TESIS DOCTORAL

2021

Finitely generated non-cocompact NEC groups

Alejandro José Monerri Molina

PROGRAMA DE DOCTORADO EN CIENCIAS

Director: Francisco Javier Cirre Torres

Codirector: Emilio Bujalance García

Departamento de Matemáticas Fundamentales, Facultad de Ciencias, UNED.

Programa de Doctorado en Ciencias.

Tesis Doctoral: Finitely generated non-cocompact NEC groups.

Año: 2021.

Autor: Alejandro José Monerri Molina.

Director: Francisco Javier Cirre Torres.

Codirector: Emilio Bujalance García.

Agradecimientos

En primer lugar, quiero expresar mi total y sincero agradecimiento a mis directores de tesis los profesores Emilio Bujalance y Javier Cirre por su decisiva ayuda y guía: ideas fundamentales sobre el contenido, correcciones y estilo, apoyo y un largo etcétera. No existen palabras para expresar la excelencia de su trabajo y el apoyo que me han brindado. No puedo dejar de incluir en esta sección al resto de personal docente y no docente de la Facultad de Ciencias y de la U.N.E.D: no es exagerar decir que es su dedicación y profesionalidad la que hace posible que muchos alumnos podamos cursar estudios de graduado, máster y doctorado compaginando con responsabilidades profesionales y familiares que de otra forma no sería posible.

A mi familia, Arancha, Alejandra y Ana, por su apoyo continuo, su comprensión y por aceptar que parte de su tiempo se lo haya dedicado a esta tesis.

Abstract

Abstract in English:

This thesis is devoted to the study of finitely generated discrete subgroups Γ of the whole group of isometries of the hyperbolic plane $\mathbf{H}$ including those which reverse the orientation (reflections and glide reflections) as well as boundary transformations (parabolic and boundary hyperbolic elements), such that the orbit space $\mathbf{H}/\Gamma$ is not compact.

Two special cases closely related to finitely generated non-cocompact NEC groups, the finitely generated discrete subgroups of orientation-preserving isometries (fuchsian groups) and the cocompact NEC groups have been extensively studied in the literature. This work presents a fairly complete introduction of the non-cocompact NEC groups, providing with proof their presentation, introducing their signatures and using them for studying their orbit spaces and the necessary and sufficient conditions of isomorphism between these groups.

We present additionally a set of invariants that classify the non-compact Klein surfaces up to homeomorphisms using the signature of the NEC group of which the Klein surface is the orbit space. The Euler characteristic of the orbit space of an NEC group is calculated. Using this we obtain the signature of the non-cocompact canonical fuchsian group linked to the signature of a given NEC group. Finally, the concept of elementary NEC groups is introduced and all the possible elementary groups deduced. Using the properties of their canonical fuchsian groups, some results describing the limit sets of NEC groups are obtained. That leads us to introduce a classification of NEC groups of first and second kind similarly as for fuchsian groups.

Abstract en español:

Esta tesis está dedicada al estudio de grupos discretos de isometrías Γ del plano hiperbólico $\mathbf{H}$ incluyendo transformaciones que revierten la orientación (reflexiones y reflexiones con desplazamiento) y elementos de contorno (parabólicos e hiperbólicos), de forma que el espacio de órbitas $\mathbf{H}/\Gamma$ es no compacto.

Dos casos específicos relacionados con los grupos NEC no cocompactos finitimante generados, los subgrupos de isometrías que preservan la orientación o grupos fuchsianos, y los grupos NEC cocompactos han sido ampliamente estudiados en la bibliografía. Este trabajo cubre una laguna que ha existido en la literatura por cierto tiempo introduciendo de forma razonablemente completa los grupos NEC finitamente generados no cocompactos. Se proporciona con demostración la presentación en forma de generadores y relaciones de estos grupos, introduciendo su signatura y usándola para estudiar sus espacios de órbitas y las condiciones necesarias y suficientes de isomorfía entre grupos NEC.

Se introduce además un conjunto de invariantes que clasifica las superficies de Klein no compactas salvo homeomorfismos a partir de la signatura del grupo NEC de la que es espacio de órbitas. Obtenemos la característica de Euler del espacio de órbitas y se usa para deducir la signatura del subgrupo fuchsiano canónico de un grupo NEC dada su signatura. Finalmente, se introduce el concepto de grupo NEC elemental y se obtiene la presentación de todos los grupos NEC elementales. Se presentan resultados relacionados con los conjuntos límite de los grupos NEC y se aplican para su clasificación en primer y segundo tipo de forma similar a como se hace con los grupos fuchsianos. Para ello se usan las propiedades del subgrupo fuchsiano canónico del grupo NEC dado.

Keywords: Hyperbolic Plane, Non-euclidean Chrystallographic Groups, Finitely generated Groups of Hyperbolic Isometries, Non-cocompact NEC Groups.

Contents

Symbols

$\mathbb{N}$	Set of natural numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Complex plane
$\mathbf{H}$	Upper half-plane
$\mathbf{R}$	Fundamental Region
$\mathbf{F}$	Fundamental Domain
$\mathbf{D}_p$	Dirichlet region centered in p
$a, b, c, etc.$	Edges of a fundamental region
$A, B, C, etc.$	Hyperbolic transformations

List of Figures

CHAPTER 0

Introduction

The main goal of this dissertation is to cover a gap in the literature providing a description with proofs of the algebraic structure and classification of non-cocompact finitely generated discrete groups of isometries of the hyperbolic plane.

The study of discrete cocompact groups of orientation preserving isometries of the hyperbolic plane was initiated in the XIX century by Poincaré in [47]. The structure of these groups, called by Poincaré fuchsian groups, was essentially solved by Fricke and Klein in [22] where the canonical form of their presentation was obtained, and the signatures, now called Fricke-Klein signatures, were introduced. The idea of this approach is to gather enough geometrical information of a group Γ to be able to reconstruct Γ as an abstract group. For that, the side pairing properties of a canonical fundamental region is used. The signature is then just a collection of combinatorial data sufficient to provide the reconstruction of the group presentation and the labelled polygon of the surface symbol. In addition, two discrete groups have the same signature if and only if they are isomorphic. The properties of the fuchsian groups and their applications have been extensively studied during the XX century turning out to be central in the study of several topics as for example compact Riemann surfaces, algebraic compact curves and many others. Several authors have contributed in this area and therefore we are going here just to outline the most relevant papers for the topics related to this thesis: properties of finitely generated fuchsian groups by Greenberg in [26], Maclachlan in [42] and Singerman in [53], hyperbolic polygons and fuchsian groups by Beardon in [4], group isomorphisms and the geometric realization of isomorphisms by Fenchel and Nielsen in [21] and Zieschang in [61], and the connection between fuchsian groups and compact Riemann surfaces started by the uniformization theorem of Riemann surfaces in the XIX century by Poincaré and Klein and followed by Hurwitz's theorem on automorphisms of compact Riemann surfaces.

The discrete subgroups of isometries of the hyperbolic plane including orientation reversing isometries with compact orbit space was worked out in the 1960s by Wilkie in [59] and Macbeath in [40]. Wilkie and Macbeath followed Fricke-Klein's idea of linking the geometry of a special fundamental region, a polygon in the hyperbolic plane, to the presentation of the group. Based on these articles, several authors have extended and applied the theory of cocompact NEC groups. Just to mention few articles linked to the content of this thesis, Singerman in [54] obtained the canonical fuchsian group and calculated the area of the compact fundamental regions given an NEC group via its signature. In [52], he initiated the study of the role of these groups in the analysis of Klein surfaces. Later on, Preston in [48] and May in [44] showed that the Klein surfaces, introduced by Klein in 1897, can be represented by actions of NEC groups similar to the uniformization of Riemann surfaces mentioned above. Finally, I would like to highlight the impact that the UNED and the Universidad Complutense de Madrid have had since the 1980s in the study of NEC groups and the groups of automorphisms of the Klein and Riemann surfaces: for example, Bujalance studied the normal subgroups of NEC groups in [8] and [9], the applications of the NEC groups to the study of Klein and Riemann surfaces have been studied by Bujalance, Cirre, Costa, Gamboa, Gromadzki, Etayo, Martínez and others in [10], [19], [12], [13] and [14]. Etayo and Martínez in [20] studied convex fundamental regions defined as hyperbolic polygons with the minimum number of sides given the signature of an NEC group, linked to the problem of obtaining the (geometric) rank of an NEC group (see for example [33]).

For the non-cocompact case, though, only very few and partial results have been given until now: Zieschang, Vogt and Coldeway in [60] provided an incomplete presentation of finitely generated non-cocompact NEC groups, without proof, using a combinatorial approach. Macbeath and Hoare in [41] gave a presentation of non-cocompact NEC groups (not necesarily finitely generated) using a purely algebraic approach that cannot be exploited in an obvious way to analyze further these groups (e.g. algebraic classification, geometric properties of the orbit space, etc.).

The approach used in this thesis follows the classical approach from Wilkie and Macbeath (i.e. Fricke-Klein's approach) linking the presentation of the group to the geometry of a fundamental domain. This opens the possiblity to use signatures and to explore the algebraic classification (via type preserving isomorphisms) and the topological classification (via homeomorphisms and diffeomorphisms) of the orbit spaces. The structure of the thesis is outlined below:

- Chapter 1 is dedicated to provide the conceptual background and motivation of this thesis. In section 1.1, we start with a short introduction on groups of isometries of metric spaces. In section 1.2, we introduce the classical Macbeath's theorem of the group presentation of groups of isometries of simply connected metric spaces. Finally, in the last section we provide an overview of the main concepts and results related to fuchsian and NEC groups.

- In chapter 2, we obtain a presentation by generators and relations of finitely generated discrete groups of hyperbolic isometries Γ with non-compact orbit space $\mathbf{H}/\Gamma$. To this end, we use the geometrical properties of a fundamental region with a canonical form, and apply Macbeath's classical theorem, see [39], on presentations of groups of isometries of simply connected spaces. The main result in this chapter provides a presentation by generators and relations which reflects the geometry underlying these groups. In particular, we include those generators and relations missing in Zieschang, Vogt and Coldeway [60, Theorem 4.11.5]. The chapter is organized as follows. Given a finitely generated non-cocompact NEC group Γ, in Section 2.1 we construct a fundamental region for Γ with a particular surface symbol that we call canonical fundamental region of Γ. We proceed in a similar way as Wilkie did in [59] for cocompact NEC groups. This surface symbol reflects geometric and topological properties of the fundamental region, and yields a canonical presentation by generators and relations of Γ. This is obtained in Section 2.2. The results of this chapter have been published in [16].

- In chapter 3, we introduce the notion of signature of non-cocompact NEC groups and based on the signature we study group isomorphisms. The main results of this chapter are the topological characterization of the orbit space $\mathbf{H}/\Gamma$ given the signature of the group Γ and the identification of the necessary and sufficent conditions on the signatures of two groups Γ and Γ' for them to be isomorphic via a type-preserving isomorphism. A key point in this thesis is related to the behaviour of the product of two reflections in the non-cocompact NEC groups that, in addition to elliptic and hyperbolic, can also be parabolic. This has a decisive impact on the group structure (far more complex as previously proposed), form of the fundamental region (appearing the so-called semi-parabolic vertices) and the orbit space (hyperbolic ends). In Section 3.1 we introduce the signature of non-cocompact NEC groups and show how it is connected to the related marked polygon and group presentation. In Section 3.2, we study the orbit space $\mathbf{H}/\Gamma$. In Section 3.3, the conditions for the existence of type-preserving isomorphisms between NEC groups based on the signatures are given. In Section 3.4, we calculate the Euler characteristic $\chi(\Gamma)$ of the orbit space of Γ and based on that we obtain the signature of the canonical fuchsian group Γ^+, namely the fuchsian subgroup of index 2 in Γ. Finally, in Section 3.5 we classify non-compact Klein surfaces up to homeomorphism using a set of invariants calculated from the signature of the NEC group Γ for which the Klein surface is the orbit space. The Sections 3.1, 3.2 and 3.3 of this chapter belong to the paper [17] that is being prepared for publication.

- Chapter 4 is dedicated to the study of additional properties of the NEC groups. We introduce in section 4.1 the elementary NEC groups: similarly to the fuchsian groups, an NEC group Γ is called elementary if there exists a finite Γ-orbit in $\mathbf{H} \cup \partial\mathbf{H}$. We then obtain all the elementary NEC groups by adding reflections and glide-reflections to the elementary fuchsian groups. We deduce in section 4.2 the form of the limit set of an NEC group, that is the same as the limit set of its canonical fuchsian group. Then, based on this result, we justify and introduce the classification of NEC groups of first and second kind as in the case of fuchsian groups and we prove different results related to finitely generated NEC groups of first kind (e.g. its signature and the measure of the fundamental region).

- Finally, in chapter 5 we discuss the main conclussions and possible further developments. A similar description as the one developed in this thesis for non-cocompact NEC groups, is well known in the case of non-cocompact fuchsian groups and cocompact NEC groups and has been an extremely useful tool for the study of non-compact Riemann surfaces and compact Klein surfaces. The results of this thesis may hopefully contribute to the study of the non-compact Klein surfaces, a topic about which not much is known.

Chapter 1

Preliminaries

In this chapter we provide an account of the most important concepts that support the results of the thesis. The aim is to show the beauty of the connections between Algebra, Geometry and Topology materialized in the study of groups of isometries of the hyperbolic plane, rather than to proceed with a formal encyclopaedic overview of the several topics involved, that otherwise are extensively and nicely available in the literature. The material in this chapter is standard and the references will be given along the way.

1.1 Motivation: groups of homeomorphisms of metric spaces

Let $(\mathbf{X}, d)$ be a metric space. The fact that we define a function $d : \mathbf{X} \times \mathbf{X} \longrightarrow \mathbb{R}$ which is a metric has deep consequences in the properties and mathematical tools that we can apply to the study of the set $\mathbf{X}$. First of all, the metric induces a topology τ, so that we can define the topological space $(\mathbf{X}, \tau)$ where the topology τ is defined as usual by the basis of open balls of X, $B(x, r) := \{y \in X | d(x, y) < r\}$. This allows us to use topological tools in the study of $\mathbf{X}$ and in analyzing its properties. A basic topologic notion that will be used throughout the thesis is the notion of *path*, i.e. a continuos map $\gamma : \mathbf{I} = [a, b] \subset \mathbb{R} \rightarrow \mathbf{X}$. The following list summarizes the most relevant properties used in this dissertation:

1. A metric space is said *connected* if it is not the union of two nonempty open sets. A metric space is *locally connected* if for each point $x \in \mathbf{X}$ and each neighborhood $\mathbf{U}$ of x,

there is a conected open set $\mathbf{V}$ such that $x \in \mathbf{V} \subseteq \mathbf{U}$.

2. A metric space is said *path-connected* if for every pair of points $x, y \in \mathbf{X}$ there is a path from x to y in $\mathbf{X}$. A metric space is *locally path-connected* if for each point $x \in \mathbf{X}$ and for each neighborhood $\mathbf{U}$ of x, there is a path-conected open set $\mathbf{V}$ such that $x \in \mathbf{V} \subseteq \mathbf{U}$.

3. A metric space is said *simply connected* if it is path connected and its fundamental group is trivial. The space is *locally simply connected* if for each point $x \in \mathbf{X}$ and for each neighborhood $\mathbf{U}$ of x, there is a simply connected open set $\mathbf{V}$ such that $x \in \mathbf{V} \subseteq \mathbf{U}$.

Secondly, the distance d enriches the algebraical properties of the set $\mathbf{X}$. An *isometry* is a map $f : \mathbf{X} \longrightarrow \mathbf{X}$ such that $d(x, y) = d(f(x), f(y)), \forall x, y \in X$; we define the group *Isom*($\boldsymbol{X}$) of isometries of the set $\mathbf{X}$ with the composition as a group operation. The elements of $\mathrm{Isom}(\mathbf{X})$ are called *motions* of the metric space. The following three definitions are central in this thesis:

Definition 1.1. Let $\Gamma \subset \mathrm{Isom}(\mathbf{X})$, a set $\mathbf{R} \subset \mathbf{X}$ is called *fundamental region* for Γ if:

1. $\mathbf{R}$ is open in $\mathbf{X}$,

2. $S\mathbf{R} \cap T\mathbf{R} = \varnothing$ for $S, T \in \Gamma$, $S \neq T$,

3. $\mathbf{X} = \cup\{T\bar{\mathbf{R}} \, : \, T \in \Gamma\}$, with $\bar{\mathbf{R}} = \mathbf{R} \cup \partial\mathbf{R}$.

If the set $\mathbf{R}$ only verifies the property 3, then we say that $\mathbf{R}$ is a Γ-*covering.*

Definition 1.2. Let $\Gamma \subset \mathrm{Isom}(\mathbf{X})$, a set $\mathbf{D} \subset \mathbf{X}$ is called *fundamental domain* for Γ if D is a connected fundamental region for Γ.

Definition 1.3. Let $\mathbf{R}$ be a fundamental region for $\Gamma \subset \mathrm{Isom}(\mathbf{X})$. $\mathbf{R}$ is called *locally finite* if the family $\{T\bar{\mathbf{R}} : T \in \Gamma\}$ is locally finite, meaning that for every $x \in \mathbf{X}$, there is a neighborhood U of x that intersects $T\bar{\mathbf{R}}$ only for finitely many T.

Of course, at this moment, the metric space is too general and we can ensure neither the existence, nor specific nice properties of the fundamental regions and domains.

The metric allows us to introduce a natural classification of the elements of $\mathrm{Isom}(\mathbf{X})$ by means of the so-called *displacement function*, $f : \mathrm{Isom}(\mathbf{X}, d) \to \mathbb{R}$, such that for all $T \in \mathrm{Isom}(\mathbf{X})$, $f(T) = \underset{x \in \mathbf{X}}{inf}\, d(Tx, x)$. Then, we classify the isometries as follows:

- *elliptic*, if the infimum is attained and is zero,
- *hyperbolic*, if the infimum is attained and greater than zero,
- *parabolic*, if the infimum is not attained.

Example 1.4. *We apply the abstract classification of motions to the metric space* $(\mathbb{R}, d_{\mathbb{R}})$*, with the usual euclidean distance in the line given by* $d_{\mathbb{R}}(x, y) = |x - y|, x, y \in \mathbb{R}$*. The orthogonal group* $O(1)$ *is* $\{\pm 1\}$ *and so the group of isometries of* $\mathbb{R}$ *is given by two motions of the form* $R(x) = x + k, S(x) = -x + k, k \in \mathbb{R} - \{0\}$*. Now,* S *has a fixed point* $\frac{k}{2}$ *and therefore the infimum is attained and is zero, so* S *is an elliptic motion. In case of* R*, there is no fixed point and the infimum is* $k \neq 0$ *and attained and so the motion is abstract hyperbolic.*

Example 1.5. *The upper half-plane model of the hyperbolic geometry is the set* $\boldsymbol{H} = \{z \in \mathbb{C} : Im(z) > 0\}$*, together with the hyperbolic distance* $d_{\boldsymbol{H}}$ *given by*

$$cosh\, d_{\boldsymbol{H}}(z, w) = 1 + \frac{|z - w|^2}{2\, Im(z).Im(w)}.$$

The group $Isom(\boldsymbol{H}, d_{\boldsymbol{H}})$ *of isometries of the hyperbolic plane is given by the orientation preserving isometries, called Möbius transformations,*

$$z \mapsto \frac{az + b}{cz + d},$$

with $a, b, c, d \in \mathbb{R}$ *and* $ad - bc = 1$*, together with the orientation reserving isometries:*

$$z \mapsto \frac{a\bar{z} + b}{c\bar{z} + d},$$

with $a, b, c, d \in \mathbb{R}$ *and* $ad - bc = -1$*.*
From the definition of the distance, it is clear that elliptic and reflections are classified as elliptic motions, since in this case a fixed point in $\boldsymbol{H}$ *means* $d_{\boldsymbol{H}}(z, Rz) = 0$*, for* R *elliptic or a reflection. Similarly, a parabolic transformation* S *has a unique fixed point in* $\mathbb{R}$ *and therefore the infimum is* 0 *but not attained in* $\boldsymbol{H}$ *and, as expected,* S *is an abstract parabolic motion. The hyperbolic motion* T *is conjugated to a transformation* $z \mapsto \lambda z, \lambda > 1$ *and so the hyperbolic*

distance between z and λz is

$$cosh\, d_{\boldsymbol{H}}(z, Tz) = 1 + |z|^2 \frac{(1-\lambda)^2}{2\lambda\, Im(z)^2}.$$

The infimum is then attained as it happens when $Re(z) = 0, Im(z) \neq 0$ and so we deduce that we have an abstract hyperbolic motion. Finally, for a glide-reflection D, the approach is similar to the hyperbolic transformations: if D is a glide reflection, then D^2 is hyperbolic and, if we take the conjugate of D^2 as $\lambda z, \lambda > 1$, then we can write $D(z) = -\sqrt{\lambda}\bar{z}$ and so the distance between z and Dz will be

$$cosh\, d_{\boldsymbol{H}}(z, Dz) = 1 + \frac{|z + \sqrt{\lambda}\bar{z}|^2}{2\sqrt{\lambda}\, Im(z)^2},$$

again we get the condition $Re(z) = 0, Im(z) \neq 0$, so the glide-reflections are abstract hyperbolic motions.

The action of a subgroup Γ of Homeo($\mathbf{X}$), in particular a subgroup of Isom($\mathbf{X}$), and its orbit spaces $\mathbf{X}/\Gamma$ are also key concepts in this thesis. The main notions are introduced below:

Definition 1.6. Let Γ be a subgroup of Homeo($\mathbf{X}$). The action of Γ on $\mathbf{X}$ is the map

$$\Gamma \times \mathbf{X} \to \mathbf{X},\ (T, x) \mapsto Tx.$$

We say that an action of Γ on $\mathbf{X}$ is *free* if $Tx \neq x$ for all $x \in \mathbf{X}, T \in \Gamma - \{e\}$. We say that Γ acts *properly discontinuosuly* on $\mathbf{X}$ if each $x \in \mathbf{X}$ has a neighborhood $\mathbf{U}$ of x in $\mathbf{X}$ such that $T\mathbf{U} \cap \mathbf{U} \neq \varnothing$ for only finitely many elements in Γ. The following definition introduces the key notion of *topological group*:

Definition 1.7. Let Γ be a group. A topological group $G = (\Gamma, \tau)$ is a topological space, with Γ a group and the topology defined such that the group operation $\Gamma \times \Gamma \to \Gamma, (S, T) \mapsto ST$ and the inversion map $\Gamma \to \Gamma, S \mapsto S^{-1}$ are continuous.

Another basic algebraical notion used systematically in this work is the concept of *orbit of a point by a group*:

Definition 1.8. Let $(\mathbf{X}, d)$ be a metric space and Γ a group acting on $\mathbf{X}$. For each $x \in \mathbf{X}$ we say that

$$\Gamma x = \{Tx \ : \ T \in \Gamma\}$$

is the Γ-*orbit* of x.

We define the relation $x \sim y$ if $\Gamma x = \Gamma y$, which can be easily proved as an equivalence relation in $\mathbf{X}$. We will denote the quotient $\mathbf{X}/\Gamma$ to the orbit space of $\mathbf{X}$ by the equivalence relation above and the elements of the quotient, the representative in the orbit of an element $x \in \mathbf{X}$ will be denoted either by $\bar{x}$ or Γx depending on which notation suits better to the context.

Thirdly, the metric has obviously a geometrical meaning: with the help of the metric, we can introduce in a set $\mathbf{X}$ the elementary geometric notions, namely *length of a path, lines (geodesic)* and *angles.* First of all, the length of a path can be defined:

Definition 1.9. Let $(\mathbf{X}, d)$ be a metric space and $\gamma : [a, b] \subset \mathbb{R} \to \mathbf{X}$ a path in X. We call length of γ to

$$l(\gamma) := sup\{\sum_{i=0}^{n} d(c(t_{i+1}), c(t_i)) \ : \ t_i \in \mathbb{R}, t_0 = a, t_{n+1} = b, t_i \leqslant t_{i+1}, i = 0, ..., n+1\}.$$

If $l(\gamma)$ is finite, the path γ is called *rectificable.* A general metric space even path-connected may not admit rectificable paths between every pair of points.

An *isometric embedding* of $[a, b] \subset \mathbb{R}$ in a metric space $(\mathbf{X}, d)$ is a map $c : \mathbf{I} = [a, b] \subseteq \mathbb{R} \to \mathbf{X}$ such that there exists $x \in \mathbb{R}$ and $d(c(x), c(y)) = \lambda|x - y|$, where $y, \lambda \in \mathbb{R}$. The role of the traditional geometrical concept of line is in the case of metric spaces played by the geodesic:

Definition 1.10. Let $(\mathbf{X}, d)$ be a metric space.

- a *geodesic* of length $L \in \mathbb{R}, L \geqslant 0$ in $\mathbf{X}$ is an isometric embedding $c : [0, L] \to \mathbf{X}$ and we call the image $c([0, L])$ a *geodesic segment* in $\mathbf{X}$ between $c(0)$ and $c(L)$. The isometric embedding $\mathbb{R} \to \mathbf{X}$ is a geodesic line.

