with the application of the order relations just mentioned instead of the set relations of Kuroiwa along with our six set oriented functions $\mathbb{D}^\alpha$ and $\widehat{\mathbb{D}}^\alpha$ to extend the results which we have presented in this thesis.

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