- a subset $\mathbf{C} \subset \mathbf{X}$ is *convex* if for each pair $x, y \in \mathbf{C}$ the geodesic segment joining x and y is contained in $\mathbf{C}$.

In a metric space the geodesics do not always exist. If every pair $x, y \in \mathbf{X}$ can be connected by a geodesic, then the metric space is called *geodesic space.*

Example 1.11. *In this example we show how the definition above can be applied in a general metric space. Let us define the metric space* $(C[0,1], d)$ *with* $C[0,1]$ *the set of continuous functions on* $[0,1]$ *taking values in* $\mathbb{R}$ *and* d *the supremum metric*

$$d(f,g) = \underset{x\in[0,1]}{sup} |f(x) - g(x)|.$$

The family of functions $f_t(x) = (1-t)x, g_t(x) = (1-t)x + t, t \in [0,1]$ *are geodesics in* $(C([0,1]), d)$*, as it can be easily checked using directly the definition:*

$$d(f_t(x), f_t(y)) = \underset{x\in[0,1]}{sup} |f_t(x) - f_t(y)| = \underset{x\in[0,1]}{sup} (1-t)|x-y| = |x-y|,$$

and similarly

$$d(g_t(x), g_t(y)) = \underset{x\in[0,1]}{sup} |g_t(x) - g_t(y)| = \underset{x\in[0,1]}{sup} (1-t)|x-y| = |x-y|.$$

Finally, we introduce the abstract notion of angle in a metric space. Let v, x, y be three distinct elements of $\mathbf{X}$. We call *comparison triangle* of (v, x, y) to a triangle in $\mathbb{R}^2$ with vertices $\bar{v}, \bar{x}, \bar{y}$ such that $|\bar{x} - \bar{y}| = d(x,y)$, $|\bar{v} - \bar{x}| = d(v,x)$ and $|\bar{v} - \bar{y}| = d(v,y)$. It can be proved that such a triangle always exists. We denote this triangle $\Delta(v,x,y)$. Applying the law of cosines to the euclidean triangle with vertices $\bar{v}, \bar{x}, \bar{y}$, we see that the angle $\angle_{\bar{v}}\bar{x}\bar{y}$ between the segments $[\bar{v}, \bar{x}]$ and $[\bar{v}, \bar{y}]$ is

$$\angle_{\bar{v}}\bar{x}\bar{y} = arccos\frac{|\bar{v}-\bar{x}|^2 + |\bar{v}-\bar{y}|^2 - |\bar{x}-\bar{y}|^2}{2|\bar{v}-\bar{x}||\bar{v}-\bar{y}|},$$

and therefore the comparison angle between x and y at v is defined as

$$\angle_v xy = arccos\frac{d(v,x)^2 + d(v,y)^2 - d(x,y)^2}{2d(v,x)d(v,y)}.$$

This leads us to define the abstract notion of angle between two geodesics of a metric space, the *Alexandrov angle*:

Definition 1.12. Let $(\mathbf{X}, d)$ be a metric space and $\alpha : [0,a] \to \mathbf{X}$ and $\beta : [0,b] \to \mathbf{X}$ be two geodesics. Given $t \in (0,a]$ and $s \in (0,b]$ consider the comparisson triangle $\Delta(v, \alpha(t), \beta(s))$ and the comparisson angle $\angle_v \alpha(t)\, \beta(s)$. The Alexandrov angle between α and β at v is the number $\angle_v(\alpha(t)\, \beta(s)) \in [0, \pi]$ defined by

$$\angle_v(\alpha, \beta) = \lim_{s,t \to 0} sup \angle_v \alpha(t)\, \beta(s).$$

Example 1.13. *Let us obtain the Alexandrov's angle for the geodesics of the metric space* $(C[0,1], d)$ *as in the example 1.10. We start with the comparison triangle at* $\Delta(f_0(x), f_t(x), g_{t'}(x))$*: we have then a triangle of sides of length* $d(f_0(x), f_t(x)) = t$, $d(f_0(x), g_{t'}(x)) = t'$ *and* $d(f_t(x), g_{t'}(x)) = max\{t, t'\}$. *A triangle with sides* $t, t', max\{t, t'\}$ *can be transformed from an equilateral one, and so the comparison angle is* $\angle_{f_0(x)} f_t(x) g_{t'}(y) = \pi/3$ *to a triangle where* $t >> t'$ *and so the triangle becomes an isosceles with one side much smaller than the other two, so that the comparisson angles approaches to* $\pi/2$. *We conclude then that the possible comparisson angles take any value in* $[\frac{\pi}{3}, \frac{\pi}{2})$. *The Alexandrov´s angle is then by the definition the lim sup of the possible values, namely*

$$\angle_{f_0(x)}(f_t(x), g_{t'}(y)) = \lim_{t,t' \to 0} sup \angle_{f_0(x)} f_t(x) g_{t'}(y) = \pi/2.$$

As just seen, the algebraical, topological and geometric properties induced by the metric in $\mathbf{X}$ are closely related. This explains the interrelations that arise in specific metric spaces that at first glance might not be obvious. Of course, these are not only materialized in the groups of isometries of the euclidean or hyperbolic spaces, $\mathbb{R}^n$ or $\mathbf{H}^n$, but has consequences and provide powerful mathematical insights in a broad collection of metric spaces.

1.2 Presentation of groups of isometries of simply connected spaces

In this section, we show how these connections mentioned above emerge in a determining way in simply connected spaces. Let $\mathbf{X}$ be such a space. Macbeath in [39] proved that the presentation of a subgroup of the group Isom($\mathbf{X}$) can be described in terms of fundamental regions $\mathbf{R} \subset \mathbf{X}$ of Γ. This reference together with Macbeath and Hoare [41] and Swan [57] are the main sources used in this section.

Let us start with a useful notion of group presentation introduced specifically for describing the group of isometries of a metric space. Let Γ be a group and let us define a Γ-*word* as a finite non-empty ordered set of elements of Γ. We denote the Γ-word $\{T_1, ..., T_n\}$ as $T_1 \cdot T_1 \cdot ... \cdot T_n$ with dots for distinguishing the Γ-word from the group element given by $T_1 T_1 ... T_n \in \Gamma$. We define the concatenation operation for the Γ-words in the usual way, such that given two Γ-words $W_1 = T_1 \cdot T_2 \cdot ... \cdot T_n$ and $W_2 = S_1 \cdot S_2 \cdot ... \cdot S_m$ by the relation $W_1 \cdot W_2 = T_1 \cdot T_2 \cdot ... \cdot T_n \cdot S_1 \cdot S_2 \cdot ... \cdot S_m$. With such an operation, the set of Γ-words is a semigroup $W(\Gamma)$, as the operation satisfies the associative property.

We define a Γ-*relation* as an unordered pair of Γ-words that we denote (W_1, W_2). Let $R(\Gamma)$ be a set of Γ-relations. We say that the relation $(S \cdot W_1 \cdot T, S \cdot W_2 \cdot T)$ with $S, T \in W(\Gamma)$ is *implied* by the relation (W_1, W_2). A relation (W_1, W_2) should be seen here as the identification of the Γ-words W_1 and W_2. Two words $S, T \in W(\Gamma)$ are said to be $R(\Gamma)$-*equivalent* (or R-equivalent) and we write $S \sim T$, if there is a finite sequence of Γ-words $S = W_1, ..., W_n = T$ such that each of the relations (W_{i-1}, W_i) is implied by a relation in $R(\Gamma)$. It is clear then that being R-equivalent is an equivalence relation:

1. $\forall S \in W(\Gamma)$, $S \sim S$, inmediate just considering the meaning of the relation as an identification,

2. $\forall S, T \in W(\Gamma)$, $S \sim T \Rightarrow T \sim S$, that again is trivially deduced from the understanding of a relation as an identification of words,

3. $\forall S, T, U \in W(\Gamma)$, $S \sim T$ and $T \sim U$, then $S \sim U$. This means that there exist two finite sequence of Γ-words $W_{1i}, W_{2j} \in R(\Gamma), i = 1, ..., n, j = 1, ..., m$ such that $S = W_{11}, ..., W_{1n} = T$ and $T = W_{21}, ..., W_{2m} = U$ and $(W_{1,i-1}, W_{1i}), (W_{2,j-1}, W_{2j})$ are implied by relations in $R(\Gamma)$, for $i = 2, ..., n, j = 2, ..., m$ and therefore $S = W_{11}, ..., W_{1n} = T, W_{22}, ..., W_{2m} = U$ is a finite sequence of words such that two consecutive relations is implied by a relation in $R(\Gamma)$, i.e. $S \sim U$.

The R-relation defines then a congruence in the semigroup $W(\Gamma)$ and we can define in the set of congruence classes the binary operation $\circ$ as $\bar{S} \circ \bar{T} = \overline{S \cdot T}$, such that the R-equivalence classes define a semigroup that we denote as $W(\Gamma)/R(\Gamma)$.

Let $\phi : W(\Gamma) \to \Gamma$ be a map such that for all relations $[W_1, W_2]$ in $R(\Gamma)$, $\phi(W_1) = \phi(W_2)$. Then, ϕ maps R-equivalent elements of $W(\Gamma)$ to the same element in Γ and it thus defines a homomorphism of $W(\Gamma)/R(\Gamma)$ into Γ. If this homomorphism is an isomorphism, then we call $\langle W(\Gamma), R(\Gamma)\rangle$ a presentation of Γ.

Let Γ be now a subgroup of Isom($\mathbf{X}$) and let us associate to each word in $W_1 \cdot W_2 \cdot ... \cdot W_n \in W(\Gamma)$ a group element $T = W_1 W_2 ... W_n \in \Gamma$. Let $\mathbf{F}$ be a locally finite Γ-covering of $\mathbf{X}$, G the subset of Γ, $G = \{T \in \Gamma \,:\, \mathbf{F} \cap T\mathbf{F} \neq \emptyset\}$ and R the set of G-relations $R = \{[S.T, ST] \,:\, S, T \in \Gamma, \mathbf{F} \cap S\mathbf{F} \cap ST\mathbf{F} \neq \emptyset\}$. Then, we have the following classical result due to Macbeath:

Theorem 1.14. *Let $(\boldsymbol{X}, d)$ a metric space and let Γ and $\boldsymbol{F}$ as above.*

1. *If $\boldsymbol{X}$ is connected, then G generates Γ,*
2. *if $\boldsymbol{F}$ is path-connected and $\boldsymbol{X}$ is connected and simply connected, then $\langle G, R\rangle$ is a presentation for Γ.*

1.3 Groups of hyperbolic isometries

The concepts and results presented in previous sections are applied in this thesis to the hyperbolic plane. For that, we mainly use the *Poincaré upper half-plane model*, which wass introduced already in the example 1.5. However, sometimes the *Poincaré disk model* will also be considered.

In summary, in the upper half-plane model of the hyperbolic geometry the points of the hyperbolic plane are the points $\mathbf{H} = \{z \in \mathbb{C} \,:\, Im(z) > 0\}$, together with the hyperbolic distance $d_{\mathbf{H}}$ given by

$$cosh\, d_{\mathbf{H}}(z, w) = 1 + \frac{|z - w|^2}{2\, Im(z) Im(w)},$$

where the Riemann metric for the distance is $(ds)^2 = \dfrac{(dx)^2 + (dy)^2}{y^2}$. The geodesic between two points $z, w \in \mathbf{H}$ is either the arc of an euclidean circle with center in $\mathbb{R}$ or a segment of an euclidean line perpendicular to $\mathbb{R}$. The set Isom($\mathbf{H}$) as defined in the example above is isomorphic to the projective linear group $PGL(2, \mathbb{R})$ and the related topology is given by the numbers (a, b, c, d) linked to each transformation.

The points of the hyperbolic plane in the Poincaré disk model are the points $\mathbf{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the hyperbolic distance given by

$$cosh\, d_{\mathbf{D}}(z,w) = 1 + 2\frac{|z-w|^2}{(1-|z|^2)(1-|w|^2)},$$

where the Riemann metric for the distance is $(ds)^2 = \dfrac{(dx)^2+(dy)^2}{1-(x^2+y^2)}$. The geodesics are again arcs of euclidean circles orthogonal to the boundary circle or diameters of the boundary circle. The set Isom($\mathbf{D}$) consists of all maps of the form

$$z \mapsto \frac{az+b}{\bar{b}z+\bar{a}},\ z \mapsto \frac{a\bar{z}+b}{\bar{b}\bar{z}+\bar{a}},$$

with $a\bar{a} - b\bar{b} = 1$. There is a one-to-one correspondence between the upper half-plane and the disk model via the Cayley transformation

$$C : \mathbf{H} \to \mathbf{D}\ :\ z \mapsto \frac{z-i}{z+i},$$

which makes Isom($\mathbf{D}$) isomorphic to Isom($\mathbf{H}$).

A non-Euclidean crystallographic (NEC) group is a discrete subgroup Γ of Isom($\mathbf{H}$). Being discrete, Γ acts discontinuously on $\mathbf{H}$. We say that Γ is cocompact if the orbit space $\mathbf{H}/\Gamma$ is compact, otherwise we say that Γ is non-cocompact. In what follows, by an NEC group we mean a finitely generated non-cocompact NEC group, unless otherwise stated. If Γ consists of orientation preserving elements then Γ is a Fuchsian group. As stated, in this thesis we focus on (non-cocompact) *proper* NEC groups, that is, groups containing orientation reversing elements.

Definition 1.15. Let Γ be an NEC group. A subset $\mathbf{F} \subset \mathbf{H}$ is called a *fundamental region of* Γ if it is closed, convex and satisfies:

1. $\bigcup_{T \in \Gamma} T\mathbf{F} = \mathbf{H}$,

2. $\mathring{\mathbf{F}} \cap T\mathring{\mathbf{F}} = \varnothing$, for all $T \in \Gamma - \{I\}$,

where $\mathring{\mathbf{F}}$ is the interior of $\mathbf{F}$ in $\mathbf{H}$. The closure of $\mathbf{F}$ in $\overline{\mathbf{H}} = \mathbf{H} \cup \mathbb{R} \cup \{\infty\}$ will be denoted by $\overline{\mathbf{F}}$, and its boundary, with a slight abuse of notation, by $\partial\mathbf{F} = \overline{\mathbf{F}} - \mathring{\mathbf{F}}$. The family of images of

$\mathbf{F}$ given by $\{T\mathbf{F} : T \in \Gamma\}$ is called a tessellation. The images of $\mathbf{F}$ by Γ are called faces of the tessellation.

Let $v \in \mathbf{H}$ be a point not fixed by any $T \in \Gamma - \{I\}$. The Dirichlet region of Γ with center v is defined as $D_v(\Gamma) = \{z \in \mathbf{H} \mid \rho(z, v) \leqslant \rho(z, Tv)$ for all $T \in \Gamma - \{I\}\}$, where ρ is the hyperbolic distance. As shown in [41], any Dirichlet region of an NEC group Γ (cocompact or not) is a closed and convex fundamental region whose associated tessellation is locally finite. Moreover $\mathbf{D}_v(\Gamma)$ is compact in $\mathbf{H}$ if and only if the orbit space $\mathbf{H}/\Gamma$ is compact.

The boundary $\partial\mathbf{D}_v$ of a Dirichlet region is a sequence of hyperbolic segments in $\mathbf{H}$ and segments in $\partial\mathbf{H} = \mathbb{R} \cup \{\infty\}$ called *edges*. The intersection between two consecutive edges is called a *vertex*. If an edge contains a fixed point of an elliptic element of order two then we also call such a point a vertex, and call edges the two hyperbolic segments of $\partial\mathbf{D}_v$ (both in the same $\mathbf{H}$-line) which are permuted by the elliptic element. A *free edge* is an edge in $\partial\mathbf{H}$, and a vertex in $\partial\mathbf{H}$ is a *vertex at infinity*. A vertex at infinity is called *improper* if it belongs to a free edge, and *proper* otherwise.

An NEC group is called geometrically finite if it admits a fundamental region $\mathbf{F}$ with finitely many edges. By Proposition 4.11.2 in [60] the group Γ is finitely generated if and only if Γ has a fundamental region with only a finite number of neighbours. A neighbour of $\mathbf{F}$ is another fundamental region of the tessellation having at least one boundary edge in common with $\mathbf{F}$. Therefore, a fundamental region $\mathbf{F}$ with a finite number of neighbours can only have a finite number of such boundary edges and so a finite number of edges. We deduce then that finitely generated groups have fundamental regions with finitely many edges.

We follow the usual method for associating surface symbols to fundamental regions. The edges of a fundamental region are paired by the elements of the NEC group, except for the edges fixed by reflections and the free edges, which are paired with no other edge. If two edges are paired by an orientation preserving element then they are labelled as p and p', and if the element reverses orientation then they are labelled as p and p^*. An edge will be labelled using a lower case letter and a sequence of consecutive edges will be labelled with a capital letter. The point of intersection of two consecutive edges a and b will be denoted by (a, b).

Going through the boundary of a fundamental region counter-clockwise, we can associate an initial and an end point to each edge. When an edge p is paired with a different edge p' by an orientation preserving transformation then the initial point of p is mapped to the end point of p' (and the end point of p to the initial point of p'). If the transformation reverses the orientation then the initial point of p is mapped to the initial point of its paired edge p^* (and the end point of p to the end point of p^*).

1.3.1 Cocompact NEC groups

In order to obtain the presentation of an NEC group Γ, we just need to apply Theorem 1.14 to a fundamental region of Γ. For doing that, Wilkie [59] built a fundamental region with a canonical surface symbol from a Dirichlet region of Γ and transforming it by cutting and pasting pieces. The detailed description of these transformations will be done for the non-cocompact case in section 2.1 including sides and vertex at infinity and will not be done here for the cocompact NEC groups. Labeling the edges of a fundamental region anticlockwise yields a surface symbol. The main results obtained by Wilkie [59], Macbeath [40] and Singerman [54] are summarized below.

Theorem 1.16. *A finitely generated cocompact NEC group admits a fundamental region with a surface symbol of one of the following forms:*

(1) $\prod_{i=1}^{r} x_i x_i' \prod_{i=1}^{g} a_i b_i' a_i' b_i \prod_{i=1}^{k} e_i C_i e_i',$

(2) $\prod_{i=1}^{r} x_i x_i' \prod_{i=1}^{g} d_i d_i^* \prod_{i=1}^{k} e_i C_i e_i',$

where each C_i is a sequence of edges fixed by reflections. Linked to this surface symbol, the presentation of the group is defined by the following theorem:

Theorem 1.17. *A finitely generated cocompact NEC group Γ admits the following presentation: it has generators*

(a) X_i, $i = 1, ..., r$ *(elliptic elements),*

(c) A_i, B_i, $i = 1, ..., g$ *(hyperbolic translations) if the orbit space* $\mathbf{H}/\Gamma$ *is orientable, or*

D_i, $i = 1, \ldots, g$ *(glide reflections) otherwise,*

(d) E_i, $i = 1, \ldots, k$, *(orientation preserving elements, usually hyperbolic elements),*

(e) C_{ij}, $i = 1, ..., k, j = 0, ..., k_i$, *(reflections),* *and relations:*

(i) $X_i^{m_i} = 1$, *for* $i = 1, \ldots, r$,

(ii) $E_i C_{ik_i} E_i^{-1} C_{i0} = 1$ *for* $i = 1, \ldots, k$,

(iii) $C_{ij}^2 = 1$, *for all the reflections,*

(iv) $(C_{i,j-1} C_{ij})^{n_{ij}} = 1$ *for* $i = 1, \ldots, k,\ j = 1, \ldots, k_i$;

(v) $\prod_{i=1}^{r} X_i \prod_{j=1}^{g} [A_j, B_j] \prod_{i=1}^{k} E_i = 1$, *if the orbit space* $\mathbf{H}/\Gamma$ *is orientable, or*

$\prod_{i=1}^{r} X_i \prod_{j=1}^{g} D_j^2 \prod_{i=1}^{k} E_i = 1$, *otherwise.*

To the presentation above we can assign a signature which is an ordered set of integers and symbols that identify the group up to isomorphisms. Macbeath introduced the signature $s(\Gamma)$ of a cocompact NEC group Γ as

$$s(\Gamma) = (g; \pm; [m_1, , ..., m_r]; \{(n_{11}, ..., n_{1k_1}), ..., (n_{k1}, ..., n_{kk_k})\}),$$

where the integers $m_i \geqslant 2$ are called proper periods, $n_{ij} \geqslant 2$ are the linked periods, $(n_{i1}, ..., n_{ik_i})$ are the period cycles and g is the orbit genus.

Two NEC groups are called *geometrically isomorphic* if there is a homeomorphism $t : z \to z'$ of $\mathbf{H}$ and a group isomorphism $\phi : \Gamma \to \Gamma'$ such that $\phi(T) = tTt^{-1}$ for all $T \in \Gamma'$. Macbeath [40, Theorem 3] showed that geometrical isomorphism and algebraical isomorphism are equivalent for cocompact NEC groups. Singerman [54] proved the following result that shows that an algebraical isomorphism between cocompact NEC groups is always type-preserving.

Theorem 1.18. *An isomorphism ϕ between two cocompact NEC groups Γ and Γ' is type-preserving, i.e. elliptics, hyperbolics, reflections and glide reflections in Γ are mapped respectively to elliptics, hyperbolics, reflections and glide reflections in Γ'.*

With the support of the signatures, Macbeath proved and presented the necessary and sufficient conditions of two cocompact NEC groups for being algebraically isomorphic:

Theorem 1.19. *Let* Γ *be a cocompact NEC group with signature*

$$s = s(\Gamma) = (g; \pm; [m_1, , ..., m_r]; \{C_1, ..., C_k\}),$$

and Γ' *another cocompact NEC group with signature*

$$s' = s'(\Gamma') = (g'; \pm; [m'_1, , ..., m'_{r'}]; \{C'_1, ..., C'_{k'}\}).$$

Then, Γ *and* Γ' *are isomorphic as abstracts groups if and only if*

1. $sign(s) = sign(s')$,

2. $g = g'$, $r = r'$, $k = k'$ *and* $k_i = k'_i$ *for* $i = 1, ..., k$,

3. $m_i = m'_{\pi(i)}$ *for a permutation* π *of* $\{1, ..., r\}$,

4. *if* $sign(s) =$ " $+$ " *then there exists a permutation* π *of* $\{1, ..., k\}$ *such that either* C'_i *is a cyclic permutation of* $C_{\pi(i)}$ *for each* $i \in \{1, ..., k\}$ *or* C'_i *is a cyclic permutation of the inverse of* $C_{\pi(i)}$ *for each* $i \in \{1, ..., k\}$,

5. *if* $sign(s) =$ " $-$ " *then there exists a permutation* π *of* $\{1, ..., k\}$ *such that* C'_i *is a cyclic permutation of either* $C_{\pi(i)}$ *or of the inverse of* $C_{\pi(i)}$, *for each* $i \in \{1, ..., k\}$.

In addition to the algebraical (group presentation) and geometrical (surface symbol) information, the signature carries also topological information of the canonical projection $f : \mathbf{H} \to \mathbf{H}/\Gamma$. For all $z \in \mathbf{H}$, the canonical projection f behaves locally $z \to z^m$, $m \in \mathbb{N}$ except for points fixed by reflections. We call m the ramification index at z. If $m > 1$, then we say that f is ramified at z. Then, the orbit space $\mathbf{H}/\Gamma$ is classified topologically up to homeomorphisms as:

Theorem 1.20. *Let* Γ *be an NEC group with signature*

$$s(\Gamma) = (g; \pm; [m_1, , ..., m_r]; \{(n_{11}, ..., n_{1k_1}), ..., (n_{k1}, ..., n_{kk_k})\}),$$

and let $S = \boldsymbol{H}/\Gamma$. *Then,*

1. $sign(s) =$ " $+$ " *if anf only if* S *is orientable,*

2. *the integers $m_1, ..., m_r$ are the ramification indices with respect to the canonical projection $\boldsymbol{H} \to \boldsymbol{H}/\Gamma$ of the r conic points lying in the interior of S,*

3. *the integers $n_{i1}, ..., n_{kk_i}$ are the ramification indices with respect to the canonical projection $\boldsymbol{H} \to \boldsymbol{H}/\Gamma$ of the k_i corner points lying on the i-th connected component of the boundary S.*

An arbitrary set of numbers and symbols s defines the signature of an NEC group Γ if and only if the rational number

$$\mu(\Gamma) = \eta g + k - 2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k_i}(1 - \frac{1}{n_{ij}}),$$

is positive, where $\eta = 2$, if $sign(s)``+"$ and $\eta = 1$ otherwise. The hyperbolic area of a fundamental region of Γ is $2\pi\mu(\Gamma)$. Also, if Γ' is a subgroup of Γ of finite index, then Γ' is an NEC group and

$$[\Gamma : \Gamma'] = \mu(\Gamma')/\mu(\Gamma),$$

which is the Riemann-Hurwitz formula associated to the covering $\mathbf{H}/\Gamma' \to \mathbf{H}/\Gamma$.

If Γ is an NEC group that contains orientation reversing elements, then it is called a proper NEC group. In a proper NEC group there is a fuchsian group Γ^+ of index two called *canonical fuchsian group*, namely $\Gamma^+ = \Gamma \cap$ Isom$^+(\mathbf{H})$, with Isom$^+(\mathbf{H})$ being the subgroup of all orientation preserving isometries of Isom$(\mathbf{H})$. We can also write $\Gamma = \Gamma^+ \cup T\Gamma^+$ where $T \in \Gamma - \Gamma^+$. The signature of the canonical fuchsian group Γ^+ of a proper NEC group Γ was obtained by Singerman [54]:

Theorem 1.21. *Let Γ be an NEC group with signature*

$$s = s(\Gamma) = (g; \pm; [m_1, , ..., m_r]; \{(n_{11}, ..., n_{1k_1}), ..., (n_{k1}, ..., n_{kk_k})\}),$$

then Γ^+ has the fuchsian signature

$$s(\Gamma^+) = (\eta g + k - 1; m_1, m_1, ..., m_r, m_r, n_{11}, ..., n_{kk_k}),$$

where, as before, $\eta = 2$, if $sign(s) = ``+"$ and $\eta = 1$ otherwise.

1.3.2 Non-cocompact NEC groups

This subsection presents the two main results published in the literature related to the topic of this thesis. The first one, due to Macbeath and Hoare [41], is a theorem that provides the algebraic structure of (not necessarily finitely generated) non-cocompact groups of hyperbolic isometries.

Theorem 1.22. *An NEC group with non-cocompact orbit space is the free product of cyclic groups, groups of the forms*

$$\langle C_0, C_1, ...; C_0^2 = C_1^2 = ... = (C_1C_0)^{n_0} = (C_2C_1)^{n_1} = ... = 1\rangle,$$

where $n_i > 1$ and the number of generators and relators is finite or infinite, and groups of the form

$$\langle ...C_{-1}, C_0, C_1, ...; ... = C_{-1}^2 = C_0^2 = C_1^2 = ... = (C_{-1}C_0)^{n_{-1}} = (C_1C_0)^{n_0} = (C_2C_1)^{n_1} = ... = 1\rangle$$

and

$$\langle C_1, ..., C_r, E; C_1^2 = ... = C_r^2 = (C_1C_2)^{n_0} = ... = (C_{r-1}C_r)^{n_{r-1}} = 1\rangle,$$

where $r \geqslant 1$ and $n_i > 1$. A fuchsian group with non-cocompact orbit space is the free product of cyclic groups.

As it can be seen in the theorem's statement, it is not straightforward to recognize the link between the generators and relations of this presentation and the geometry of the hyperbolic plane (for example a marked polygon), or the properties of the related orbit space. This implies that the usage of this algebraic result cannot be used easily to get further properties of these groups (e.g. topology of the orbit space).

The second result existing in the literature is due to Zieschang, Vogt and Coldewey [60, Theorem 4.11.5] and provides a presentation of finitely generated non-cocompact NEC groups. While it can be easier linked to the geometry of the orbit space, the theorem as well as additional properties were stated without proof and after the results presented in this work it turned that the presentation is incomplete.

Theorem 1.23. *A finitely generated NEC group* Γ *admits the following presentation: it has generators*

(a) X_i, $i = 1, ..., s$ *(elliptic elements),*

(b) A_i, B_i, $i = 1, ..., g$ *(hyperbolic translations) if the orbit space* $\mathbf{H}/\Gamma$ *is orientable, or*

D_i, $i = 1, \ldots, g$ *(glide reflections) otherwise,*

(c) E_i, $i = 1, \ldots, r_1$ *(parabolic elements),*

E_i, $i = r_1 + 1, \ldots, r_2$ *(hyperbolic translations),*

E_i, $i = r_2 + 1, \ldots, r_3$ *(hyperbolic translations),*

E_i, $i = r_3 + 1, \ldots, r$ *(boundary hyperbolic translations),*

(e) C_i, $i = r_1 + 1, ..., r_2$ *(reflections),*

C_{ij}, $i = r_2 + 1, ..., r_3$, $j = 1, \ldots, t_i$ *(reflections),*

C_{ijk}, $i = r_3 + 1, ..., r$, $j = 1, \ldots, u_i$, $k = 1, \ldots, u_{i,j}$ *(reflections),*

and relations:

(i) $X_i^{m_i} = 1$, *for* $i = 1, \ldots, s$,

(ii) $E_i C_{ik_i} E_i^{-1} C_{i0} = 1$ *for* $i = 1, \ldots, k$,

(iii) $C_i^2 = C_{ij}^2 = C_{ikj}^2 = 1$, *for all the reflections,*

(iv) $(C_{i,j,k-1} C_{ij})^{\check{n}_{i,j,k}} = 1$ *for* $i = r_1 + 1, \ldots, r_1 + r_2 + r_3,\ \ j = 2, \ldots, u_i;$

$(C_{i,j,k-1} C_{ijk})^{\hat{n}_{ijk}} = 1$ *for* $i = r_3 + 1, \ldots, r,\ \ j = 1, \ldots, t_i, \underset{c}{=}2, \ldots, t_i j$

(v) $\prod_{i=1}^{s} X_i \prod_{i=1}^{r} E_i \prod_{i=1}^{g} [A_i, B_i]$, *if the orbit space* $\mathbf{H}/\Gamma$ *is orientable, or*

$\prod_{i=1}^{s} X_i \prod_{i=1}^{r} E_i \prod_{i=1}^{g} [D_i^2]$, *otherwise.*

However, the presentation is incomplete since the authors did not take into account products of reflections with a common fixed point at infinity. In the orbit space these reflections yield non-compact boundary components with *semi-cusps*, that is, boundary components with either just one point at infinity or with two points at infinity each of which belongs to another boundary component. We will come to this point in Chapter 3. This changes totally the form of the group presentation, symbol surface and orbit space, as well as the related properties (signatures, group isomorphisms, etc.).

Chapter 2

Presentation of finitely generated non-cocompact NEC groups

This chapter is organized as follows. Given a generic finitely generated non-cocompact NEC group Γ, in section 2.1 we construct a fundamental region for Γ with a particular surface symbol. This surface symbol reflects geometric and topological properties of the fundamental region, and provides a canonical presentation by generators and relations of Γ. This is obtained in Section 2.2.

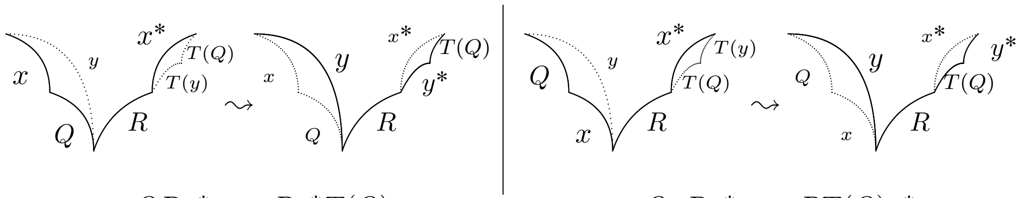
2.1 Surface symbols

A fundamental region of an NEC group can be transformed conveniently by cutting and pasting pieces. This yields a new fundamental region with a new surface symbol. We follow James in [30] and proceed in a similar way as Wilkie did in [59] for cocompact NEC groups to obtain a fundamental region with a canonical surface symbol. We will follow the notation as explained in section 1.3.

2.1.1 Transformation rules

Transformation rules I. Let x, x^* be a pair of labels, and let us write $x^* = T(x)$ with T an orientation reversing element. Then a sequence Q of consecutive edges on one side of x can be removed, provided $T(Q)$ is put on the same side of x^*. So if Q is on the right side of x then Q

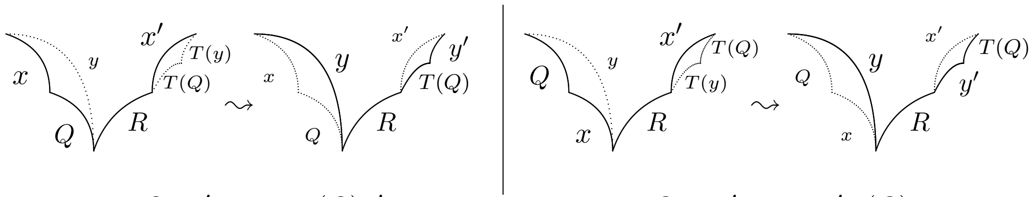
can be moved to the right side of x^*, and if Q is on the left side of x then Q can be moved to the left side of x^*, see Figure 21.



$xQRx^* \rightsquigarrow yRy^*T(Q)$ $\qquad$ $QxRx^* \rightsquigarrow yRT(Q)y^*$

Figure 21: Transformation rules Ia and Ib

Transformation rules II. Let x, x' be a pair of labels, and let us write $x' = T(x)$ with T an orientation preserving element. Then a sequence Q of consecutive edges on one side of x can be moved (without inversion) to the other side of x'. So if Q is on the right side of x then Q can be moved to the left side of x', and if Q is on the left side of x then Q can be moved to the right side of x', see Figure 22.



$xQRx' \rightsquigarrow yRT(Q)y'$ $\qquad$ $QxRx' \rightsquigarrow yRy'T(Q)$

Figure 22: Transformation rules IIa and IIb

The inverse of transformation rules I and II also apply.

Remark 2.1. *We observe that under transformation rules I and II, the initial and end points of the original sequence coincide with the initial and end points respectively of the transformed sequence.*

The canonical surface symbol is obtained in a series of steps by applying the above transformation rules. Although the method is familiar, the existence of vertices and edges at infinity makes it more cumbersome. We have to keep track of these vertices and edges.

First step. *Assemble all the pairs y, y' with no other label between them and obtain a surface symbol of the form*

$$\prod_i x_i x_i' \prod_i p_i p_i' L \tag{1.1}$$

where (x_i, x_i') and (p_i, p_i') are elliptic and parabolic vertices respectively, and L contains no sequence xx' or pp'. The surface symbol may contain labels x, x' with no other label between them. We use transformation rule IIb (with $R = \varnothing$) to move all such pairs to the front of the symbol. Observe that the vertex (x, x') is fixed by the orientation preserving transformation X which pairs x with x'. If (x, x') lies in $\mathbf{H}$ then X is elliptic; otherwise X is parabolic. In the latter case we write such pair as pp'. We first move the pairs such that the vertex (x, x') lies in $\mathbf{H}$ and then the pairs pp', where the vertex (p, p') is proper. This way we get (1.1).

Second step. *Assemble all the pairs d, d^* together and obtain a surface symbol of the form*

$$\prod_i x_i x_i' \prod_i p_i p_i' \prod_i d_i d_i^* \; M \tag{1.2}$$

where M contains no pair xx', pp' or dd^*. If L in (1.1) contains a pair of labels d, d^* then we assemble them together by using the transformation rule Ia (with $R = \varnothing$). Successive applications of transformation rules Ia and Ib allow us to assemble together all pairs d, d^* and write them in the front of L.

The sequence M in (1.2) may contain a sequence of the form $aBb'Ca'Db$, where B, C and D are labels of consecutive edges of the fundamental region. Such pair a, b is called a linked pair. In the third step we deal with these pairs.

Third step. *Assemble all the linked pairs and obtain a surface symbol of the form*

$$\prod_i x_i x_i' \prod_i p_i p_i' \prod_i d_i d_i^* \; \prod_i a_i b_i' a_i' b_i \; N \tag{1.3}$$

where N contains no pair xx', pp', dd^* and no linked pair. Applying successively transformation rule II (and its inverses) changes a sequence $aBb'Ca'Db$ with a linked pair in the following way

$$aBb'Ca'Db \rightsquigarrow ab'Ca'DbB \rightsquigarrow ab'a'DCbB \rightsquigarrow DCab'a'bB,$$

where we first have moved B (from the left of b' to the right of b), then C (from the right of b'

to the left of b), and finally DC (from the right of a' to the left of a). To lighten notation, we have kept the letters a, b', a', b and also B, C, D since no confusion may arise. The same is done in the following movements. Using again transformation rule II, the linked pair $ab'a'b$ is moved to the front:

$$Xab'a'b \rightsquigarrow ab'a'Xb \rightsquigarrow ab'Xa'b \rightsquigarrow aXb'a'b \rightsquigarrow ab'a'bX,$$

where we have moved X first from the left of a to the right of a', then from the left of b to the right of b', then from the left of a' to the right of a and finally from the left of b' to the right of b. Repeating this process we assemble all linked pairs together in the front of M to get a surface symbol of the form (1.3).

It is well known that a pair d, d^* together with a linked pair can be turned into three pairs d_1, d_1^*, d_2, d_2^*, d_3, d_3^*. Continuing this process we may eliminate the linked pairs. We do not perform this transformation now but at the end, to avoid considering different cases.

Therefore, after step three we obtain a surface symbol of the form (1.3) where N contains no pair xx', pp', dd^* and no linked pair. The sequence N consists of edges paired with no other edge and pairs e, e' with at least one more label between them. We deal with these sequences in step fourth.

Fourth step. *Assemble the sequences eCe' with C non-empty and obtain a surface symbol of the form*

$$\prod_i x_i x_i' \prod_i p_i p_i' \prod_i d_i d_i^* \prod_i a_i b_i' a_i' b_i \prod_i e_i C_i e_i'. \tag{1.4}$$

where each C_i is a sequence of labels representing free edges or edges fixed by reflections.

If the sequence N in (1.3) contains a pair of labels e, e' then there exists at least one more label between e and e'. Choose a label e such that there are as few labels between e and e' as for any other such paired symbol. Then each label between e and e' is paired with no other edge and so it represents either a free edge or an edge fixed by a reflection. We move such a sequence eCe' to the front of N by using transformation rule IIb. Repeating this process transforms N into a sequence of the form $\prod_i e_i C_i e_i'\ T$, where each C_i is a sequence of labels representing free edges and edges fixed by reflections. We get

$$\prod_i x_i x_i' \prod_i p_i p_i' \prod_i d_i d_i^* \prod_i a_i b_i' a_i' b_i \prod_i e_i C_i e_i'\ T, \tag{1.5}$$

where the remaining sequence T has no pair of labels e, e' and so it is also a sequence of labels representing free edges or edges fixed by reflections. Lemma 2.2 below shows that this surface symbol can be changed to another with empty T. Eliminating the sequence T makes it easier to obtain in Section 2.2 a presentation of the NEC group Γ.

Lemma 2.2. *The surface symbol* (1.5) *can be transformed into another of the same form with one more sequence eCe' and empty T.*

Proof. Assume first that the surface symbol contains a sequence eCe'. To lighten notation, we denote by eCe' its last sequence and divide e into two edges e_1e_2. We can write the surface symbol as $Ae_1e_2Ce_2'e_1'T$ with $A = \prod xx' \prod pp' \prod dd^* \prod ab'a'b \prod eCe'$, where in the last product we have omitted the sequence $e_1e_2Ce_2'e_1'$. Using transformation rule IIb we move A from the left of e_1 to the right of e_1' without inversion, and obtain $ye_2Ce_2'y'A_1T$. Observe that A_1 has the same form as A, that is, $\prod xx' \prod pp' \prod dd^* \prod ab'a'b \prod eCe'$. We write this surface symbol as $e_2Ce_2'zA_1Tz'$, which has one more sequence of the form eCe' than (1.5). We now repeat the same transformations as in the previous steps and move successively the pairs xx', pp', dd^* and $ab'a'b$ in A_1 to the front, obtaining therefore a surface symbol of the form

$$\prod_i x_ix_i' \prod_i p_ip_i' \prod_i d_id_i^* \prod_i a_ib_i'a_i'b_i \prod_i e_iC_ie_i' \; zTz'.$$

This shows the Lemma in the case where the surface symbol contains a sequence eCe'.

We assume now that there is no sequence eCe'. If there exists a pair xx' or pp' then we repeat movements similar to the above one. Let xx' be the first pair in the above surface symbol, and let us divide x into two parts x_1 and x_2 so that the above surface symbol can be written as $x_1x_2x_2'x_1'AT$, or, starting at T, as $Tx_1x_2x_2'x_1'A$. Transformation rule IIb allows us to move T from the left of x_1 to the right of x_1', obtaining $yx_2x_2'y'T'A = x_2x_2'zT'Az'$. The same transformations as in the above steps move the sequence A frontwards, yielding therefore a surface symbol of the form

$$\prod_i x_ix_i' \prod_i p_ip_i' \prod_i d_id_i^* \prod_i a_ib_i'a_i'b_i \; zTz'.$$

We assume now that there is no pair xx' or pp' but there is a pair dd^*. Let dd^* be the first such pair and divide d into two parts d_1 and d_2 so that the surface symbol (1.5) can be written as $d_1 d_2 d_1^* d_2^* AT$, or, starting at d_2^*, as $d_2^* AT d_1 d_2 d_1^*$. Let us denote by D the orientation reversing transformation such that $D(d) = d^*$, so $D(d_i) = d_i^*$ for $i = 1, 2$. Transformation rule Ib allows us to move $d_2^* AT$ from the left of d_1 to the left of d_1^* by means of D:

$$\begin{aligned} d_2^* AT\, d_1\, d_2\, d_1^* &\rightsquigarrow y\, d_2\, D(d_2^* AT)\, y^* = y d_2 D(AT) D(d_2^*) y^* = y d_2 D(AT) D^2(d_2) y^* = \\ &= y d_2 D(AT) d_2' y^*, \end{aligned}$$

where $d_2' := D^2(d_2)$ is the image of d_2 under the orientation preserving element D^2. Let G be the orientation reversing transformation such that $G(y) = y^*$. Transformation rule Ib allows us to use G to move $d_2 D(AT) d_2'$ from the right of y to the right of y^*:

$$y\, d_2 D(AT) d_2'\; y^* \rightsquigarrow zz^*\, G(d_2 D(AT) d_2') = zz^*\, G(d_2') G D(AT) G(d_2).$$

Since GD preserves orientation, the structure of $GD(AT)$ is the same as that of AT. In addition, if $e := G(d_2')$ then $G(d_2) = GD^{-2}(d_2') = GD^{-2}G^{-1}(e)$, which we may write as e' because it is the image of e under an orientation preserving element. So we may write $zz^*\, G(d_2') GD(AT) G(d_2)$ as $zz^* eATe'$. We now move the sequence A frontwards, using the movements done in the above steps, obtaining therefore a surface symbol of the form

$$\prod_i d_i d_i^* \prod_i a_i b_i' a_i' b_i\; eTe'.$$

We finally consider the case of a surface symbol of the form $\prod_i a_i b_i' a_i' b_i\; T$. Let $ab'a'b$ be the first linked period and let us divide a into two edges a_1 and a_2 so that the surface symbol can be written as $a_1 a_2 b' a_2' a_1' b\, A\, T$ or, starting at T, as $T a_1 a_2 b' a_2' a_1' b\, A$. We first move T from the left of a_1 to the right of a_1' to get $a_1 a_2 b' a_2' a_1' T b\, A$. We now move the sequence $Q = a_1' T$ (without inversion) in the following way:

$$a_1 a_2 b' a_2'\; a_1' T\; b\, A \rightsquigarrow a_1 a_2 b'\; a_1' T\; a_2' b\, A \rightsquigarrow a_1 a_2\; a_1' T\; b' a_2' b\, A \rightsquigarrow a_1 a_2 b' a_2' b\; a_1' T\; A,$$

where we first have moved $a_1' T$ from the left of b to the right of b', then to the right of a_2 and finally to the right of b. Starting at a_2 we may write this surface symbol as $a_2 b' a_2' b\, eT\, Ae'$, where $e := a_1'$ and e' is the image of e under an orientation preserving element. We finally move A frontwards to obtain a surface symbol of the prescribed form $\prod ab'a'b\; eTe'$. $\square$

Summarizing, after the fourth step we obtain a surface symbol of the form (1.4) where each C_i is a sequence of labels representing free edges or edges fixed by reflections. In the sixth and final step we will arrange these sequences according to the types of vertices at infinity they have. Before that, we eliminate some vertices at infinity. This is done in step five.

A vertex at infinity in the surface symbol (1.4) may be of different types, as *parabolic*, which is the fixed point of a parabolic element, *semi-parabolic*, which is either the common vertex of two consecutive edges fixed by reflections or a vertex at infinity of the form (e_i, c) or (c, e_i') in a sequence $e_i C e_i'$, and *improper*, which is the initial or end point of a free edge. But there may be other vertices at infinity, as Examples 2.3 and 2.4 show. In the next step we transform the surface symbol to eliminate these other vertices, leaving the parabolic, semi-parabolic and improper vertices as the unique types of vertices at infinity.

Example 2.3. *Let Γ be the group generated by the glide reflection $D(z) = -2\overline{z}$. Clearly, Γ is a finitely generated discrete NEC group with non-compact orbit space (an unbounded Möbius band). A fundamental domain for Γ is the following. Let d be the hyperbolic line joining the real points -2 and 1, and let $d^* = D(d)$ be the hyperbolic line joining the real points -2 and 4. These two edges together with the free edge f joining 1 and 4 determine a fundamental domain for Γ, with surface symbol dd^*f. It has two improper vertices (1 and 4) and one proper vertex, $v = -2$, where the two consecutive edges d and d^* meet. However, v is not a fixed point of D. It is a proper vertex but it is neither parabolic, nor semi-parabolic, nor improper.*

Example 2.4. *Let C_1, C_2, C_3 and C_4 be the hyperbolic lines with endpoints $(-2,-1)$, $(-1,1)$, $(1,3)$ and $(3,-3)$ respectively. It is easy to see that C_2 and C_4 are paired by the hyperbolic transformation $T_a : z \mapsto 3z$, whilst C_1 and C_3 are paired by the hyperbolic transformation $T_b : z \mapsto (-3z+5)/(z-4)$. It follows that the interior of the region bounded by $C_1 \cup C_2 \cup C_3 \cup C_4 \cup [-3,-2]$ is a fundamental domain for the action of the group $\langle T_a, T_b \rangle$. We claim that this is a discrete group. The axis of the hyperbolic transformation T_a, with endpoints $Fix(T_a) = \{0, \infty\}$, intersects the axis of the hyperbolic transformation T_b, with endpoints $Fix(T_b) = \{(1 \pm \sqrt{41})/4\}$. Moreover, the commutator $[T_a, T_b] : z \mapsto (-27z-45)/(8z+13)$ is also a hyperbolic transformation. It follows from the Discreteness Theorem [24, Thm. 3.1.1], that the group $\langle T_a, T_b \rangle$ is discrete. A surface symbol for the above fundamental domain is $ab'a'bf$, where a, b', a', b are the above hyperbolic lines C_1, C_2, C_3, C_4 and f is the free edge joining the improper* *vertices* -3 *and* -2*. We again have vertices at infinity which are neither parabolic, nor semi-parabolic, nor improper.*

Fifth step. *Change the surface symbol* (1.4) *so that the unique vertices at infinity are parabolic, semi-parabolic or improper vertices.*

If this is not so then the initial point of an edge x_i, p_i, d_i, d_i^*, a_i, b_i', a_i', b_i or e_i lies at infinity. Lemma 2.5 below shows that the initial points of all these edges are in the same Γ-orbit.

Lemma 2.5. *Assume F has surface symbol $\prod_i x_i x_i' \prod_i p_i p_i' \prod_i d_i d_i^* \prod_i a_i b_i' a_i' b_i \prod_i e_i C_i e_i'$ as in* (1.4)*. Then the initial points of the edges x_i, p_i, d_i, d_i^*, a_i, b_i', a_i', b_i and e_i are all in the same Γ-orbit. In particular all of them lie either in $\boldsymbol{H}$ or at infinity.*

Proof. Let us use capital letters to denote the elements of Γ which pair edges of F, so we have $X_i(x_i') = x_i$, $P_i(p_i') = p_i$, $D_i(d_i^*) = d_i$, $A_i(a_i') = a_i$, $B_i(b_i') = b_i$ and $E_i(e_i') = e_i$. The initial point of x_i is mapped by the transformation X_i^{-1} to the end point of x_i', which is the initial point of x_{i+1}. It follows that the initial points of all the edges x_i are in the same Γ-orbit. The initial point of the last edge x_i is mapped to the end point of x_i', which is the initial point of p_1. Starting with p_1, we repeat the same arguments changing x_i and X_i by p_i and P_i respectively, to conclude that the initial points of all the edges p_i are in the same Γ-orbit as the initial points of the edges x_i. The end point of the last edge p_i' is the initial point of d_1. This point is mapped by D_1^{-1} to the initial point of d_1^*, which is mapped, again by D_1^{-1}, to the initial point of d_2. Continuing with this process we see that the initial points of the edges d_i and d_i^* are in the same Γ-orbit as the above initial points. The end point of the last edge d_i^* is the initial point of a_1. This point is mapped by A_1^{-1} to the end point of a_1'. This is the initial point of b_1, which is mapped by B_1^{-1} to the end point of b_1'. This is the initial point of a_1', which is mapped by A_1 to the end point of a_1. Finally, B_1 maps this point, which is the initial point of b_1', to the final point of b_1. So the five vertices of $a_1 b_1' a_1' b_1$ are in the same Γ-orbit. The end point of b_1 is the initial point of the sequence $a_2 b_2' a_2' b_2$. We repeat the argument to conclude that all the vertices in the sequence $\prod_i a_i b_i' a_i' b_i$ are in the same Γ-orbit. The end point of the last edge b_i is the initial point of e_1, which is mapped by E_1^{-1} to the end point of e_1'. This is the initial point of e_2'. Repeating the above arguments we see that the initial points of all the edges e_i are in the same Γ-orbit as the above initial points. This shows the lemma. We also observe that the end point of the last edge e_i' is the initial point of the edge x_1 that we started with. $\square$

In order to show step five, assume first that the initial point of an edge x_i or p_i lies at infinity. Then we may write the surface symbol (1.4) as $zz'C$ where the initial point of z lies at infinity and its end point (z, z') is either elliptic or parabolic. If C is a free edge then the initial point of z is an improper vertex and there is nothing to do. In this case the surface symbol can be written as eCe' where $e := z'$ and $e' := z$ are paired by an orientation preserving element. This surface symbol has one of the two forms described in Theorem 2.9, where the sequence eCe' is called μ-sequence and denoted as $\tilde{e}\widetilde{C}\tilde{e}'$. This corresponds to a cyclic NEC group Γ generated either by an elliptic or by a parabolic transformation.

If C is not a free edge then we write $C = C_1C_2$ with the vertex between C_1 and C_2 lying in $\mathbf{H}$, so the surface symbol can be written as $C_2zz'C_1$. Applying transformation rule IIb with $R = \varnothing$ and $Q = C_2$ changes the surface symbol to $yy'T(C_2)C_1$, where T pairs z with z'. The initial point of y is the initial point of C_2, which lies in $\mathbf{H}$. We repeat this transformation with the next pair zz', assembling the initial point of z with the end point of y' which lies in $\mathbf{H}$. Continuing with this process we assemble all sequences zz', dd^*, $ab'a'b$ and eCe' together as done in previous steps. We end up with a sequence of the same form as (1.4) above, where all its vertices lie in $\mathbf{H}$ except the parabolic, semi-parabolic or improper vertices.

Assume now that (1.4) contains no pair xx' or pp' but does contain some pair dd^* with its first vertex at infinity. To lighten notation, let dd^* the first such pair, and let us divide d into two parts δ_1 and δ_2 with the vertex (δ_1, δ_2) lying in $\mathbf{H}$. So the surface symbol can be written as $\delta_1\delta_2\delta_1^*\delta_2^*B$. Applying transformation rule Ia with $Q = \delta_1^*$ and $R = \varnothing$ we obtain a surface symbol of the form $\delta_1\, yy^*T(\delta_1^*)\, B$, where T is the orientation reversing transformation pairing d and d^*. Since the end point of δ_1 is finite, the same happens to the three vertices of the pair yy^*. Starting with the edge y we may write this surface symbol as yy^*eBe' where $e' := \delta_1$ is the image of $e := T(\delta_1^*)$ under an orientation preserving element. We next move frontwards all the edges dd^*, $ab'a'b$ and eCe' occurring in the sequence B as done in previous steps. We again end up with a sequence of the same form as (1.4) but with the initial vertex of y lying in $\mathbf{H}$.

We now consider the case where (1.4) has no pair xx', pp' or dd^* but has a linked pair with its vertices at infinity. We deal with this case in Lemma 2.6.

Lemma 2.6. *A surface symbol $ab'a'bC$ where the vertices of the linked pair $ab'a'b$ lie at infinity, can be transformed into a surface symbol $uv'u'v\, eCe'$ where the linked pair $uv'u'v$ has no vertex at infinity.*

Proof. The proof consists of cutting and pasting the fundamental domain in an appropriate way, keeping track of the vertices at infinity. Let us denote by A and B the orientation preserving transformations such that $A(a) = a'$ and $B(b) = b'$. We divide each edge a and b into two edges a_1, a_2 and b_1, b_2 respectively, with the vertices (a_1, a_2) and (b_1, b_2) finite, so that the surface symbol can be written as $a_1a_2b_2'b_1'a_2'a_1'b_1b_2\,C$ or, starting at a_2, as $a_2b_2'b_1'a_2'a_1'b_1b_2\,C\,a_1$. Here, $a_i' = A(a_i)$ and $b_i' = B(b_i)$. We first apply twice transformation rule IIa, one with $x = a_2$, $Q = b_2'$ and $R = b_1'$ (here $x' = a_1' = A(a_1)$), and another with $x = a_1'$, $Q = b_1$ and $R = b_2C$ (here, $x' = a_1 = A^{-1}(x)$). We obtain

$$a_2b_2'b_1'a_2'\ a_1'b_1b_2Ca_1 \overset{A}{\rightsquigarrow} yb_1'A(b_2')y'\ a_1'b_1b_2Ca_1 \overset{A^{-1}}{\rightsquigarrow} yb_1'A(b_2')y'\ zb_2CA^{-1}(b_1)z'.$$

Observe that the four edges y, y', z and z' lie in $\mathbf{H}$. We now apply transformation rule IIb with $Q = b_1'$, $x = A(b_2')$ and $R = y'z$ (here $x' = b_2 = (AB)^{-1}(x)$). We obtain

$$y\ \ b_1'A(b_2')y'zb_2\ \ CA^{-1}(b_1)z' \overset{(AB)^{-1}}{\rightsquigarrow} y\ \ vy'zv'(AB)^{-1}(b_1')\ \ CA^{-1}(b_1)z'.$$

We now substitute each primed edge by its description as the image of the unprimed edge. Namely, $y' = A(y)$, $v' = (AB)^{-1}(v)$, $b_1' = B(b_1)$ and $z' = A^{-1}(z)$. So the surface symbol, starting with z', is

$$A^{-1}(z)\ y\ v\ A(y)\ z\ (AB)^{-1}(v)\ (AB)^{-1}(B(b_1))\,C\,A^{-1}(b_1).$$

The pair of consecutive edges $A^{-1}(z)y$ are mapped by A to $A(y)z$. So we may change the surface symbol to

$$u\ v\ A(u)\ (AB)^{-1}(v)\ (AB)^{-1}(B(b_1))\,C\,A^{-1}(b_1).$$

Denoting the edge $(AB)^{-1}(B(b_1))$ by e we have that $A^{-1}(b_1) = A^{-1}B^{-1}AB(e)$ is the image, say e', of e under an orientation preserving transformation. This shows that the surface symbol can be written as $uvu'v'\,eCe'$. Now, the initial vertex of u is the initial point of $A^{-1}(z) = z'$, which lies in $\mathbf{H}$ as pointed out above. It follows that the vertices of the linked pair $uvu'v'$ lie in $\mathbf{H}$, and this concludes the proof. □

Remark 2.7. *Observe that the sequence C in Lemma 2.6 cannot be empty since otherwise the fundamental domain would be a four sided polygon whose four vertices would be fixed by* *the commutator $[A, B]$ of the two transformations A and B pairing the edges. This would give $[A, B] = 1$. But this is impossible because there is no Fuchsian group isomorphic to $\mathbb{Z} \times \mathbb{Z}$, see, for instance, [31, Thm 5.7.4].*

We finally consider the case where the surface symbol (1.4) has no pair xx', pp', dd^* and no linked pair $ab'a'b$ but has a sequence eCe' with the initial point of e lying at infinity. We divide e into two parts e_1 and e_2 with the vertex (e_1, e_2) lying in $\mathbf{H}$. Starting with e_2, the surface symbol can be written as $e_2Ce_2'\,e_1'Ae_1$ where A consists of sequences eCe'. We now move these sequences frontwards as done above and assemble them to e_2Ce_2'. We obtain a surface symbol of the same form as (1.4) where the initial point of e_2 lies in $\mathbf{H}$.

Summarizing, after the fifth step we obtain a surface symbol of the form (1.4) where the unique vertices at infinity are parabolic, semi-parabolic or improper vertices. In the final step we arrange the sequences $e_iC_ie_i'$ according to the types of vertices at infinity they have.

Sixth step. *Arrange the sequences eCe' with C non-empty.*

We first move to the front those sequences with no vertex at infinity. We call them *o-sequences*, see Figure 23. In the orbit space $\mathbf{H}/\Gamma$, an o-sequence corresponds to a compact boundary component, which is usually known as *oval*.

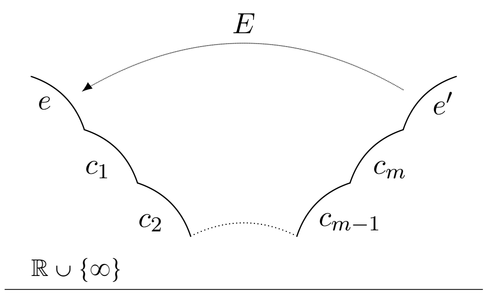


Figure 23: An o-sequence: a compact boundary component.

The remaining sequences eCe' contain vertices at infinity. Lemma 2.8 below shows that we may assume that the end point of e lies at infinity.

Lemma 2.8. *Let eCe' be a sequence of edges such that C has a vertex at infinity. We may change the sequence to another of the form yDy' where the end point of y is a vertex at infinity.*

Proof. If the initial vertex of C does not lie at infinity (otherwise we are done) then we may write $C = QR$ where the vertex between Q and R lies at infinity and Q is non-empty. Transformation rule IIa changes $eQRe'$ into $yRE(Q)y'$ where E pairs e and e'. Since E preserves points at infinity, the initial point of R, which is the final point of y, is a vertex at infinity. $\square$

The final point of y may be either proper or improper. If the first edge in R is free (so that the last edge of Q is fixed by a reflection) then the final point of y is improper and the initial point of y' is proper. That is, the first edge of D is free and its last edge is fixed by a reflection.

We now arrange these sequences eCe' with vertices at infinity. We first write those sequences whose unique vertices at infinity are the end point of e and the initial point of e'. These vertices are semi-parabolic. We call such C a *ν-sequence* and denote it by $\breve{C}$, see Figure 24. These two vertices are in the same Γ-orbit, so in the orbit space $\mathbf{H}/\Gamma$ a ν-sequence gives rise to a non-compact boundary component with just one point at infinity. This is a *semi-cusp*.

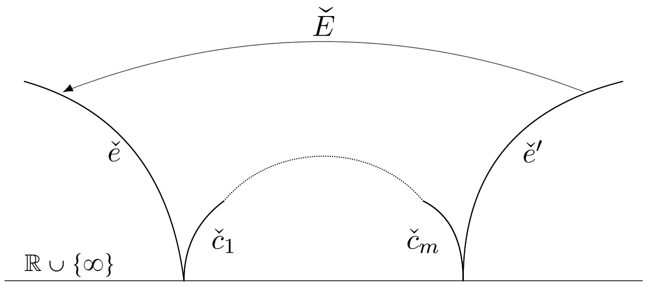


Figure 24: A ν-sequence $\breve{C}$: a boundary component with one semi-cusp.

We then write those eCe' such that C has more than two vertices at infinity but no free edge. These proper vertices are semi-parabolic. We call such C an *η-sequence* and denote it by $\widehat{C}$, see Figure 31. In the orbit space $\mathbf{H}/\Gamma$ an η-sequence gives rise to (at least two) non-compact boundary components, each of them having two points at infinity. These two points are also semi-cusps but, unlike the case of ν-sequences, each such semi-cusp is a point at infinity of two different non-compact boundary components.

We finally write those sequences eCe' with improper vertices. We call such C a *μ-sequence* and denote it by $\widetilde{C}$. If $\widetilde{C}$ consists of a single free edge then in the orbit space $\mathbf{H}/\Gamma$ the μ-sequence gives rise to a funnel with no boundary component. Assume $\widetilde{C}$ has more than one edge. As

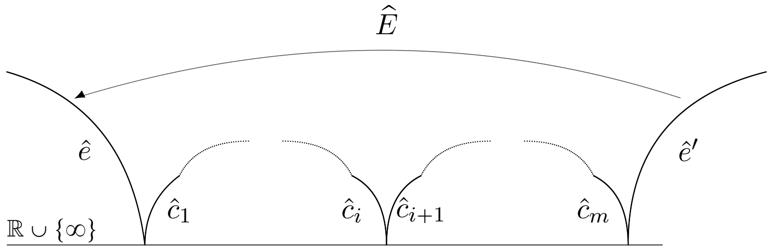


Figure 25: An η-sequence $\widehat{C}$: each boundary component has two semi-cusps.

pointed out after Lemma 2.8 we may assume that the end point of $\tilde{e}$ is improper and the initial point of $\tilde{e}'$ is proper, see Figure 32. In the orbit space $\mathbf{H}/\Gamma$ such μ-sequence gives rise to a funnel with non-compact boundary components. Each of these boundary components has two points at infinity, which may be semi-cusps or not. A μ-sequence corresponds to a hyperbolic end with infinite area.

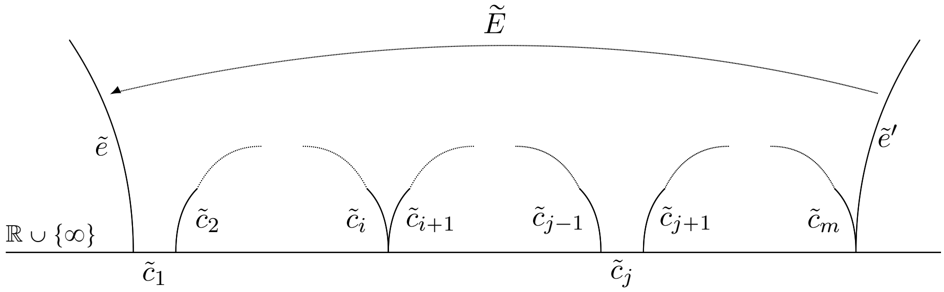


Figure 26: A μ-sequence $\widetilde{C}$: a hyperbolic end with infinite area

We therefore get a surface symbol of the form

$$\prod_{i=1}^{r} x_i x_i' \prod_{i=1}^{s} p_i p_i' \prod_{i=1}^{n} d_i d_i^* \prod_{i=1}^{m} a_i b_i' a_i' b_i \prod_{i=1}^{k} e_i C_i e_i' \prod_{i=1}^{l} \check{e}_i \check{C}_i \check{e}_i' \prod_{i=1}^{q} \hat{e}_i \widehat{C}_i \hat{e}_i' \prod_{i=1}^{t} \tilde{e}_i \widetilde{C}_i \tilde{e}_i',$$

where C_i, $\check{C}_i$, $\widehat{C}_i$ and $\widetilde{C}_i$ are o-, ν-, η- and μ-sequences respectively.

Assume now that there exists a pair dd^*. It is well known that this pair together with a linked pair can be turned into three pairs $d_1 d_1^*$, $d_2 d_2^*$, $d_3 d_3^*$. Continuing this process we eliminate the linked pairs. This gives the two types of surface symbols collected in the following Theorem 2.9.

Theorem 2.9. *A finitely generated NEC group admits a fundamental region with a surface symbol of one of the following forms:*

(1) $\prod_{i=1}^{r} x_i x_i' \prod_{i=1}^{s} p_i p_i' \prod_{i=1}^{g} a_i b_i' a_i' b_i \prod_{i=1}^{k} e_i C_i e_i' \prod_{i=1}^{l} \check{e}_i \check{C}_i \check{e}_i' \prod_{i=1}^{q} \hat{e}_i \widehat{C}_i \hat{e}_i' \prod_{i=1}^{t} \tilde{e}_i \widetilde{C}_i \tilde{e}_i',$

(2) $\prod_{i=1}^{r} x_i x_i' \prod_{i=1}^{s} p_i p_i' \prod_{i=1}^{g} d_i d_i^* \prod_{i=1}^{k} e_i C_i e_i' \prod_{i=1}^{l} \check{e}_i \check{C}_i \check{e}_i' \prod_{i=1}^{q} \hat{e}_i \widehat{C}_i \hat{e}_i' \prod_{i=1}^{t} \tilde{e}_i \widetilde{C}_i \tilde{e}_i',$

where C_i, $\check{C}_i$, $\widehat{C}_i$ and $\widetilde{C}_i$ are o-, ν-, η- and μ-sequences respectively, and the unique vertices at infinity are parabolic, semi-parabolic or improper vertices.

2.2 Generators and relations of non-cocompact NEC groups

In this section we obtain a presentation by generators and relations of a finitely generated non-cocompact NEC group Γ. We follow Wilkie in [59] where the analogous result for cocompact groups is done.

Let $\mathbf{F}$ be a fundamental region for Γ. It follows from [39, Theorem 2 and Corollary] that a set of generators of Γ is the set $\{T \in \Gamma : \mathbf{F} \cap T\mathbf{F} \neq \varnothing\}$. Moreover, it is shown in [59, Theorem 2] (see also Lemma 13 in [41]) that the above set can be restricted to the set

$$E = \{T \in \Gamma : \mathbf{F} \cap T\mathbf{F} \text{ is an edge}\}.$$

Each finite vertex v of a fundamental domain F has N faces meeting at v, say $F = T_0\mathbf{F}$, $T_1\mathbf{F}, \ldots, T_{N-1}\mathbf{F}$. Then the elements $G_i := T_{i-1}^{-1}T_i$ $1 \leqslant i \leqslant N-1$ and $G_N = T_{N-1}^{-1}$ satisfy $T_1 \cdots T_N = 1$. This is called the *canonical relation associated to v*. It can be expressed in terms of the elements in E.

The following Theorem 2.10 gives a presentation of Γ in terms of the elements of E. It is the non-compact version of [59, Theorem 2], and it follows from [39] and [41].

Theorem 2.10. *Let R_1 be the set of relations $C^2 = 1$ where C runs over the reflections fixing an edge of F. Let R_2 be the set of canonical relations which includes one for each set of congruent finite vertices of F. Then $R_1 \cup R_2$ is a complete set of defining relations in terms of the elements in E.*

Observe that proper and improper vertices do not contribute to the set R_2 with any relation.

Our goal now is to analyze the canonical relation associated to each type of vertex. Since proper and improper vertices do not contribute, the canonical relations are the same as in the cocompact case. However, the different ways in which vertices at infinity occur in the surface symbol lead to different ways in which products of consecutive reflections are described.

We first consider vertices of the form $v = (x, x')$. Let X be the elliptic element pairing the edges x and x'. Then the canonical relation of the vertex (x, x') is

$$X^m = 1,$$

where m equals the number of faces meeting at (x, x').

We next consider vertices of the form $v = (c_i, c_{i+1})$ where c_i and c_{i+1} are consecutive edges fixed by the reflections C_i and C_{i+1} respectively. These vertices occur in o-, ν-, η- and μ-sequences. Then the canonical relation of the vertex (c_i, c_{i+1}) is

$$(C_i C_{i+1})^n = 1,$$

where $2n$ equals the number of faces meeting at (c_i, c_{i+1}).

We next consider vertices of the form (e, c_1), which is congruent to a vertex (c_k, e'), where c_1 and c_k are edges fixed by the reflections C_1 and C_k respectively, and e and e' are edges paired by a hyperbolic transformation E. The canonical relation of this congruent pair of vertices is

$$(E C_k E^{-1} C_1)^n = 1,$$

where $4n$ is the number of faces meeting at (e, c_1). By Lemma 2.8, we are assuming that these vertices lie in **H** only in o-sequences. So for ν-, η- and μ-sequences we do not have this relation.

Before dealing with the canonical relation corresponding to the set of congruent vertices which have not been considered yet, we examine the set of canonical relations of the vertices of the different types of sequences eCe'.

If C is an o-sequence with m edges $c_1, \ldots, c_m$ fixed by the reflections $C_1, \ldots, C_m$, then its $m-1$ vertices (c_i, c_{i+1}) for $i = 1, \ldots, m-1$ are all in **H**, see Figure 23. The canonical relations of these vertices are $(C_i C_{i+1})^{n_{i+1}} = 1$ for $i = 1, \ldots, m-1$. In addition, the canonical relation of the vertex (e, c_1) (or its congruent vertex (c_m, e')) is $(C_0 C_1)^{n_1} = 1$, where $C_0 = E C_m E^{-1}$.

So an o-sequence provides the following relations:

$$C_i^2 = 1 \text{ for } i = 0, \ldots, m, \qquad (C_{i-1}C_i)^{n_i} = 1 \text{ for } i = 1, \ldots, m, \text{ and } EC_mE^{-1}C_0 = 1.$$

If $\check{C}$ is a ν-sequence with m edges $\check{c}_1, \ldots, \check{c}_m$ fixed by the reflections $\check{C}_1, \ldots, \check{C}_m$, then it has one proper vertex, which we may assume is the final point of e, see Figure 24. The vertex $(\check{e}, \check{c}_1)$ (and its congruent vertex $(\check{c}_m, \check{e}'))$ lies at infinity and hence provides no canonical relation. So a ν-sequence provides the following relations:

$$\check{C}_i^2 = 1 \text{ for } i = 1, \ldots, m, \qquad \text{and} \qquad (\check{C}_{i-1}\check{C}_i)^{\check{n}_i} = 1 \text{ for } i = 2, \ldots, m.$$

If $\widehat{C}$ is an η-sequence with m edges $\hat{c}_1 \ldots, \hat{c}_m$ fixed by the reflections $\widehat{C}_1, \ldots, \widehat{C}_m$, then it has more than one proper vertex, see Figure 31. Write

$$\widehat{V} = \{i \in \{2, \ldots, m\} : \text{ the vertex } (\hat{c}_{i-1}, \hat{c}_i) \text{ is proper}\}.$$

The vertices which lie in $\mathbf{H}$ are $(\hat{c}_{i-1}, \hat{c}_i)$ for $i \notin \widehat{V}$, and their corresponding canonical relations are therefore $(\widehat{C}_{i-1}\widehat{C}_i)^{\hat{n}_i} = 1$ for $i \notin \widehat{V}$. By Lemma 2.8, we may assume that the vertices $(\hat{e}, \hat{c}_1)$ and $(\hat{c}_m, \hat{e}')$ lie at infinity. So they provide no canonical relation. Therefore an η-sequence provides the following relations:

$$\widehat{C}_i^2 = 1 \text{ for } i = 1, \ldots, m, \qquad \text{and} \qquad (\widehat{C}_{i-1}\widehat{C}_i)^{\hat{n}_i} = 1 \text{ for } i \in \{2, \ldots, m\} - \widehat{V}.$$

If $\widetilde{C}$ is a μ-sequence with m edges $\tilde{c}_1, \ldots, \tilde{c}_m$ then some of them are free edges. If $m = 1$ (hence $\widetilde{C}$ consists of a single free edge) then the only generator associated to a μ-sequence is the hyperbolic transformation $\widetilde{E}$ pairing $\tilde{e}$ and $\tilde{e}'$ (there is no reflection) and there is no canonical relation because there is no vertex in $\mathbf{H}$. Assume $\widetilde{C}$ has more than one edge. As pointed out after Lemma 2.8, we may assume that the vertex $(\tilde{e}, \tilde{c}_1)$ is improper (so $\tilde{c}_1$ is a free edge) and that $(\tilde{c}_m, \tilde{e}')$ is proper (so $\tilde{c}_m$ is fixed by a reflection), see Figure 32. Let U be the set of indices of free edges:

$$U = \{i \in \{1, \ldots, m\} : \tilde{c}_i \text{ is a free edge}\}.$$

The canonical reflections associated to a μ-sequence are therefore $\widetilde{C}_i$ for $i \in \{1, \ldots, m\} - U$.

Observe that $1 \in U$ and that it contains no pair of consecutive indices (so $2 \notin U$). In addition, if $m > 1$ then $m \notin U$ by our assumption based on Lemma 2.8. Let $\widetilde{V} = \{i \in \{3, \ldots, m\}$: the vertex $(\tilde{c}_{i-1}, \tilde{c}_i)$ is proper$\}$ be the set of indices of proper vertices. Then a μ-sequence provides the following relations:

$$\begin{aligned} \widetilde{C}_i^2 &= 1 \quad \text{for } i \in \{1, \ldots, m\} - U, \quad \text{and} \\ (\widetilde{C}_{i-1}\widetilde{C}_i)^{\tilde{n}_i} &= 1 \quad \text{for } i \in \{2, \ldots, m\} - \widetilde{V} \quad \text{and} \quad \{i-1, i\} \cap U = \varnothing. \end{aligned}$$

It remains to deal with the canonical relation corresponding to the set of congruent vertices which have not been considered yet. As usual, we will use capital letters to denote the elements of Γ which pair edges of F, so we have $X_i(x_i') = x_i$, $P_i(p_i') = p_i$, $D_i(d_i^*) = d_i$, $A_i(a_i') = a_i$, $B_i(b_i') = b_i$, $E_i(e_i') = e_i$, $\check{E}_i(\check{e}_i') = \check{e}_i$, $\widehat{E}_i(\hat{e}_i') = \hat{e}_i$, $\widetilde{E}_i(\tilde{e}_i') = \tilde{e}_i$.

Assume first that F has a surface symbol of the form

$$\prod_{i=1}^{r} x_i x_i' \prod_{i=1}^{s} p_i p_i' \prod_{i=1}^{g} a_i b_i' a_i' b_i \prod_{i=1}^{k} e_i C_i e_i' \prod_{i=1}^{l} \check{e}_i \check{C}_i \check{e}_i' \prod_{i=1}^{q} \hat{e}_i \widehat{C}_i \hat{e}_i' \prod_{i=1}^{t} \tilde{e}_i \widetilde{C}_i \tilde{e}_i'.$$

In this case, the set of vertices which have not been considered yet are the initial points of the edges x_i, p_i, a_i, b_i', a_i', b_i, e_i, $\check{e}_i$, $\hat{e}_i$ and $\tilde{e}_i$. These points are all congruent, as shown in Lemma 2.5. Moreover, it follows from the proof of this lemma that applying successively the letters of the word

$$W := \left(\prod_{i=1}^{r} X_i \prod_{i=1}^{s} P_i \prod_{i=1}^{g} [A_i, B_i] \prod_{i=1}^{k} E_i \prod_{i=1}^{l} \check{E}_i \prod_{i=1}^{q} \widehat{E}_i \prod_{i=1}^{t} \widetilde{E}_i \right)^{-1}$$

we go through all these points, starting and ending at the initial point of the edge x_1. Therefore W is an orientation preserving transformation which fixes this point. So W is either the identity or a non-trivial elliptic element. In the first case the canonical relation associated to this set of congruent vertices is

$$\prod_{i=1}^{r} X_i \prod_{i=1}^{s} P_i \prod_{i=1}^{g} [A_i, B_i] \prod_{i=1}^{k} E_i \prod_{i=1}^{l} \check{E}_i \prod_{i=1}^{q} \widehat{E}_i \prod_{i=1}^{t} \widetilde{E}_i = 1.$$

This is called the “long relation”. If W has order $m_0 \neq 1$ then we take it as a new elliptic generator $X_0 = W$ and obtain the relations

$$X_0^{m_0} = 1 \qquad \text{and} \qquad X_0 \cdot \prod_{i=1}^{r} X_i \prod_{i=1}^{s} P_i \prod_{i=1}^{g} [A_i, B_i] \prod_{i=1}^{k} E_i \prod_{i=1}^{l} \breve{E}_i \prod_{i=1}^{q} \widehat{E}_i \prod_{i=1}^{t} \widetilde{E}_i = 1.$$

Observe that the presentations in both cases ($W = 1$ or not) just differ in the number of elliptic generators: they are $X_1, \ldots, X_r$, in the first case, and $X_0, \ldots, X_r$ in the second.

Assume now that the surface symbol is of the form

$$\prod_{i=1}^{r} x_i x_i' \prod_{i=1}^{s} p_i p_i' \prod_{i=1}^{g} d_i d_i^* \prod_{i=1}^{k} e_i C_i e_i' \prod_{i=1}^{l} \check{e}_i \check{C}_i \check{e}_i' \prod_{i=1}^{q} \hat{e}_i \widehat{C}_i \hat{e}_i' \prod_{i=1}^{t} \tilde{e}_i \widetilde{C}_i \tilde{e}_i'.$$

In this case, the set of vertices which have not been considered yet are the initial points of the edges x_i, p_i, d_i, d_i^*, e_i, $\check{e}_i$, $\hat{e}_i$ and $\tilde{e}_i$, which are all congruent. Here we consider the word

$$W := \left(\prod_{i=1}^{r} X_i \prod_{i=1}^{s} P_i \prod_{i=1}^{g} D_i^2 \prod_{i=1}^{k} E_i \prod_{i=1}^{l} \breve{E}_i \prod_{i=1}^{q} \widehat{E}_i \prod_{i=1}^{t} \widetilde{E}_i \right)^{-1}$$

to go through all these points, starting and ending at the initial point of the edge x_1. Again, W preserves orientation, so it is either the identity or a non-trivial elliptic element. If W is the identity then the canonical relation associated to these vertices is

$$\prod_{i=1}^{r} X_i \prod_{i=1}^{s} P_i \prod_{i=1}^{g} D_i^2 \prod_{i=1}^{k} E_i \prod_{i=1}^{l} \breve{E}_i \prod_{i=1}^{q} \widehat{E}_i \prod_{i=1}^{t} \widetilde{E}_i = 1.$$

If W has order $m_0 \neq 1$ then we take W as a new elliptic generator $X_0 = W$ and obtain the relations

$$X_0^{m_0} = 1 \qquad \text{and} \qquad X_0 \cdot \prod_{i=1}^{r} X_i \prod_{i=1}^{s} P_i \prod_{i=1}^{g} D_i^2 \prod_{i=1}^{k} E_i \prod_{i=1}^{l} \breve{E}_i \prod_{i=1}^{q} \widehat{E}_i \prod_{i=1}^{t} \widetilde{E}_i = 1.$$

Again, the presentations just differ in the number of elliptic generators.

There are no more sets of congruent vertices lying in $\mathbf{H}$. So there are no more relations. We summarize the results obtained in this section in the following theorem.

Theorem 2.11. *A finitely generated NEC group* Γ *admits the following presentation: it has generators*

(a) X_i, $i = 1, ..., r$ *(elliptic elements),*

(b) P_i, $i = 1, ..., s$ *(parabolic elements),*

(c) A_i, B_i, $i = 1, ..., g$ *(hyperbolic translations) if the orbit space* $\mathbf{H}/\Gamma$ *is orientable, or*

D_i, $i = 1, \ldots, g$ *(glide reflections) otherwise,*

(d) E_i, $i = 1, \ldots, k$, $\breve{E}_i$, $i = 1, \ldots, l$, $\widehat{E}_i$, $i = 1, \ldots, q$, $\widetilde{E}_i$, $i = 1, \ldots, t$ *(orientation preserving elements, usually hyperbolic elements),*

(e) C_{ij}, $i = 1, ..., k, j = 0, ..., k_i$, *(reflections)*

$\breve{C}_{ij}$, $i = 1, \ldots, l$, $j = 1, \ldots, l_i$, *(reflections)*

$\widehat{C}_{ij}$, $i = 1, \ldots, q$, $j = 1, \ldots, q_i$, *(reflections)*

$\widetilde{C}_{ij}$, $i = 1, \ldots, t$, $j \in \{1, \ldots, t_i\} - U_i$ *for some* U_i *with* $1 \in U_i$, $t_i \notin U_i$ *(if* $t_i > 1$*) and containing no pair* $j, j+1$ *of consecutive elements, (these* $\widetilde{C}_{ij}$ *are reflections),*

and relations:

(i) $X_i^{m_i} = 1$, *for* $i = 1, \ldots, r$,

(ii) $E_i C_{ik_i} E_i^{-1} C_{i0} = 1$ *for* $i = 1, \ldots, k$,

(iii) $C_{ij}^2 = \breve{C}_{ij}^2 = \widehat{C}_{ij}^2 = \widetilde{C}_{ij}^2 = 1$, *for all the reflections,*

(iv) $(C_{i,j-1}C_{ij})^{n_{ij}} = 1$ *for* $i = 1, \ldots, k$, $j = 1, \ldots, k_i$;

$(\breve{C}_{i,j-1}\breve{C}_{ij})^{\breve{n}_{i,j}} = 1$ *for* $i = 1, \ldots, l$, $j = 2, \ldots, l_i$;

$(\widehat{C}_{i,j-1}\widehat{C}_{ij})^{\hat{n}_{ij}} = 1$ *for* $i = 1, \ldots, q$, $j \in \{2, \ldots, q_i\} - \widehat{V}_i$ *for some non-empty* $\widehat{V}_i \subset \{2, \ldots, q_i\}$;

$(\widetilde{C}_{i,j-1}\widetilde{C}_{ij})^{\tilde{n}_{ij}} = 1$ *for* $i = 1, \ldots, t$, $j \in \{2, \ldots, t_i\} - \widetilde{V}_i$ *for some* $\widetilde{V}_i \subset \{2, \ldots, t_i\}$ *(maybe empty) whenever the reflections* $\widetilde{C}_{i,j-1}$ *and* $\widetilde{C}_{ij}$ *exist, that is, whenever* $\{j-1, j\} \cap U_i = \emptyset$.

(v) $\prod_{i=1}^{r} X_i \prod_{i=1}^{s} P_i \prod_{j=1}^{g} [A_j, B_j] \prod_{i=1}^{k} E_i \prod_{i=1}^{l} \breve{E}_i \prod_{i=1}^{q} \widehat{E}_i \prod_{i=1}^{t} \widetilde{E}_i = 1$, *if the orbit space* $\mathbf{H}/\Gamma$ *is orientable, or*

$$\prod_{i=1}^{r} X_i \prod_{i=1}^{s} P_i \prod_{j=1}^{g} D_j^2 \prod_{i=1}^{k} E_i \prod_{i=1}^{l} \breve{E}_i \prod_{i=1}^{q} \widehat{E}_i \prod_{i=1}^{t} \widetilde{E}_i = 1, \text{ otherwise.}$$

CHAPTER 3

Signature of finitely generated non-cocompact NEC groups

In this chapter we introduce the notion of the signature of a finitely generated non-cocompact NEC group Γ. The signature provides more than just a symbolic structure for defining the presentation of a group: with the help of the signature we can assign additionally a marked polygon to a canonical fundamental region and describe the geometric structure of the orbit space $\mathbf{H}/\Gamma$. Additionally, it is a highly efficient approach to deal with isomorphic groups as well as to represent a broad set of properties linked to the group. This chapter is organized in the following way. We introduce the signature in Section 3.1. In Section 3.2, we study the orbit space of an NEC group. In Section 3.3, we provide the necessary and sufficient conditions for the existence of a type-preserving isomorphism between two NEC groups. In Section 3.4 we obtain the signature of the canonical fuchsian subgroup of an NEC group given its signature. Finally, in Section 3.5, we present a topological classification of the orbit space of $\mathbb{H}$ under the action of an NEC group.

3.1 Signature of finitely generated NEC groups

Let Γ be an NEC group and $\mathbf{F}$ a fundamental region of Γ. As shown in Chapter 2, the vertices of an η-sequence $(\hat{c}_1, \ldots, \hat{c}_q)$ can be labelled with integers starting from one from left to right, where the first and last vertices of the sequence are proper vertices in the same orbit and the sequence might include additional proper vertices, see Figure 31.

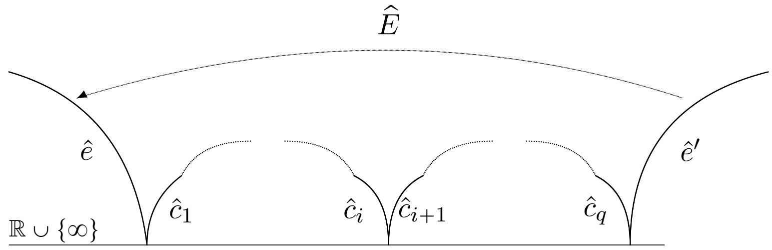


Figure 31: An η-sequence $\widehat{C}$: in the orbit space each boundary component has two semi-cusps.

Given the group presentation, we define the signature of the $\eta-$sequence as the ordered set $\hat{N}$ consisting of the orders of the products of two consecutive canonical reflections such that in case of elliptic products we have integers $\hat{n}_j \geqslant 2$, or the symbol "∞", in case the product is parabolic. We define the parabolic data of the η-sequence as $\hat{V} = \{\hat{v}_2, ..., \hat{v}_{\hat{v}}\}$, i.e. the set of integers j such that the product of reflections $\hat{C}_{j-1}\hat{C}_j$ is parabolic. The parabolic data $\hat{V}$ gives the position of the symbols "∞" in the signature $\hat{N}$ of the $\eta-$sequence, $\hat{N}$.

We may also write the signature of the η-sequence in the following way:

$(I_1, ..., I_{|\hat{V}|}) = ((\hat{n}_2, ..., \hat{n}_{\hat{v}_2-1}), (\hat{n}_{\hat{v}_2+1}, ..., \hat{n}_{\hat{v}_3-1}), ..., (\hat{n}_{\hat{v}_{\hat{v}}+1}, ..., \hat{n}_q)),$

we call each I_j a component of the signature of the η-sequence, where the symbols "∞" are removed. If $\hat{N}$ has two consecutive symbols "∞", which happens when the η-sequence includes an edge whose initial and end points are both proper vertices, then we write $I_j = (-)$ and say that the component is empty.

A μ-sequence $(\tilde{c}_1, \ldots, \tilde{c}_t)$ is a sequence of t edges where, as shown in Chapter 2, we may assume that the first one is a free-side and, if $t > 1$, the last one is an edge fixed by a reflection, see Figure 32.

Given the group presentation of Γ, we define the signature of a $\mu-$sequence as the set $\tilde{N}$, which is an ordered set of integers and the symbol "∞", such that in the j-th position we have either the order of the elliptic product of the reflections $\tilde{C}_{j-1}\tilde{C}_j$, or the symbol "$\infty$" if the product $\tilde{C}_{j-1}\tilde{C}_j$ is parabolic, or the number 0, if $\{j-1, j\} \cap U \neq \varnothing$, i.e. if the product $\tilde{C}_{j-1}\tilde{C}_{j+1}$ is hyperbolic and the edge $\tilde{c}_j$ is free. In summary, the set $\tilde{N}$ has as many elements as vertices are in the μ-sequence (considering the first and last one as paired), with as many

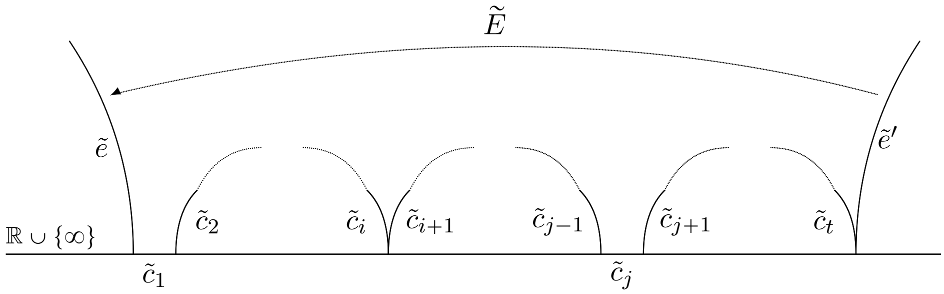


Figure 32: A μ-sequence $\widetilde{C}$: a hyperbolic end with infinite area.

"0"s as improper vertices, as many times times the symbol "∞" as proper vertices, and as many integers as the number of elliptic products $\tilde{C}_{j-1}\tilde{C}_j$.

We define the sets: U, the hyperbolic data of the $\mu-$sequence, as the set of integers $\{1, u_2, ..., u_u\}$ consisting of the labels of the free-sides, $\tilde{V}$, the parabolic data of the $\mu-$sequence, as the set of integers $\{\tilde{v}_1, ..., \tilde{v}_{\tilde{v}}\}$ consisting of the number j if the product of reflections $\tilde{C}_{j-1}\tilde{C}_j$ is parabolic, and the set L, as the set of $j \in \{1, ..., t\}$ of the μ-sequence such that $\tilde{C}_{j-1}\tilde{C}_j$ is elliptic. Then we have the following equality:

$$t = |L| + |\tilde{V}| + 2|U| = |L| + \tilde{v} + 2u.$$

The expanded form of the signature of the μ-cycle is:

$$(I_1, ..., I_{|U|}) = ((\tilde{n}_3, ..., \tilde{n}_{u_2-1}), (\tilde{n}_{u_2+2}, ..., \tilde{n}_{u_3-1}), ..., (\tilde{n}_{u_u+2}, ..., \tilde{n}_t)),$$

where we call each I_j a *component* of the signature of the μ-sequence, which is delimited by the 0s removed from $\tilde{N}$. If a component is empty, namely $\tilde{N}$ has four consecutive 0s, then we will write $I_j = (-)$.

We are now in a position to introduce the signature of an NEC group:

Definition 3.1. The signature of a non-cocompact NEC group Γ with presentation as in Theorem 2.9 is a collection of symbols and non-negative integers of the form:

$$sg(\Gamma) = (g; \pm; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_q)\}; \{\tilde{C}_1, ..., \tilde{C}_t\}),$$

where we write, with a slight abuse of notation, the signatures and the sequences with the same symbols $C_i, \check{C}_i, \hat{C}_i$ and $\tilde{C}_i$. Additionally,

1. g is a non-negative integer called orbit genus of sg,
2. the signs " + ", " − ", that we write $sign(sg)$, where $sign(sg) =$ " + " if the orbit space $\boldsymbol{H}/\Gamma$ is orientable and " − " otherwise,
3. s is a non-negative integer,
4. $m_1, ..., m_r$ are non-negative integers $\geqslant 2$ called proper periods of sg,
5. each symbol C_i, that we call o-cycle is an ordered set of integers $(n_{i2}, n_{i3}, ..., n_{ik_i})$, called linked periods of the o-cycle or o-periods,
6. each symbol $\check{C}_i$, that we call ν-cycle, is an ordered set of integers $(\check{n}_{i2}, \check{n}_{i3}, ..., \check{n}_{iq_i})$, that we call linked periods of the ν-cycle or ν-periods,
7. each symbol $\hat{C}_i$, that we call η-cycle, is the expanded form of the signature of an η-sequence, i.e. an ordered set of integers $((\hat{n}_{i2}, ..., \hat{n}_{\hat{v}_{i2}-1}), (\hat{n}_{\hat{v}_{i2}+1}, ..., \hat{n}_{\hat{v}_{i3}-1}), ..., (\hat{n}_{\hat{v}_{i\hat{v}_i}+1}, ..., \hat{n}_{iq_i}))$. We call its elements linked periods of the η-cycle or η-periods,
8. each symbol $\tilde{C}_i$, that we call μ-cycle, is the expanded form of the signature of a μ-sequence, i.e. an ordered set of integers $((\tilde{n}_{i3}, ..., \tilde{n}_{u_{i2}-1}), (\tilde{n}_{u_{i2}+2}, ..., \tilde{n}_{u_{i3}-1}), ..., (\tilde{n}_{u_{iu_i}+2}, ..., \tilde{n}_{t_i}))$ We call its elements linked periods of the μ-cycle or μ-periods.

Remarks

1. The signature can be represented in the following ways:
   - short form:
     $sg = (g; \pm; s; [m_1, ..., m_r];$
     $\{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}_t\}),$
   - expanded form, that is used for the theorems 3.4 and 3.9:
     $sg = (g; \pm; s; [m_1, ..., m_r];$
     $\{(n_{11}, ..., n_{1k_1}), ..., (n_{k1}, ..., n_{kk_k})\}; \{(\check{n}_{12}, ..., \check{n}_{1l_1}), ..., (\check{n}_{l2}, ..., \check{n}_{ll_l})\};$
     $\{((\hat{n}_{12}, ..., \hat{n}_{1,\hat{v}_{12}-1}), (\hat{n}_{1,\hat{v}_{12}+1}, ..., \hat{n}_{1,\hat{v}_{13}-1}), ..., (\hat{n}_{1,\hat{v}_{1,\hat{v}_1}+1}, ..., \hat{n}_{1q_1})), ...,$
     $((\hat{n}_{q2}, ..., \hat{n}_{q,\hat{v}_{q2}-1}), (\hat{n}_{q,\hat{v}_{q2}+1}, ..., \hat{n}_{q,\hat{v}_{q3}-1}), ..., (\hat{n}_{q,\hat{v}_{q,\hat{v}_q}+1}, ..., \hat{n}_{qq_q}))\};$

$\{((\tilde{n}_{13},...,\tilde{n}_{1,u_{12}-1}),(\tilde{n}_{1,u_{11}+2},...,\tilde{n}_{1,u_{11}-1}),...,(\tilde{n}_{1,u_{1u_1}+2},...,\tilde{n}_{1t_1})),...,$
$((\tilde{n}_{t3},...,\tilde{n}_{t,u_{t1}-1}),(\tilde{n}_{t,u_{t1}+2},...,\tilde{n}_{t,u_{t2}-1}),...,(\tilde{n}_{t,u_{tu_t}+2},...,\tilde{n}_{tt_t}))\})$

2. The sets of o-, ν-, η- and μ-cycles may be empty, that is $r=0, k=0, l=0, q=0$ and/or $t=0$. In such a case we write $[-], \{-\}$, $\{-\}$ and $\{-\}$ respectively. As usual, we can have a finite number n of empty cycles, and in such case we write $\{(-),\underline{\ n\ },(-)\}$.

3. Additionally, the sets $\tilde{N}_i, \tilde{V}_i$ may be empty and in case that a μ-sequence $\widetilde{C}_i$ is such that $\tilde{N}_i=\varnothing, U_i=\{1\}, \tilde{V}_i=\varnothing$, we write the signature of the μ-sequence just as $(-)$. If there are t of such μ-sequences, then we write $\{(-),\underline{\ t\ },(-)\}$.

4. The signature of a cocompact Fuchsian group can be represented in the following way, $(g;+;0;[m_1,...,m_r];\{-\};\{-\};\{-\};\{-\})$ which is equivalent to the classical Fuchsian signature $(g;m_1,...,m_r)$.

5. The signature of a cocompact NEC group can be represented in the following way $(g;\pm;0;[m_1,...,m_r];\{(n_{11},...,n_{1k_1}),...,(n_{k1},...,n_{kk_k})\};\{-\};\{-\};\{-\})$ which is equivalent to Wilkie's signature $(g;\pm;[m_1,...,m_r];\{(n_{11},...,n_{1k_1}),...,(n_{k1},...,n_{kk_k})\})$.

6. The signature of a non-cocompact Fuchsian group can be represented in the following way $(g;+;s;[m_1,...,m_r];\{-\};\{-\};\{-\};\{(-),\underline{\ t\ },(-)\})$ which is equivalent to the Fuchsian signature $(g;m_1,...,m_r;s;t)$.

3.2 Orbit space of non-cocompact NEC groups

The orbit-space $\mathbf{H}/\Gamma$ of $\mathbf{H}$ under the action of an NEC group Γ can be obtained in the usual way by identifying the paired edges of a fundamental region. For the NEC groups considered in this paper (finitely generated), the orbit space is a non-compact surface of finite genus and with finitely many boundary components compact or not. In this section we describe the topological and geometrical properties of the orbit space $\mathbf{H}/\Gamma$ and the canonical projection $\mathbf{H} \longrightarrow \mathbf{H}/\Gamma$ in terms of the signature of the group Γ defined above.

In the orbit space $\mathbf{H}/\Gamma$ there exist finitely many simple closed curves $l_1,...,l_n$ which decompose $\mathbf{H}/\Gamma$ into connected surfaces $S_0,S_1,...,S_n$ where S_0 is a compact surface called compact core. The topological and geometrical properties of S_0 are well-known and described for example in [59]. The surfaces $S_1, ..., S_n$ are non-compact and satisfy $S_0 \cap S_i = l_i, 1 \leqslant i \leqslant n$ and $S_i \cap S_j = \varnothing, 1 \leqslant i \neq j \leqslant n$. Each S_i is topologically an unbounded cylinder with one compact boundary component (the curve l_i) and a hyperbolic end. This can be easily seen by means of a fundamental region whose surface symbol is canonical as the ones described in Section 2. For instance, a geodesic joining the initial point of an edge p_i with the end point of p_i', both paired by a parabolic element, yields a simple closed curve l_i in the orbit space $\mathbf{H}/\Gamma$. This curve separates S_0 from a non-compact surface S_i which is topologically a cylinder with one compact boundary component (the curve l_i) and a cusp. A geodesic joining the initial point of an edge $\check{e}_i$ with the end point of $\check{e}_i'$ of a $\nu-$sequence, now paired by a boundary hyperbolic element, yields a simple closed curve l_i in the orbit space $\mathbf{H}/\Gamma$. This curve separates S_0 from a non-compact surface S_i which, in this case, is topologically a cylinder with one compact boundary component and a hyperbolic end which we call of type ν. Similar descriptions can be done with η- and μ-sequences, with the corresponding cylinders having different hyperbolic ends. In summary, each parabolic generator corresponds to a hyperbolic end called cusp, each hyperbolic boundary generator to a funnel, each $\nu-$sequence to a hyperbolic end of type ν, each $\eta-$sequence to a hyperbolic end of type η and each $\mu-$sequence to a hyperbolic end of type μ .

A non-compact boundary component of $\mathbf{H}/\Gamma$ corresponds, in the canonical surface symbol of the group, to a sequence of edges fixed by reflections whose unique vertices at infinity are its initial and end points. The types of vertices at infinity (proper or improper) classify the non-compact boundary components into four types as follows, see Figure 33:

- Type I: non-compact boundary components with a unique point at infinity, which we call semi-cusp of type I. This occurs just in $\nu-$sequences.

- Type II: non-compact boundary components with two points at infinity (both corresponding to proper vertices) each of which is also a point at infinity of another non-compact boundary component. We call them semi-cusps of type II.

- Type III: non-compact boundary components with two points at infinity and just one of them is a semi-cusp (corresponding to a proper vertex), that is, just one of them is the point at infinity of another non-compact boundary component.

- Type IV: non-compact boundary components with two points at infinity (both corresponding to improper vertices) none of which is a semi-cusp. For convenience, this is also the case of an $\eta-$sequence with only one free-side (fuchsian case).

A $\mu-$sequence may provide non-compact boundary components of types II, III and IV. We claim that the total number of them is $|U| + |\tilde{V}|$, where $U := \{i \in \{1, ..., t\} : \tilde{c}_i$ is a free edge$\}$ is the hyperbolic data of the μ-sequence and $\tilde{V}$ is the parabolic data defined above. In fact, there are $2|U|$ improper vertices - the last vertex $(\tilde{c}_m, \tilde{e'})$ is in the same orbit as the first vertex $(\tilde{e}, \tilde{c}_1)$ - and $|\tilde{V}|$ proper vertices. Each improper vertex belongs to a unique boundary component, whilst each proper vertex belongs to two. So the total number of boundary components is $|U| + |\tilde{V}|$ as claimed. We call the boundary components associated to a $\mu-$sequence *cuts.* Two cuts sharing a proper vertex are called *parallel cuts.* Finally, two consecutive cuts not sharing a proper vertex will be called *ultraparallel cuts.*

In order to count the number of components of type II, III and IV in a $\mu-$sequence, we use the concept of c-decomposition of a finite subset A of natural numbers. The ordered set of non-empty subsets of A, $(M_1, ..., M_n)$ such that its elements are the biggest subsets of A consisting of consecutive numbers is called $c-$decomposition of A, see Examples 3.3.

Let us define $F_i = \{1, 2, ..., t_i\} - U_i, i = 1, ..., t$ for each $\mu-$sequence of the fundamental region, $M_i = \bigcup_{j=1}^{|U_i|} M_{ij}$ be the $c-$decomposition of F_i and let us define the sets $\Lambda_{ij} = M_{ij} \cap \tilde{V}_i$, for $i = 1, ..., t$, $j = 1, ..., |U_i|$.

Now, it is easy to count the specific number of boundary components associated to a μ-sequence. We have two main cases, when $t_i = 1$, that we discuss in Example 3.2 below, and the general case with $t_i > 1$ as shown below:

- The number IV_i of components of type IV associated to the μ-sequence i equals the number of elements of the c-decomposition M_i such that $\Lambda_{ij} = \varnothing$, that is $IV_i = card\{j : |\Lambda_{ij}| = 0\}$.
- The number III_i of components of type III of the μ-sequence i is the same as the number of improper vertices not corresponding to IV-components, that is $III_i = 2(|U_i| - IV_i)$.

- Finally, the number II_i of components of type II associated to the μ-sequence i is the total number of its boundary components minus the components III_i and IV_i, i.e. $II_i = |U_i| + |\tilde{V}_i| - III_i - IV_i = |\tilde{V}_i| - |U_i| + IV_i$.

Example 3.2. *We call a $\mu-$sequence with a unique edge, simple-funnel, i.e. the fuchsian funnel. In this case we have $t_i = 1$, so $F = \varnothing$ and therefore the c-decomposition is empty. This means that there are no sets Λ. As stated above in the classification of the non-compact boundaries, the number of components of type IV in this case is* 1 *by definition, and for the types II and II we have, $III = 2(1-1) = 0$ and $II = 0 - 1 + 1 = 0$.*

Examples 3.3.

(1) For the $\mu-$sequence $\tilde{c}_1, \tilde{c}_2, ..., \tilde{c}_9$ with $U = \{1, 3, 5\}$ and $V = \{7, 8, 9\}$, we have $F = \{1, ..., 9\} - \{1, 3, 5\} = \{2, 4, 6, 7, 8, 9\}$ with the c$-$decomposition given by (I_1, I_2, I_3) with
$I_1 = \{2\}$, $I_2 = \{4\}$, $I_3 = \{6, 7, 8, 9\}$,
such that
$\Lambda_1 = I_1 \cap \tilde{V} = \varnothing$, $\Lambda_2 = I_2 \cap \tilde{V} = \varnothing$, $\Lambda_3 = I_3 \cap \tilde{V} = \{7, 8, 9\}$.
So the number of components of type II, III and IV equal
$IV = 2$, $III = 2(3-2) = 2$, $II = 3 - 3 + 2 = 2$.

(2) For a $\mu-$sequence $\tilde{c}_1, \tilde{c}_2, ..., \tilde{c}_9$ with $U = \{1, 4, 7\}$ and $V = \{3, 6, 9\}$ we get $F = \{1, ..., 9\} - \{1, 4, 7\} = \{2, 3, 5, 6, 8, 9\}$ with the c$-$decomposition given by (I_1, I_2, I_3) with
$I_1 = \{2, 3\}$, $I_2 = \{5, 6\}$, $I_3 = \{8, 9\}$,
such that
$\Lambda_1 = I_1 \cap \tilde{V} = \{3\}$, $\Lambda_2 = I_2 \cap \tilde{V} = \{6\}$, $\Lambda_3 = I_3 \cap \tilde{V} = \{9\}$.
So the number of components of type II, III and IV equal
$IV = 0$, $III = 2(3-0) = 6$, $II = 3 - 3 + 0 = 0$.

These examples show that the number of proper and improper vertices is not enough for determining the types of the components. For that we need additionally to know their distribution, an information that as seen above can be traslated into an algorithm which computes the number of empty Λ_{ij}.

Before describing the form of the orbit space of a finitely generated non-cocompact NEC group, we summarize briefly the idea of a covering which is both branched and folded: let $\mathbf{S}$ denote the unit disc $|z| < 1$ and let n be a positive integer. Let $\mathbf{S}^*$ be the subset of $\mathbf{S}$ consisting of those $re^{i\theta}$ such that $0 < r < 1$ and $0 \leqslant \theta \leqslant \frac{\pi}{n}$. We define the mapping $\varphi : \mathbf{S} \to \mathbf{S}^*$ by $\varphi(re^{i\theta}) = re^{i|\theta^*|}$, where $|\theta^*|$ satisfies $-\frac{\pi}{n} < \theta^* \leqslant \frac{\pi}{n}$, $\theta \equiv \theta^*(mod\, 2\pi/n)$. The mapping $f : X \to Y$ is said to be folded and branched to order n at a point $x \in X$ if x has a neighbourhood V and there exist homeomorphisms $h_1 : V \to \mathbf{S}^*$, $h_2 : \mathbf{S}^* \to f(V)$ satisfying $f = h_2 \circ \varphi \circ h_1$. Finally we have:

Theorem 3.4. *Let Γ be a finitely generated non-cocompact NEC group with the signature as in Definition 3.1 and let the numbers II_i, III_i, IV_i of components of the different types as defined above. Then the orbit space $S = \boldsymbol{H}/\Gamma$ is a surface:*

1. *of topological genus g, the genus of the compact core of S,*
2. *orientable if the signature of Γ has sign " + ", and non-orientable otherwise,*
3. *with s cusps,*
4. *with r conic points lying in the interior of the compact core of S with the branching orders of the projection $\boldsymbol{H} \longrightarrow \boldsymbol{H}/\Gamma$ given by the integers $m_1, ..., m_r$,*
5. *with k boundary components on the compact core of S,*
6. *with k_i corner points lying on the i-th boundary component of the compact core of S, where the canonical projection $\Gamma \longrightarrow \boldsymbol{H}/\Gamma$ is branched and folded with branching orders n_{ij},*
7. *with l hyperbolic ends of type ν each one consisting of one boundary component of type I,*
8. *with l_i corner points lying on the i-th boundary component of type I, where the canonical projection $\Gamma \longrightarrow \boldsymbol{H}/\Gamma$ is branched and folded with branching orders $\check{n}_{ij}$,*
9. *with q hyperbolic ends of type η each one consisting of $|V_i| + 1$ boundary components of type II,*
10. *with q_i corner points lying on the i-th non-compact boundary component of type II, where the canonical projection $\Gamma \longrightarrow \boldsymbol{H}/\Gamma$ is branched and folded with branching orders $\hat{n}_{ij}$,*

11. *with t hyperbolic ends of funnel type each one consisting of II_i, III_i and IV_i boundary components of type II, III and IV respectively,*

12. *with t_i corner points lying on the i-th hyperbolic end of type funnel, where the canonical projection $\Gamma \longrightarrow \boldsymbol{H}/\Gamma$ is branched and folded with branching orders $\tilde{n}_{ij} < \infty$.*

In the example below we describe the orbit-space of $\mathbf{H}$ under the action of a non-cocompact NEC group in terms of its signature.

Example 3.5. *Let Γ be an NEC group with signature*

$$(3;+;1;[m_1,m_2,m_3];\{(n_{11},n_{12})\};\{(\check{n}_{12},\check{n}_{13},\check{n}_{14})\};\{-\};\{(-),((\tilde{n}_{23},\tilde{n}_{24})),((\tilde{n}_{33},\tilde{n}_{34}),(\tilde{n}_{36}))\}).$$

Theorem 3.4 above shows that the orbit space $\boldsymbol{H}/\Gamma$ has the following data:

1. *it has topological genus $g = 3$,*

2. *is orientable,*

3. *with $s = 1$ cusps,*

4. *with $r = 3$ conic points lying in the interior of the compact core of S with the branching orders given by the integers m_1, m_2, m_3,*

5. *with $k = 1$ boundary component on the compact core of S,*

6. *with $k_1 = 2$ semi-conic points lying on the boundary component of the compact core of S, where the canonical projection $\Gamma \longrightarrow \boldsymbol{H}/\Gamma$ is branched with branching orders n_{11} and n_{12},*

7. *with $l = 1$ boundary components of type I, that is, one non-compact boundary component with one point at infinity,*

8. *with $l_i = 3$ corner points lying on the component of type I, where the canonical projection $\Gamma \longrightarrow \boldsymbol{H}/\Gamma$ is branched with branching orders $\check{n}_{12}, \check{n}_{13}$ and $\check{n}_{14}$,*

9. *with $t = 3$ funnels,*

10. *with* $0, 2$ *and* 3 *corner points lying on the 3 funnels* $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$ *respectively, where the canonical projection* $\Gamma \longrightarrow \boldsymbol{H}/\Gamma$ *is branched and folded with branching orders* $\tilde{n}_{23}, \tilde{n}_{24}, \tilde{n}_{33}, \tilde{n}_{34}$ *and* $\tilde{n}_{36}$.

Finally, we calculate explicitely the numbers II_i, III_i, IV_i *of components of type II, III and IV in each funnel* $i = 1, 2, 3$. *We have then,*
$F_1 = \{1\} - U_1 = \varnothing$, $F_2 = \{1, 2, 3, 4\} - U_2 = \{2, 3, 4\}$, $F_3 = \{1, 2, 3, 4, 5, 6\} - U_3 = \{2, 3, 4, 5, 6\}$, $I_1 = \varnothing, I_2 = I_{21} = \{2, 3, 4\}, I_3 = I_{31} = \{2, 3, 4, 5, 6\}, \Lambda_{11} = \varnothing, \Lambda_{21} = I_{21} \cap \tilde{V}_2 = \varnothing, \Lambda_{31} = I_{31} \cap \tilde{V}_3 = \{4\}$.
For the first case, we have obviously $II_1 = III_1 = IV_1 = 0$ *as we have a funnel without cuts. For the other cases:*
$IV_2 = 1$, $III_2 = 2(|U_2| - IV_2) = 2(1-1) = 0$ *and* $II_2 = |\tilde{V}_2| - |U_2| + IV_2 = 0 - 1 + 1 = 0$,
$IV_3 = 1$, $III_3 = 2(|U_3| - IV_3) = 2(1-1) = 0$ *and* $II_3 = |\tilde{V}_3| - |U_3| + IV_3 = 1 - 1 + 1 = 1$.

Example 3.6. *Figure* 33 *illustrates the non-compact orbit-space of an NEC group of signature*

$$(3; +; 1; [m_1, m_2, m_3]; \{(n_{11}, n_{12})\}; \{(\check{n}_{12}, \check{n}_{13}, \check{n}_{14})\}; \{((\hat{n}_{12}), (\hat{n}_{14}))\};$$
$$\{(-), ((\tilde{n}_{23}, \tilde{n}_{24}), (\tilde{n}_{26})), ((\tilde{n}_{33}, \tilde{n}_{34}, \infty, \tilde{n}_{36}))\}).$$

of genus 3 with three funnels including one with two cuts and other with two parallel cuts, one cusp, one hole with a semi-cusp, one hole with two semi-cusps, an additional compact boundary and conic points lying in the surface and in its different borders, corresponding to the NEC group of the example.

3.3 Algebraic classification: type-preserving isomorphisms

In this section, we determine necessary and sufficient conditions for two non-cocompact NEC groups to be isomorphic via a type-preserving isomorphism. In case of cocompact NEC groups, the concept of algebraic isomorphism and type-preserving isomorphism are equivalent, as shown by Macbeth in [40]. However, this property cannot be extended to the non-cocompact case as the example below shows:

Example 3.7. *Let* Γ *and* Γ' *be two NEC groups with signatures* $(1; +; 0; [-]; \{-\}; \{-\}; \{((-), (-))\}; \{-\})$ *and* $(1; +; 0; [-]; \{-\}; \{-\}; \{-\}; \{((-), (-))\})$, *respectively. Let us define the assignment* $\phi : \Gamma \rightarrow$

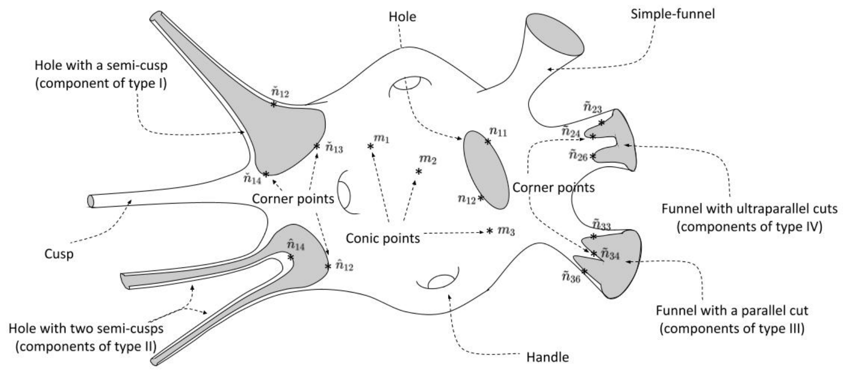


Figure 33: Non-compact orbit-space.

Γ', *by:*

$A_1 \to A'_1$,

$B_1 \to B'_1$,

$C_1 \to C'_1$,

$C_2 \to C'_2$,

$E \to H$.

It is clear that ϕ defines a group isomorphism. Let us suppose it is type-preserving. Then, the parabolic element C_1C_2 is transformed into the element $\phi(C_1)\phi(C_2)$, which is by the definition of ϕ the product of two reflections, that should be parabolic. However, from the signature above this product is hyperbolic and therefore the isomorphism assigns a parabolic transformation into a hyperbolic transformation which is a contradiction.

We extend now Macbeath's definition of directly and inversely equivalence of period-cycles in the compact case to ν-, η-, μ-cycles in the non-compact case. For the period-cycles, Macbeath's definition says that the period-cycle $C' = (n'_1, ..., n'_{k'})$ is directly equivalent to the period-cycle $C = (n_1, ..., n_k)$ if $k = k'$ and $(n'_1, ..., n'_k)$ is a cyclic permutation of $(n_1, ..., n_k)$. Analogously C' is inversely equivalent to C if $k = k'$ and $(n'_1, ..., n'_k)$ is a cyclic permutation of $(n_1, ..., n_k)$ reversed, i.e. $(n_k, ..., n_1)$. Similarly, we introduce the following definitions:

Definition 3.8. Directly and reversely equivalence of ν-, η-, μ-cycles:

1. The ν-cycle $\check{C}' = (\check{n}'_1, ..., \check{n}'_{l'})$ is directly equivalent to the ν-cycle $\check{C} = (\check{n}_1, ..., \check{n}_l)$ if they are identical, that is $l = l'$ and $\check{n}_i = \check{n}'_i$ for $i = 1, ..., l$. They are reversely equivalent if $l = l'$ and $\check{C}'$ is the same as $\check{C}$ reversed.

2. The η-cycle $\hat{C}'$ that we write using the expanded form $(I'_1, ..., I'_{|\hat{V}'|}) = ((\hat{n}'_2, ..., \hat{n}'_{\hat{v}'_2-1}), ..., (\hat{n}'_{|\hat{V}'|+1}, ..., \hat{n}'_{q'}))$ is directly equivalent to the η-cycle $\hat{C} = (I_1, ..., I_{|\hat{V}|}) = ((\hat{n}_2, ..., \hat{n}_{\hat{v}_2-1}), ..., (\hat{n}_{|\hat{V}|+1}, ..., \hat{n}_q))$, if $q = q', |\hat{V}| = |\hat{V}'|$ and the components $(I'_1, ..., I'_{|\hat{V}|})$ of $\hat{C}'$ are a cyclic permutation of the components $(I_1, ..., I_{|\hat{V}|})$ of $\hat{C}$. We say that $\hat{C}'$ is reversely equivalent to $\hat{C}$ if the components of $\hat{C}'$ are a cyclic permutation of the reversed components of $\hat{C}$, reversed, i.e. if $(I'_1, ..., I'_{|\hat{V}|})$ is a cyclic permutation of $(I^*_{|\hat{V}|}, ..., I^*_1)$, where we define I^*_i such that for a given component $I_i = (\hat{n}_{\hat{v}_i+1}, ..., \hat{n}_{\hat{v}_{i+1}-1})$, $I^*_i = (\hat{n}_{\hat{v}_{i+1}-1}, ..., \hat{n}_{\hat{v}_i+1})$.

3. The μ-cycle $\tilde{C}'$ that we write using the components $((\tilde{n}'_2, ..., \tilde{n}'_{\tilde{u}'_2-1}); ...; (\tilde{n}'_{|\tilde{U}'|+2}, ..., \tilde{n}'_{t'})) = (I'_1, ..., I'_{|\tilde{U}'|})$ is directly equivalent to the μ-cycle $\tilde{C} = ((\tilde{n}_2, ..., \tilde{n}_{\tilde{u}_2-1}); ...; (\tilde{n}_{|\tilde{U}|+2}, ..., \tilde{n}_t)) = (I_1, ..., I_{|\tilde{U}|})$ if $t = t', |\tilde{U}| = |\tilde{U}'|$ and the components $(I'_1, ..., I'_{|\tilde{U}|})$ of $\tilde{C}'$ are a cyclic permutation of the components $(I_1, ..., I_{|\tilde{U}|})$ of $\tilde{C}$. We say that $\tilde{C}'$ is reversely equivalent to $\tilde{C}$ if the components of $\tilde{C}'$ are a cyclic permutation of the reversed components of $\tilde{C}$, reversed, i.e. if $(I'_1, ..., I'_{|\tilde{U}|})$ is a cyclic permutation of $(I^*_{|\tilde{U}|}, ..., I^*_1)$, where as defined previously, if $I_i = (\tilde{n}_{\tilde{u}_i+2}, ..., \tilde{n}_{\tilde{u}_{i+1}-1})$, then $I^*_i = (\tilde{n}_{\tilde{u}_{i+1}-1}, ..., \tilde{n}_{\tilde{u}_i+2})$.

Theorem 3.9. *An NEC group* Γ *with signature*

$sg = (g; \pm; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}_t\}),$

is isomorphic to an NEC group Γ' *with signature*

$sg' = (g'; \pm; s'; [m'_1, ..., m'_{r'}]; \{C'_1, ..., C'_{k'}\}; \{\check{C}'_1, ..., \check{C}'_{l'}\}; \{\hat{C}'_1, ..., \hat{C}'_{q'}\}; \{\tilde{C}'_1, ..., \tilde{C}'_{t'}\}),$

via a type-preserving isomorphism if and only if $g = g'$*, sign(sg) = sign(sg'),* $s = s'$*,* $r = r'$*,* $k = k'$*,* $l = l'$*,* $q = q'$*,* $t = t'$*, the proper periods* $[m'_1, ..., m'_{r'}]$ *are a permutation of the proper periods* $[m_1, ..., m_r]$ *and there exist permutations* π *of* $\{1, ..., k\}$*,* $\check{\pi}$ *of* $\{1, ..., l\}$*,* $\hat{\pi}$ *of* $\{1, ..., q\}$ *and* $\tilde{\pi}$ *of* $\{1, ..., t\}$ *such that:*

1. *in the orientable case, all the cycles* C'_i*,* $\check{C}'_i$*,* $\hat{C}'_i$ *and* $\tilde{C}'_i$ *are either directly equivalent to the corresponding cycles* C_{π_i}*,* $\check{C}_{\check{\pi}(i)}$*,* $\hat{C}_{\hat{\pi}(i)}$ *and* $\tilde{C}_{\tilde{\pi}(i)}$ *or all are reversely equivalent.*

2. *in the non-orientable case, each cycle* C'_i*,* $\check{C}'_i$*,* $\hat{C}'_i$ *and* $\tilde{C}'_i$ *is either directly equivalent to the corresponding cycles* $C_{\pi(i)}$*,* $\check{C}_{\check{\pi}(i)}$*,* $\hat{C}_{\hat{\pi}(i)}$ *and* $\tilde{C}_{\tilde{\pi}(i)}$ *or reversely equivalent.*

Similarly to the cocompact case, we see that in the orientable case, corresponding pairs of cycles are all paired in the same way -either all directly or all inversely. In the non-orientable case, some may be paired directly and some inversely.

Proof of neccesary conditions. Associated to the presentation of Γ as in Theorem 2.11 we have a marked polygon $\mathbf{F}$ with a canonical surface symbol as described in Section 2. As we are assuming that Γ and Γ' are isomorphic via a type preserving isomorphism, Γ' admits exactly the same presentation as Γ, its associated marked polygon $\mathbf{F}'$ has the same amount of edges and vertices, and it can be built such that they are written in the same order. Moreover, the side-pairing is carried out by the same type of transformations as in $\mathbf{F}$.

More specifically, for the orientable case the marked polygon of Γ can be defined as:

$$\mathbf{F} = \prod_{i=1}^{r} x_i x_i' \prod_{i=1}^{s} p_i p_i' \prod_{i=1}^{g} a_i b_i' a_i' b_i \prod_{i=1}^{k} e_i C_i e_i' \prod_{i=1}^{l} \check{e}_i \check{C}_i \check{e}_i' \prod_{i=1}^{q} \hat{e}_i \hat{C}_i \hat{e}_i' \prod_{i=1}^{t} \tilde{e}_i \tilde{C}_i \tilde{e}_i',$$

and in the way we have defined the isomorphic presentation, we can build a fundamental region of Γ' so that the labelled polygon has the form:

$$\mathbf{F}' = \prod_{i=1}^{r} x_i'' x_i''' \prod_{i=1}^{s} p_i'' p_i''' \prod_{i=1}^{g} a_i'' b_i''' a_i''' b_i'' \prod_{i=1}^{k} e_i'' C_i e_i''' \prod_{i=1}^{l} \check{e}_i'' \check{C'}_i \check{e}_i''' \prod_{i=1}^{q} \hat{e}_i'' \hat{C'}_i \hat{e}_i''' \prod_{i=1}^{t} \tilde{e}_i'' \tilde{C'}_i \tilde{e}_i''',$$

with the same amount of edges, free-sides, vertices in $\mathbf{H}$ and at infinity, of the same type and order. The same construction above as well as the discussion in the coming paragraphs can be done for the non-orientable case without changes in the argument. As mentioned above, the labelled polygons $\mathbf{F}$ and $\mathbf{F}'$ can be built such that the paired edges $x_i x_i'$, $p_i p_i'$, $a_i b_i' a_i' b_i$, $e_i C_i e_i'$, $\check{e}_i \check{C}_i \check{e}_i'$, $\hat{e}_i \hat{C}_i \hat{e}_i'$ and $\tilde{e}_i \tilde{C}_i \tilde{e}_i'$ of $\mathbf{F}$ and $x_i'' x_i'''$, $p_i'' p_i'''$, $a_i'' b_i''' a_i''' b_i''$, $e_i'' C_i' e_i'''$, $\check{e}_i' \check{C}_i' \check{e}_i'''$, $\hat{e}_i'' \hat{C}_i' \hat{e}_i'''$ and $\tilde{e}_i'' \tilde{C}_i' \tilde{e}_i'''$ of $\mathbf{F}'$ follow the same order in both labelled polygons with ν-, η-, μ-sequences including the same free-edges, edges and vertices in $\mathbf{H} \cup \partial\mathbf{H}$ in the same order. This is so due to the fact that the group Γ and the presentation of Γ' injected by the type-preserving isomorphism have identical number, type and order of transformations, fact that cannot be stated in general for the given presentation of the group Γ' as Example 3.7 shows. Two hyperbolic polygons with the same amount and order of edges and vertices in $\mathbf{H} \cup \partial\mathbf{H}$ are homeomorphic. If we identify the corresponding points on the paired edges, we obtain $\mathbf{F}/\Gamma$ and $\mathbf{F}'/\Gamma'$, two non-compact surfaces with boundaries and in the way we have defined $\mathbf{F}, \mathbf{F}'$, it follows then that $\mathbf{F}/\Gamma$ and $\mathbf{F}'/\Gamma'$ are homemeomorphic.

In the diagram below, we represent the different marked polygons and surfaces described up to now, where i, i' are inclusion maps, j_1, j_2, j_1', j_2' are natural projections, r, r' the homeomorphisms mentioned above and θ, θ' are the maps which make the squares I and III below commutative.

$$\begin{tikzcd}
\mathbf{H} \arrow[d, "j_2"] \arrow[dr, phantom, "\mathbf{I}"] & \mathbf{F} \arrow[l, "i"'] \arrow[r, "r"] \arrow[d, "j_1"] \arrow[dr, phantom, "\mathbf{II}"] & \mathbf{F}' \arrow[r, "i'"] \arrow[d, "j_1'"] \arrow[dr, phantom, "\mathbf{III}"] & \mathbf{H} \arrow[d, "j_2'"] \\
\mathbf{H}/\Gamma & \mathbf{F}/\Gamma \arrow[l, "\theta"'] \arrow[r, "r'"] & \mathbf{F}'/\Gamma' \arrow[r, "\theta'"] & \mathbf{H}/\Gamma'
\end{tikzcd}$$

Figure 34: Homeomorphism between $\mathbf{H}/\Gamma$ and $\mathbf{H}/\Gamma'$

It is well known that the maps θ, θ' are homeomorphisms, as described for example in [23, Th. 5.9.6], where the proof is done for Fuchsian groups though, but it can be extended to NEC groups without changes. Therefore, the map defined by $r^* = \theta' r' \theta^{-1} : \mathbf{H}/\Gamma \to \mathbf{H}/\Gamma'$ is also a homeomorphism. Additionally, if $[C_i], [\check{C}_i], [\hat{C}_i]$ and $[\tilde{C}_i]$ denote the image in $\mathbf{H}/\Gamma$ of an o-, ν-, η- or an μ-sequence respectively, then $r^*([C_i]) = [C_j'], r^*([\check{C}_i]) = [\check{C}_j'], r^*([\hat{C}_i]) = [\hat{C}_j'], r^*([\tilde{C}_i]) = [\tilde{C}_j']$ are respectively the corresponding images in $\mathbf{H}/\Gamma'$ of an o-, ν-, η- or an μ-sequence of Γ' of the same type, due to the fact that Γ and Γ' are isomorphic via a type-preserving assignation. Taking into consideration that r^* is a homeomorphism, we have the following cases:

(i) $[C] \to [C']$:
this is the compact case and we have a homeomorphism $r^* : S^1 \to S^1$ that has either degree $+1$ or -1. In the orientable case, all the boundaries are mapped with the same degree and so the sets of period-cycles are transformed either all to a cyclic permutation, or all to a cyclic permutation of the other reversed. In the non-orientable case, each boundary can be mapped with either degree $+1$ or -1 and so each period cycles can be permuted cyclically or cyclically reversed.

(ii) $[\check{C}] \to [\check{C}']$:
in this case the boundaries are homeomorphic to S^1 minus one point and so we can

write $r^* : S^1 - \{a\} \to S^1 - \{a'\}$. The extension of r^* to the compactification, which we still denote r^*, has degree $+1$ or -1. However, as $[\check{C}], [\check{C}']$ have one point removed, the homeomorphism leaves the ν-periods as they are or reversed. This situation is explained in Figure 35, where the ovals are the non-compact borders, and $N_1, N_2, ..., N_l$ are distinguished points on the border related to the ν-periods. Let γ be a path on the border from the distinguished point N_1 related to the ν-period n_1 to the point N_l related to the ν-period n_l and let γ' be the image by r^* of γ. As we have removed one point on the borders, there is just a unique continuous path possible between N_1 and N_l and the same happens between $r^*(N_1)$ and $r^*(N_l)$. But if there were cyclic permutations of the periods, then both paths γ and γ' would be homeomorphic, which is not possible because the path γ contains all the other distinguished points $N_2, ..., N_{l-1}$ and the path γ' contains none of their images. We conclude then, that the ν-periods in the case of type-preserving isomorphisms can only be either the same or the same reversed.

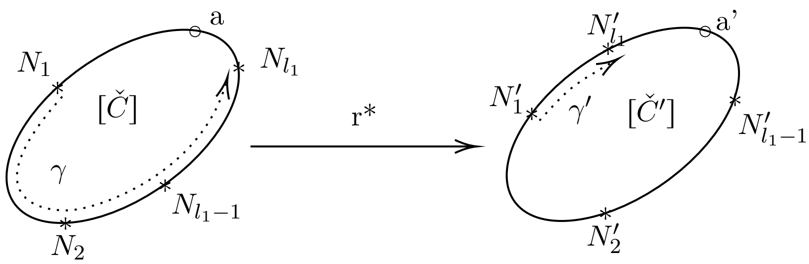


Figure 35: Paths γ and γ' on non-compact borders cannot be homeomorphic

(iii) $[\hat{C}] \to [\hat{C}']$:

in this case the boundaries are homeomorphic to S^1 minus n points and so we can write $r^* : S^1 - \{a_1, ..., a_n\} \to S^1 - \{a'_1, ..., a'_n\}$. Again, the extension of r^* to the compactification has degree $+1$ or -1. The η-cycle can be written in the expanded form $((\hat{n}_2, ..., \hat{n}_{v_2-1}), (\hat{n}_{v_2+1}, ..., \hat{n}_{v_3-1}), ..., (\hat{n}_{\hat{v}_{\hat{v}}+1}, ..., \hat{n}_q))$. Let us denote $I_j = (\hat{n}_{v_{j-1}+1}, ..., \hat{n}_{v_j-1})$. In this way the η-cycle can be represented in the form $(I_1, ..., I_{|\hat{V}|})$. Using exactly the same argument as the discussion about paths on the border in the case of ν-cycles, we conclude that the η-periods cannot be exchanged between components or permuted cyclically within the components. Only cyclic permutations of the components or cyclic permutations of the reversed cycles I_j^* are possible, i.e. only the cyclic permutations $(I_2, ..., I_{|\hat{V}|}, I_1)$, $(I_3, ..., I_{\hat{v}_{|\hat{V}|}}, I_1, I_2)$, etc. and cyclic permutations of the reversed periods $(I^*_{|\hat{V}|}, I^*_{|\hat{V}|-1}, ..., I^*_1)$,$(I^*_{|\hat{V}|-1}, ..., I^*_1, I^*_{|\hat{V}|})$, etc. are compatible with the homeomorphism induced by the type-preserving isomorphism.

(iv) $[\tilde{C}] \to [\tilde{C}']$:

in this case the boundaries are homeomorphic to S^1 minus n points and minus m closed arcs and so we can write $r^* : S^1 - \{a_1, ..., a_n\} - \sum_{i=1}^{m}[b_i, c_i] \to S^1 - \{a'_1, ..., a'_n\} - \sum_{i=1}^{m}[b'_i, c'_i]$, where $[b_i, c_i], [b'_i, c'_i]$ stand for closed arcs removed on the borders. Removing a closed arc from S^1 yields a homeomorphic space as removing a point from S^1. Then, we can compactify adding $m + n$ points and we get that r^* has degree $+1$ or -1. Now, the boundary by the μ-cycle can be written in the form $((\tilde{n}_3, ..., \tilde{n}_{u_2-1}), (\tilde{n}_{u_2+2}, ..., \tilde{n}_{u_3-1}), ..., (\tilde{n}_{u_u+2}, ..., \tilde{n}_t)$. Let us denote $I_j = (\tilde{n}_{u_j+2}, ..., \tilde{n}_{u_{j+1}-1})$. In this way the μ-cycle can be represented in the form $(I_1, ..., I_{|U|})$. Using exactly the same argument as the discussion about paths on the border as in the case of η- and ν-cycles, we conclude that the μ-periods cannot be exchanged between components or permuted cyclically within the components and only cyclic permutations of the components or cyclic permutations of the reversed cycles I_j^* are possible, i.e. only the cyclic permutations $(I_2, ..., I_{|U|}, I_1)$, $(I_3, ..., I_{|U|}, I_1, I_2)$, etc. and cyclic permutations of the reversed cycles $(I^*_{|U|}, ..., I^*_1)$, $(I^*_{|U|-1}, ..., I^*_1, I^*_{|U|})$, $(I^*_{|U|-2}, ..., I^*_1, I^*_{|U|}, I^*_{|U|-1})$, etc. are compatible with the homeomorphism induced by the type-preserving isomorphism.

□

Before dealing with the proof of the sufficient conditions we show in the following lemmas that some basic transformations between the signatures of the groups Γ and Γ' can be represented as type-preserving isomorphisms between them. We will apply these lemmas to finalize the proof of the Theorem 3.9.

Lemma 3.10. *An NEC group* Γ *with signature*

$(g; \pm; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}_t\})$,

is isomorphic via a type-preserving isomorphism to an NEC group Γ' *with signature*

$$(g; \pm; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_{\tilde{\pi}(1)}, ..., \check{C}_{\tilde{\pi}(l)}\}; \{\hat{C}_1, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}_t\}),$$

where $\tilde{\pi}$ is a permutation of $\{1, ..., l\}$.

Proof. It is enough to prove that two consecutive ν-cycles can be permuted, as any other permutation is a finite composite of this basic one. To lighten notation, we show that $\check{C}_1$ and $\check{C}_2$ can be permuted. Then, the presentation of Γ includes the relations:

(*i*) $\check{C}_{1j}^2 = \check{C}_{2j}^2 = 1$, for all the reflections,

(*ii*) $(\check{C}_{1,j-1}\check{C}_{1j})^{\check{n}_{1j}} = 1$ for $j = 2, \ldots, l_1$; $(\check{C}_{2,j-1}\check{C}_{2,j})^{\check{n}_{2j}} = 1$ for $j = 2, \ldots, l_2$;

(*iii*) $\prod_{i=1}^{r} X_i \prod_{i=1}^{s} P_i \prod_{j=1}^{g} [A_j, B_j] \prod_{i=1}^{k} E_i \check{E}_1 \check{E}_2 ... \check{E}_l \prod_{i=1}^{q} \hat{E}_i \prod_{i=1}^{t} \tilde{E}_i = 1.$

where we are assuming that the orbit space $\boldsymbol{H}/\Gamma$ is orientable. In the non-orientable case the proof is the same and will not be repeated. The presentation of Γ' includes:

(*i'*) ${\check{C}'}_{1j}^{2} = {\check{C}'}_{2j}^{2} = 1$, for all the reflections,

(*ii'*) $(\check{C}'_{1,j-1}\check{C}'_{1j})^{\check{n}'_{1j}} = 1$ for $j = 2, \ldots, l_1$; $(\check{C}'_{2,j-1}\check{C}'_{2j})^{\check{n}'_{2j}} = 1$ for $j = 2, \ldots, l_2$;

(*iii'*) $\prod_{i=1}^{r} X'_i \prod_{i=1}^{s} P'_i \prod_{j=1}^{g} [A'_j, B'_j] \prod_{i=1}^{k} E'_i \check{E}'_1 \check{E}'_2 ... \check{E}'_l \prod_{i=1}^{q} \hat{E}'_i \prod_{i=1}^{t} \tilde{E}'_i = 1,$

where we assume that $\check{n}'_{1j} = \check{n}_{2j}$ and $\check{n}'_{2j} = \check{n}_{1j}$. It is easy to show that the isomorphism $\phi : \Gamma \to \Gamma'$ defined by:

1. $\phi : \check{E}_1 \to \check{E}'_1 \check{E}'_2 \check{E}'^{-1}_1$,

2. $\phi : \check{E}_2 \to \check{E}'_1$,

3. $\phi : \check{C}_{1j} \to \check{E}'_1 \check{C}'_{2j} \check{E}'^{-1}_1$,

4. $\phi : \check{C}_{2j} \to \check{C}'_{1j}$,

5. all the other generators of Γ correspond in the same manner to those in Γ',

is a type-preserving isomorphism. It is relevant to remark that the parabolic elements $\check{E}_1 \check{C}_{1l_1} \check{E}_1^{-1} \check{C}_{11}$ and $\check{E}_2 \check{C}_{2l_2} \check{E}_2^{-1} \check{C}_{21}$ are transformed into parabolic elements. □

The proof is similar for the η- and μ-cycles and will not be repeated. In these cases we just need to consider that there are more parabolic and/or hyperbolic products of reflections. The corresponding lemmas are:

Lemma 3.11. *An NEC group* Γ *with signature*

$$(g; \pm; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}_t\}),$$

is isomorphic via a type-preserving isomorphism to an NEC group Γ' *with signature*

$$(g; \pm; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_{\hat{\pi}(1)}, ..., \hat{C}_{\hat{\pi}(q)}\}; \{\tilde{C}_1, ..., \tilde{C}_t\}),$$

where $\hat{\pi}$ *is a permutation of* $\{1, ..., q\}$*.*

Lemma 3.12. *An NEC group* Γ *with signature*

$$(g; \pm; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}_t\}),$$

is isomorphic via a type-preserving isomorphism to an NEC group Γ' *with signature*

$$(g; \pm; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_q\}; \{\tilde{C}_{\tilde{\pi}(1)}, ..., \tilde{C}_{\tilde{\pi}(t)}\}),$$

where $\tilde{\pi}$ *is a permutation of* $\{1, ..., t\}$*.*

Recall that two ν-cycles are directly equivalent if they are the same and therefore there is nothing to prove for them in the case of direct equivalence. We now deal with cyclic permutations of the components of an η- or a μ-cycle, see Lemmas 3.13 and 3.14 below:

Lemma 3.13. *The group* Γ *with signature*

$$(g; \pm; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_i, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}_t\}),$$

is isomorphic via a type-preserving isomorphism to an NEC group Γ' *with signature*

$$(g; \pm; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}'_i, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}_t\}),$$

where the η*-cycle* $\hat{C}'_i$ *is directly equivalent to the cycle* $\hat{C}_i$*, i.e. the components of* $\hat{C}'_i$ *are a cyclic permutation of the components of* $\hat{C}_i$*.*

Proof. Let us assume that Γ and Γ' have all the same relations except for the i-th η-cycle, and let us use the component representation of the η-cycles, so that the i-th η-cycle has the form $(I_{i1}, ..., I_{i|\hat{V}_i|})$. To lighten notation we drop the first subscript "i", where the context makes it obvious the distinction between the cycles and the reflections. We write $((\hat{n}_2, ..., \hat{n}_{\hat{v}_2-1}), (\hat{n}_{\hat{v}_2+1}, ..., \hat{n}_{\hat{v}_3-1}), ..., (\hat{n}_{\hat{v}_{\hat{v}_i}+1}, ..., \hat{n}_{q_i}))$, the j-th component being $I_j = (\hat{n}_{\hat{v}_j+1}, ..., \hat{n}_{\hat{v}_{j+1}-1})$, where by convention we write $\hat{v}_1 = 1$. The corresponding relations, either in the orientable or in the non-orientable case, can be written:

(*i*) $(\hat{C}_{j-1}\hat{C}_j)^{\hat{n}_j} = 1$ for $j \in \{2, \ldots, q_i\} - \hat{V}_i$ and a non-empty $\hat{V}_i = \{\hat{v}_2, ..., \hat{v}_{\hat{v}_i}\} \subset \{2 \ldots, q_i\}$,

(*ii*) the product $\hat{C}_{j-1}\hat{C}_j$ is parabolic for $j \in \hat{V}_i$,

(*iii*) the product $\hat{E}\hat{C}_{q_i}\hat{E}^{-1}\hat{C}_1$ is parabolic.

The i-th η-cycle of Γ' is assumed by hypothesis to be a cyclic permutation of $(I_1, ..., I_{|\hat{V}_i|})$, that we may take $(I_2, ..., I_{|\hat{V}_i|}, I_1)$, as any other cyclic permutation is the succesive application of this one. The corresponding relations include then:

(i') $(\hat{C'}_{j-1}\hat{C'}_j)^{\hat{n'}_j} = 1$ for $j \in \{2, \ldots, q_i\} - \hat{V'}_i$ for some non-empty $\hat{V'}_i \subset \{2, \ldots, q_i\}$,

(ii') the product $\hat{C'}_{j-1}\hat{C'}_j$ is parabolic for $j \in \hat{V'}_i$,

(iii') the product $\hat{E}'\hat{C}'_{q_i}\hat{E}'^{-1}\hat{C}'_1$ is parabolic.

As defined before, the components of the i-th η-cycle of Γ' are the cyclic permutation of the components of the i-th η-cycle of Γ such that the $\hat{v}_2$-th reflection in Γ is transformed into the first reflection in Γ', and therefore the j-th reflection in Γ is transformed in the $(j - \hat{v}_2 + 1)$-th reflection in Γ', taking the indexes modulo q_i. Similarly, the set $\{1\} \cup \hat{V}_i = \{1, \hat{v}_2, ..., \hat{v}_{\hat{v}_i}\}$ is transformed after the cyclic permutation in the set $\{1\} \cup \hat{V'}_i = \{1, \hat{v}'_2, ..., \hat{v}'_{\hat{v}_i}\} = \{(1 - \hat{v}_2 + 1), 1, ..., (\hat{v}_{\hat{v}_i} - \hat{v}_2 + 1)\}$ with the integers read module q_i. Reordering, we finally get $\{1\} \cup \hat{V'}_i = \{1, \hat{v}_3 - \hat{v}_2 + 1, ..., \hat{v}_{\hat{v}_i} - \hat{v}_2 + 1, q_i - \hat{v}_2 + 2\}$. The isomorphism $\phi : \Gamma \to \Gamma'$ is defined by:

1. $\phi : \hat{C}_j \to \hat{C}'_{j-\hat{v}_2+1}$, $j - \hat{v}_2 + 1 > 0$,
2. $\phi : \hat{C}_j \to \hat{E}'\hat{C}'_{j-\hat{v}_2+1}\hat{E}'^{-1}$, otherwise,
3. all the other generators of Γ correspond in the same manner to those in Γ'.

Now, it is clear that ϕ is an isomorphism, it is type-preserving and $(i'), (ii'), (iii')$ are the images of $(i), (ii), (iii)$ via ϕ:

- Let us assume first that $j \notin \{1\} \cup \hat{V}_i$. For $j > v_2 - 1$, we have $\phi((\hat{C}_{j-1}\hat{C}_j)^{\hat{n}_j}) = (\hat{C}'_{j-\hat{v}_2}\hat{C}'_{j-\hat{v}_2+1})^{\hat{n}_j} = 1$. The periods verify then $\hat{n'}_j = \hat{n}_{j-\hat{v}_2+1}$ and we finally get $(\hat{C}'_{j-1}\hat{C}'_j)^{\hat{n'}_j} = 1$. In the case $j \leqslant \hat{v}_2 - 1$, we have $\phi((\hat{C}_{j-1}\hat{C}_j)^{\hat{n}_j}) = (\hat{E}'^{-1}\hat{C}'_{j-\hat{v}_2}\hat{C}'_{j-\hat{v}_2+1}\hat{E}')^{\hat{n}_j} = 1$ and $(\hat{C}'_{j-1}\hat{C}'_j)^{\hat{n}'_j} = 1$ with $\hat{n}'_j = \hat{n}_{j-\hat{v}_2+1}$.
- Let us assume that $j \in \{1\} \cup \hat{V}_i$. Then, for $j > v_2 - 1$ the image $\hat{C'}_{j-\hat{v}_2}\hat{C'}_{j-\hat{v}_2+1}$ of the parabolic element $\hat{C}_{j-1}\hat{C}_j$ is parabolic. For $j < v_2 - 1$, the image $E'\hat{C'}_{j-\hat{v}_2}\hat{C'}_{j-\hat{v}_2+1}E'^{-1}$ of the parabolic element $\hat{C}_{j-1}\hat{C}_j$ is parabolic and so is the product $\hat{C'}_{j-\hat{v}_2}\hat{C'}_{j-\hat{v}_2+1}$. The image $\hat{E}'\hat{C'}_{q_i-\hat{v}_2+1}\hat{C'}_{q_i-\hat{v}_2+2}\hat{E}^{-1}$ of the parabolic element $\hat{E}\hat{C}_{q_i}\hat{E}^{-1}\hat{C}_1$ is parabolic and

so must be $\hat{C}'_{q_i-\hat{v}_2+1}\hat{C}'_{q_i-\hat{v}_2+2}$. Finally, the image $\hat{E}'\hat{C}'_{q_i}\hat{E}'^{-1}\hat{C}'_1$ of the parabolic element $\hat{C}_{\hat{v}_2-1}\hat{C}_{\hat{v}_2}$ is again parabolic. The index calculations are done module q_i.

□

Lemma 3.14. *An NEC group* Γ *with signature*

$$(g; \pm; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}_i, ..., \tilde{C}_t\}),$$

is isomorphic via a type-preserving isomorphism to an NEC group Γ' *with signature*

$$(g; \pm; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}'_i, ..., \tilde{C}_t\}),$$

where the μ*-cycle* $\tilde{C}'_i$ *is directly equivalent to the cycle* $\tilde{C}_i$*, i.e. the components of* $\tilde{C}'_i$ *are a cyclic permutation of the components of* $\tilde{C}_i$*.*

Proof. The proof of this lemma is very similar to the lemma before with the following difference: the components in the previous case of η-cycles are delimited by the parabolic product of reflections. Now the hyperbolic products of reflections delimit the components of the μ-cycles. Let us assume that Γ and Γ' have the same relations except for the i-th μ-cycle, and let us use the component representation of the μ-cycles, so that the i-th μ-cycle has the form $(I_1, ..., I_{|U_i|})$, where we have dropped again the first subscript "i". The corresponding relations either in the orientable or the non-orientable case can be written:

- *(i)* $(\tilde{C}_{j-1}\tilde{C}_j)^{\tilde{n}_j} = 1$ whenever the reflections $\tilde{C}_{j-1}$ and $\tilde{C}_j$ exist, that is, whenever $\{j-1, j\} \cap U_i = \varnothing$, where for simplifying the proof, the parabolic products $\tilde{C}_{j-1}\tilde{C}_j$ where $j \in \tilde{V}_i$ are assumed with period infinite,
- *(ii)* the product $\tilde{C}_{j-1}\tilde{C}_{j+1}$ is hyperbolic for $j \in U_i$,
- *(iii)* the product $\tilde{E}\tilde{C}_{t_i}\tilde{E}^{-1}\tilde{C}_2$ is hyperbolic.

The i-th μ-cycle of Γ' is assumed by hypothesis to be a cyclic permutation of $(I_1, ..., I_{|U_i|})$, that we may take $(I_2, ..., I_{|U_i|}, I_1)$, as any other cyclic permutation is the succesive application of this one. The corresponding relations include then:

- *(i′)* $(\tilde{C}'_{j-1}\tilde{C}'_j)^{\tilde{n'}_j} = 1$ whenever the reflections $\tilde{C}'_{j-1}$ and $\tilde{C}'_j$ exist, that is, whenever $\{j-1, j\} \cap U'_i = \varnothing$, where we use again infinite periods for parabolic products,
- *(ii′)* the product $\tilde{C}'_{j-1}\tilde{C}'_{j+1}$ is hyperbolic for $j \in U'_i$,

(iii') the product $\tilde{E}'\tilde{C}'_{t_i}\tilde{E}'^{-1}\tilde{C}'_2$ is hyperbolic.

As defined before, the components of the i-th μ-cycle of Γ' are the cyclic permutation of the components of the i-th μ-cycle of Γ such that the free-side at the u_2-th position is transformed into the first free-side of the i-th μ-cycle of Γ', and therefore the j-th reflection in Γ is transformed in the $(j - u_2 + 1)$-th reflection in Γ', taking the indexes modulo t_i. Similarly, the set $U_i = \{1, u_2, ..., u_{u_i}\}$ is transformed after the cyclic permutation in the set $U'_i = \{1, u'_2, ..., u'_{u'_i}\} = \{1 - u_2 + 1, 1, ..., u_{u_i} - u_2 + 1\}$ and reordering we finally get $U'_i = \{1, u_3 - u_2 + 1..., u_{u_i} - u_2 + 1, t_i - u_2 + 2\}$. The isomorphism $\phi : \Gamma \to \Gamma'$ is defined by:

1. $\phi : \tilde{C}_j \to \tilde{C}'_{j-u_2+1}$, $j - u_2 + 1 > 0$,
2. $\phi : \tilde{C}_j \to \tilde{E}'\tilde{C}'_{j-u_2+1}\tilde{E}'^{-1}$, otherwise,
3. all the other generators of Γ correspond in the same manner to those in Γ'.

It is clear that ϕ is an isomorphism, it is type-preserving and $(i'), (ii'), (iii')$ are the images of $(i), (ii), (iii)$ via ϕ. For the elliptic and parabolic products of reflections, the proof is exactly the same as in the lemma before and will not be repeated here. Let us assume then that $j > u_2 - 1, j \in \tilde{U}_i$. We see that the image $\tilde{C}'_{j-u_2}\tilde{C}'_{j-u_2+2} = \tilde{C}'_{j'-1}\tilde{C}'_{j'+1}$ of the hyperbolic element $\tilde{C}_{j-1}\tilde{C}_{j+1}$ is hyperbolic. If $j \leqslant u_2 - 1, j \in \tilde{U}_i$, the image $\tilde{E}'\tilde{C}'_{j-u_2}\tilde{C}'_{j-u_2+2}\tilde{E}'^{-1}$ of the hyperbolic element $\tilde{C}_{j-1}\tilde{C}_{j+1}$ is hyperbolic and so is the product $\tilde{C}'_{j-u_2}\tilde{C}'_{j-u_2+2} = \tilde{C}'_{j'-1}\tilde{C}'_{j'+1}$. The image $\tilde{E}'\tilde{C}'_{t_i-u_2+1}\tilde{C}'_{t_i-u_2+3}\tilde{E}'^{-1}$ of the hyperbolic element $\tilde{E}\tilde{C}_{t_i}\tilde{E}^{-1}\tilde{C}_2$ is hyperbolic and so is $\tilde{C}'_{t_i-u_2+1}\tilde{C}'_{t_i-u_2+3}$. Finally, the image $\tilde{E}'\tilde{C}'_{t_i}\tilde{E}'^{-1}\tilde{C}'_2$ of the hyperbolic element $\tilde{C}_{\tilde{v}_2-1}\tilde{C}_{\tilde{v}_2+1}$ is also hyperbolic. The index calculations are done module t_i. □

We now deal with reverse equivalence of cycles. In the next lemma we show that an NEC group with a signature with $sign =$ " + ", where all the periods and components of the η- and ν-cycles, are the same reversed as the ones of another NEC group with the same $sign$ in the signature, is isomorphic via a type-preserving isomorphism:

Lemma 3.15. *An NEC group Γ with signature*

$(g; +; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}_t\})$,

is isomorphic via a type-preserving isomorphism to an NEC group Γ' with signature

$(g; +; s; [m_1, ..., m_r]; \{C_1^*, ..., C_k^*\}; \{\check{C}_1^*, ..., \check{C}_l^*\}; \{\hat{C}_1^*, ..., \hat{C}_q^*\}; \{\tilde{C}_1^*, ..., \tilde{C}_t^*\})$,
where the cycles $C_i^*, \check{C}_i^*, \hat{C}_i^*, \tilde{C}_i^*$ are inversely equivalent to the cycles $C_i, \check{C}_i, \hat{C}_i, \tilde{C}_i$ respectively, i.e. the periods of C_i^* are a cyclic permutation of the periods of C_i^* reversed, the periods of $\check{C}_i^*$ are the periods of $\check{C}_i$ reversed and the components of $\hat{C}_i^*, \tilde{C}_i^*$ are a cyclic permutation of the reversed components of $\hat{C}_i, \tilde{C}_i$, reversed.

Proof. Using the properties of the proper periods (cocompact NEC case) and the lemmas above, we may assume Γ' to have the signature:
$(g; +; s; [m_r, ..., m_1]; \{C_k^*, ..., C_1^*\}; \{\check{C}_l^*, ..., \check{C}_1^*\}; \{\hat{C}_q^*, ..., \hat{C}_1^*\}; \{\tilde{C}_t^*, ..., \tilde{C}_1^*\})$.
Therefore, we have to prove that the group Γ is isomorphic to a group which includes the relations:

(i') $X_i^{'m_{r-i+1}} = 1$ for $i = 1, \dots, r$,

(ii') $E_i' C_{ik_i}' E_i^{'-1} C_{i0}' = 1$ for $i = 1, \dots, k$,

(iii') $C_{ij}^{'2} = \check{C}_{ij}^{'2} = \hat{C}_{ij}^{'2} = \tilde{C}_{ij}^{'2} = 1$, for all the reflections,

(iv') $(C_{i,j-1}' C_{ij}')^{n_{i,k_i-j+1}} = 1$ for $i = 1, \dots, k,\ j = 1, \dots, k_i$;

$(\check{C}_{i,j-1}' \check{C}_{ij}')^{\check{n}_{i,l_i-j+2}} = 1$ for $i = 1, \dots, l,\ j = 2, \dots, l_i$;

$(\hat{C}_{i,j-1}' \hat{C}_{ij}')^{\hat{n}_{i,q_i-j+2}} = 1$ for $i = 1, \dots, q$, $j \in \{2, \dots, q_i\} - \hat{V}'_i$ for some non-empty $\hat{V}'_i \subset \{1, \dots, q_i\}$;

$(\tilde{C}_{i,j-1}' \tilde{C}_{ij}')^{\tilde{n}_{i,t_i-j+3}} = 1$ for $i = 1, \dots, t$, $j \in \{3, \dots, t_i\} - \tilde{V}'_i$ for some $\tilde{V}'_i \subset \{3, \dots, t_i\}$ (maybe empty) whenever the reflections $\tilde{C}_{i,j-1}'$ and $\tilde{C}_{ij}'$ exist, that is, whenever $\{j-1, j\} \cap U_i' = \varnothing$.

(v') $\prod_{i=1}^{r} X_{r-i+1}' \prod_{i=1}^{s} P_i' \prod_{j=1}^{g} [A_j', B_j'] \prod_{i=1}^{k} E_{k-i+1}' \prod_{i=1}^{l} \check{E}'_{l-i+1} \prod_{i=1}^{q} \hat{E}'_{q-i+1} \prod_{i=1}^{t} \tilde{E}'_{t-i+1} = 1.$

The isomorphism $\phi : \Gamma \to \Gamma'$ is defined by:

1. $\phi : A_i \to B_{g-i+1}^{'}$,
2. $\phi : B_i \to A_{g-i+1}^{'}$,
3. $\phi : X_i \to F X_{r-i+1}^{'-1} F^{-1}$,
4. $\phi : P_i \to F \prod_{i=1}^{r} (X_i') P_{s-i+1}^{'-1} \prod_{i=1}^{r} (X_{r-i+1}^{'-1}) F^{-1}$,

5. $\phi : E_i \to E_i^{\prime -1}$,

6. $\phi : \check{E}_i \to \check{F}\check{E}_i^{\prime -1}\check{F}^{-1}$,

7. $\phi : \hat{E}_i \to \hat{F}\hat{E}_i^{\prime -1}\hat{F}^{-1}$,

8. $\phi : \tilde{E}_i \to \tilde{F}\tilde{E}_i^{\prime -1}\tilde{F}^{-1}$,

9. $\phi : C_{ij} \to C'_{i,k_i-j}$, $\phi : \check{C}_{ij} \to \check{F}\check{C}'_{i,l_i-j+1}\check{F}^{-1}$, $\phi : \hat{C}_{ij} \to \hat{F}\hat{C}'_{i,q_i-j+1}\hat{F}^{-1}$, $\phi : \tilde{C}_{ij} \to \tilde{F}\tilde{C}'_{i,t_i-j+2}\tilde{F}^{-1}$,

where $F = \prod_{i=1}^{k} E'_{k-i+1} \prod_{i=1}^{l} \check{E}'{}_{l-i+1} \prod_{i=1}^{q} \hat{E}'{}_{q-i+1} \prod_{i=1}^{t} \tilde{E}'{}_{t-i+1}$, $\check{F} = \prod_{i=1}^{k} E'_{k-i+1}$, $\hat{F} = \prod_{i=1}^{k} E'_{k-i+1} \prod_{i=1}^{l} \check{E}'_{l-i+1}$ and $\tilde{F} = \prod_{i=1}^{k} E'_{k-i+1} \prod_{i=1}^{l} \check{E}'_{l-i+1} \prod_{i=1}^{q} \hat{E}'_{q-i+1}$. It is clear that ϕ is an isomorphism. Additionally, the relations in Γ are transformed into:

1. $\phi(E_i C_{ik_i} E_i^{-1} C_{i0}) = E_i^{\prime -1} C'_{i0} E'_i C'_{ik_i} = 1$, so $E'_i C'_{ik_i} E_i^{\prime -1} C'_{i0} = 1$.

2. The image of the parabolic element $\check{E}'_i \check{C}'_{il_i} \check{E}_i^{\prime -1} \check{C}'_{i1}$ is $\phi(\check{E}_i \check{C}_{il_i} \check{E}_i^{-1} \check{C}_{i1}) = \check{F}\check{E}_i^{\prime -1} \check{C}'_{i1} \check{E}'_i \check{C}'_{il_i} \check{F}^{-1}$, which is also parabolic and therefore $\check{E}'_i \check{C}'_{il_i} \check{E}_i^{\prime -1} \check{C}'_{i1}$ is parabolic too. The same happens to the image of $\hat{E}_i \hat{C}_{iq_i} \hat{E}_i^{-1} \hat{C}_{i1}$.

3. Similarly, the image of the hyperbolic element $\tilde{E}_i \tilde{C}_{it_i} \tilde{E}_i^{-1} \tilde{C}_{i2}$ is $\pi(\tilde{E}_i \tilde{C}_{it_i} \tilde{E}_i^{-1} \tilde{C}_{i2}) = \tilde{F}\tilde{E}_i^{\prime -1} \tilde{C}'_{i2} \tilde{E}'_i \tilde{C}'_{it_i} \tilde{F}^{-1}$, which is also hyperbolic and so $\tilde{E}'_i \tilde{C}'_{it_i} \tilde{E}_i^{\prime -1} \tilde{C}'_{i1}$ is hyperbolic too.

4. Lengthy but easy computations show that the image of
$$\prod_{i=1}^{r} X_i \prod_{i=1}^{s} P_i \prod_{j=1}^{g} [A_j, B_j] \prod_{i=1}^{k} E_i \prod_{i=1}^{l} \check{E}_i \prod_{i=1}^{q} \hat{E}_i \prod_{i=1}^{t} \tilde{E}_i$$
is $F \prod_{i=1}^{s} P_i^{\prime -1} \prod_{i=1}^{r} X_i^{\prime -1} F^{-1} \prod_{j=1}^{g} [B'_{g-j+1}, A'_{g-j+1}] F^{-1}$, whose inverse is
$$\prod_{i=1}^{r} X'_i \prod_{i=1}^{s} P'_i \prod_{j=1}^{g} [A'_j, B'_j] \prod_{i=1}^{k} E'_{k-i+1} \prod_{i=1}^{l} \check{E}'{}_{l-i+1} \prod_{i=1}^{q} \hat{E}'{}_{q-i+1} \prod_{i=1}^{t} \tilde{E}'{}_{t-i+1},$$
which is the identity.

□

The following last three lemmas deal with groups with non-orientable orbit space and show that ν-, η- and μ-cycles can be reversed individually. The lemmas are the non-cocompact equivalent to Macbeath's [40, Lemma 5]. We prove all of them in the same way at the end.

Lemma 3.16. *An NEC group* Γ *with signature*

$$(g; -; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_i, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}_t\}),$$

is isomorphic via a type-preserving isomorphism to an NEC group Γ' *with signature*

$$(g; -; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_i^*, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}_t\}),$$

where the cycle $\check{C}_i^*$ *is inversely equivalent to the cycle* $\check{C}_i$*, i.e. the periods of* $\check{C}_i^*$ *are the periods of* $\check{C}_i$ *reversed.*

Lemma 3.17. *An NEC group* Γ *with signature*

$$(g; -; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_i, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}_t\}),$$

is isomorphic via a type-preserving isomorphism to an NEC group Γ' *with signature*

$$(g; -; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_i^*, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}_t\}),$$

where the cycle $\hat{C}_i^*$ *is inversely equivalent to the cycle* $\hat{C}_i$*, i.e. the components of* $\hat{C}_i^*$ *are a cyclic permutation of the components of* $\hat{C}_i$ *reversed.*

Lemma 3.18. *An NEC group* Γ *with signature*

$$(g; -; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}_i, ..., \tilde{C}_t\}),$$

is isomorphic via a type-preserving isomorphism to an NEC group Γ' *with signature*

$$(g; -; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}_i^*, ..., \tilde{C}_t\}),$$

where the cycle $\tilde{C}_i^*$ *is inversely equivalent to the cycle* $\tilde{C}_i$*, i.e. the components of* $\tilde{C}_i^*$ *are a cyclic permutation of the components of* $\tilde{C}_i$ *reversed.*

Proof. Let us assume that Γ and Γ' have the same relations except for one of the i-th cycle, where in Γ' the components and periods are as in Γ but reversed. Then, the presentation of Γ includes:

(*i*) $\overline{C}_{ij}^2 = 1$, for all the reflections, $(\overline{C}_{i,j-1}\overline{C}_{ij})^{\overline{n}_{ij}} = 1$,

(*ii*) $\prod_{i=1}^{r} X_i \prod_{i=1}^{s} P_i \prod_{j=1}^{g} D_i^2 \overline{E}_1 ... \overline{E}_{i-1} \overline{E}_i \overline{E}_{i+1} ... \overline{E}_{k+l+q+t}$,

where we use now the notation $\overline{E}_i$ for all the hyperbolic elements $E_i, \check{E}_i, \hat{E}_i, \tilde{E}_i$ and $\overline{C}_i$ for all the reflections $C_i, \check{C}_i, \hat{C}_i, \tilde{C}_i$ with the aim to unify the proof of the lemmas above. The presentation of Γ' includes:

(*i'*) $\overline{C}_{ij}'^2 = 1$, for all the reflections, $(\overline{C}_{i,j-1}'\overline{C}_{ij}')^{\overline{n}_{ij}'} = 1$,

(*iii'*) $\prod_{i=1}^{r} X_i' \prod_{i=1}^{s} P_i' \prod_{j=1}^{g} D_i'^2 \overline{E}_1' ... \overline{E}_{i-1}' \overline{E}_i' \overline{E}_{i+1}' ... \overline{E}_{k+l+q+t}'$, where the components and periods of the i-th boundary component are reversed and $\overline{n}'_{i,j} = \overline{n}_{i,o_i+1-j}$ where o_i is k_i, l_i, q_i or t_i depending on the type of the i-th boundary component. The isomorphism $\phi : \Gamma \to \Gamma'$ is defined by:

1. $\phi : \overline{E}_i \to F\overline{E}_i'^{-1}F^{-1}$,
2. $\phi : D_g \to D'_g\overline{E}'_i$,
3. $\phi : \overline{E}_j \to \overline{E}_i'^{-1}\overline{E}'_j\overline{E}'_i, j = 1, ..., i-1$,
4. $\phi : \overline{C}_{i,j} \to F\overline{C}'_{i,o_i+1-j}F^{-1}$, $F = (\overline{E}'_1...\overline{E}'_i)^{-1}D_g'^{-1}$,
5. all the other generators of Γ correspond in the same manner to those in Γ'.

It is clear that ϕ is an isomorphism, however in order to show that it is type-preserving, we have to prove that $D'_g\overline{E}'_i$ is a glide reflection. In the cocompact case, this is an indirect result of [40, Lemma 5]: as an isomorphism between cocompact NEC groups is necessarily a type preserving isomorphism, the products $D'_g\overline{E}'_i$ in these isomorphisms are glide reflections. As the type of transformation resulting of the product depends only on the relative positions of the axis of the translation $\overline{E}'_i$, the axis of the glide reflection D'_g and its line of reflection, then we just need to build a compact fundamental region from the non-compact one where the edges paired by D'_g preceed in the surface symbol to the sides paired by $\overline{E}'_i$. By just leaving all the sides identic except for the edges between the paired by hyperbolic elements that are removed, and joining them via a new side in $\mathbf{H}$ fixed by a reflection we get the needed compact fundamental region. In this way we prove that the products $D'_g\overline{E}'_i$ are glide-reflections and so the isomorphism is type-preserving.

Now, we prove that $(i'), (ii'), (iii')$ are the images of $(i), (ii), (iii)$ via ϕ:

(i) $\phi(\overline{C}^2_{ij}) = F\overline{C}'_{i,o_i+1-j}F^{-1}F\overline{C}'_{i,o_i+1-j}F^{-1} = 1$ and this implies $\overline{C}'^2_{i,o_i+1-j} = 1$,

(ii) $\phi((\overline{C}_{i,j-1}\overline{C}_{ij})^{\overline{n}_{i,j}}) = (F\overline{C}'_{i,o_i+1-j}F^{-1}F\overline{C}'_{i,o_i+1-j}F^{-1})^{\overline{n}_{i,j}} = 1$ so $(\overline{C}'_{i,o_i+1-j}\overline{C}'_{i,o_i+1-j})^{\overline{n}_{i,j}} = 1$ and $(\overline{C}'_{i,j}\overline{C}'_{i,j})^{\overline{n}_{i,o_i+1-j}} = 1$,

(iii) $\phi(\prod_{i=1}^r X_i \prod_{i=1}^s P_i \prod_{j=1}^g D_i^2\overline{E}_1...\overline{E}_{i-1}\overline{E}_i\overline{E}_{i+1}...\overline{E}_{k+l+q+t}) =$
$= \prod_{i=1}^r X'_i \prod_{i=1}^s P'_i \prod_{j=1}^{g-1} D_i'^2 D'_g\overline{E}'_i D'_g\overline{E}'_i\overline{E}_i'^{-1}\overline{E}'_1\overline{E}'_i...\overline{E}_i'^{-1}\overline{E}'_{i-1}\overline{E}'_i F\overline{E}_i'^{-1}F^{-1}\overline{E'}_{i+1}...\overline{E'}_{k+l+q+t} =$
$= \prod_{i=1}^r X'_i \prod_{i=1}^s P'_i \prod_{j=1}^g D_i'^2\overline{E}'_1...\overline{E}'_{i-1}\overline{E}'_i\overline{E'}_{i+1}...\overline{E'}_{k+l+q+t} = 1.$

Finally, depending on the type of the i-th boundary, we can have in Γ:

1. an additional relation in case of compact boundaries: $\overline{E}_i\overline{C}_{i,k_i}\overline{E}_i^{-1}\overline{C}_{i,0} = 1$,

2. in the case of a ν- or a η-cycle, the product $\overline{E}_i\overline{C}_{i,o_i}\overline{E}_i^{-1}\overline{C}_{i,1}$ is parabolic,

3. in the case of a μ-cycle, the product $\overline{E}_i\overline{C}_{i,o_i}\overline{E}_i^{-1}\overline{C}_{i,1}$ is hyperbolic,

that should be verified in Γ' by the corresponding relations. The first case is the cocompact case and will not be discussed here. In the second and third cases, if the product $\overline{E}_i\overline{C}_{i,o_i}\overline{E}_i^{-1}\overline{C}_{i,1}$ is parabolic (hyperbolic), then so will be $\phi(\overline{E}_i\overline{C}_{i,o_i}\overline{E}_i^{-1}\overline{C}_{i,1}) = F\overline{E}_i^{'-1}\overline{C}'_{i,1}\overline{E}_i^{'}\overline{C}'_{i,o_i}F^{-1}$ and therefore $\overline{E}_i^{'-1}\overline{C}'_{i,1}\overline{E}_i^{'}\overline{C}'_{i,o_i}$ is parabolic (hyperbolic), and so is its conjugate $\overline{C}'_{i,1}\overline{E}_i^{'}\overline{C}'_{i,o_i}\overline{E}_i^{'-1}$ and finally the inverse $\overline{E}_i^{'}\overline{C}'_{i,o_i}\overline{E}_i^{'-1}\overline{C}'_{i,1}$ is then also parabolic (hyperbolic). □

Proof of the sufficient conditions of Theorem 3.9. Let us assume that we have an NEC group Γ with signature $sg = (g; \pm; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}_t\})$ and that Γ' has a signature $sg' = (g; \pm; s; [m'_1, ..., m'_r]; \{C'_1, ..., C'_k\}; \{\check{C}'_1, ..., \check{C}'_l\}; \{\hat{C}'_1, ..., \hat{C}'_q\}; \{\tilde{C}'_1, ..., \tilde{C}'_t\})$, with sign($sg$)=sign($sg'$), the proper periods $[m'_1, ..., m'_r]$ are a permutation of the proper periods $[m_1, ..., m_r]$ and there exist permutations ϕ of $\{1, ..., k\}$, $\check{\phi}$ of $\{1, ..., l\}$, $\hat{\phi}$ of $\{1, ..., q\}$ and $\tilde{\phi}$ of $\{1, ..., t\}$ such that in the orientable case, all the cycles C'_i, $\check{C}'_i$, $\hat{C}'_i$ and $\tilde{C}'_i$ are either directly equivalent to the corresponding cycles $C_{\phi(i)}$, $\check{C}_{\check{\phi}(i)}$, $\hat{C}_{\hat{\phi}(i)}$ and $\tilde{C}_{\tilde{\phi}(i)}$ or all are reversely equivalent. In the non-orientable case, each cycle C'_i, $\check{C}'_i$, $\hat{C}'_i$ and $\tilde{C}'_i$ is either directly equivalent to the corresponding cycles $C_{\phi(i)}$, $\check{C}_{\check{\phi}(i)}$, $\hat{C}_{\hat{\phi}(i)}$ and $\tilde{C}_{\tilde{\phi}(i)}$ or reversely equivalent.

The general sufficient conditions stated in the theorem are then the concatenation of the following basic operations applied in the signatures:

1. permuting the ν-, η- and μ-cycles (Lemmas 3.10, 3.11 and 3.12),

2. cyclically permuting the components of the η- and μ-cycles (Lemmas 3.13 and 3.14),

3. cyclically permuting the reversed components of all the η- and μ-cycles reversed (Lemma 3.15),

4. cyclically permuting the reversed components of a ν, an η- and a μ-cycle in the non-orientable case (Lemmas 3.16, 3.17 and 3.18),

plus the well known conditions on the proper- and cycle-periods corresponding to the cocompact part already proved by Wilkie [59] and Macbeath [40].

Then, Lemmas 3.10 to 3.18 prove finally that the conditions in Theorem 3.9 are sufficient. □

3.4 Canonical fuchsian subgroup

Given the signature of a finitely generated NEC group in this section we compute the signature of its canonical fuchsian subgroup. A non-cocompact fuchsian group has signature of the form $(g; m_1, m_2, ..., m_r; s; t)$, where s is the number of cusps and t the number of funnels of the corresponding orbit space.

In a finitely generated fuchsian group each conjugacy class of maximal finite cyclic subgroups gives rise to a period, the number of conjugacy classes of maximal parabolic cyclic subgroups gives rise to the parabolic part of the signature (number of cusps) and the number of conjugacy classes of maximal boundary hyperbolic cyclic subgroups gives rise to the hyperbolic part of the signature (number of funnels). A boundary hyperbolic element is characterized by the fact that it leaves a unique interval of discontinuity σ in $\mathbb{R}$ invariant such that two boundary elements with the same interval of discontinuity σ are powers of a unique boundary element that stabilizes σ, see [5, pag. 262 and 266].

We are now in the situation of characterizing the canonical fuchsian group of an NEC group:

Theorem 3.19. *Let* Γ *be a finitely generated NEC group with signature:*

$$sg = (g; \pm; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_q\}; \{\tilde{C}_1, ..., \tilde{C}_t\}),$$

in full

$$sg = (g; \pm; s; [m_1, ..., m_r]; \{(n_{11}, ..., n_{1k_1}), ..., (n_{k1}, ..., n_{kk_k})\}; \{(\check{n}_{12}, ..., \check{n}_{1l_1}), ..., (\check{n}_{l2}, ..., \check{n}_{ll_l})\};$$
$$\{((\hat{n}_{12}, ..., \hat{n}_{1,\hat{v}_{12}-1}), ..., (\hat{n}_{1,\hat{v}_{1,\hat{v}_1}+1}, ..., \hat{n}_{1,q_1})), ..., ((\hat{n}_{q2}, ..., \hat{n}_{q,\hat{v}_{q2}-1}), ..., (\hat{n}_{q,\hat{v}_{q,\hat{v}_q}+1}, ..., \hat{n}_{q,q_q}))\};$$
$$\{((\tilde{n}_{13}, ..., \tilde{n}_{1,u_{12}-1}), ..., (\tilde{n}_{1,u_{1u_1}+1}, ..., \tilde{n}_{1,t_1})), ..., ((\tilde{n}_{t3}, ..., \tilde{n}_{t,u_{t1}-1}), ..., (\tilde{n}_{t,u_{tu_t}+1}, ..., \tilde{n}_{t,t_t}))\}).$$

Then the canonical Fuchsian subgroup of Γ *has signature*

$$sg(\Gamma^+) = (\eta g + k + l + q - 1; m_1, m_1, m_2, m_2, ..., m_r, m_r, n_{11}, ..., n_{kk_k}, \check{n}_{12}, ..., \check{n}_{ll_l},$$
$$\hat{n}_{12}, ..., \hat{n}_{qq_q}, \tilde{n}_{13}, ..., \tilde{n}_{tt_t}; 2s + l + \sum_{i=1}^{q} \hat{v}_i + \sum_{i=1}^{t} \tilde{v}_i; 2t + \sum_{i=1}^{t} |U_i|),$$

where $\eta = 2$ *for sign* " $+$ " *and* $\eta = 1$ *for sign* " $-$ ".

Proof. We first look for conjugacy classes in Γ^+ of elliptic, parabolic and boundary hyperbolic elements of Γ. By [54, Lemma 4], for an elliptic generator $X \in \Gamma$, the groups conjugate to $\langle X \rangle$ in Γ fall into two conjugacy classes inside Γ^+. For a parabolic generator $P \in \Gamma$, following the proof of this lemma, we show that for any $G \in \Gamma - \Gamma^+$, the group $G\langle P \rangle G^{-1}$, which is conjugate in Γ to $\langle P \rangle$ is not conjugate in Γ^+ to $\langle P \rangle$. Otherwise, there would exist $G' \in \Gamma^+$ such that $G'G\langle P \rangle G^{-1}G'^{-1} = \langle P \rangle$. As $\langle P \rangle$ is the stabilzer of a unique point $p \in \partial\mathbf{H}$, we deduce $GG'(p) = p$ and therefore $G'G \in stab(p)$, which is a contradiction as $P \in \Gamma^+$.

For a boundary hyperbolic element H, if the groups $\langle H \rangle$ and $G\langle H \rangle G^{-1}$ with $G \in \Gamma - \Gamma^+$, were conjugate in Γ^+ by G', then if σ is the unique interval of discontinuity of H, the elements of $G\langle H \rangle G^{-1}$ and $G'G\langle H \rangle G^{-1}G^{-1}$ would have the same interval of discontinuity, i.e. $G\sigma = G'G\sigma = \sigma$.

Claim G and $G'G$ are reflections.
Proof. Let us assume that G is a glide reflection such that G^2 is a boundary element that stabilizes σ and therefore the boudary elements G^2 and in $\langle H \rangle$ are powers of a unique boundary element that stabilizes σ. This means that there exists a canonical fundamental region where the glide reflection G links two edges with at least a vertex at infinity. However, as the edges linked by a glide reflection have the orientation reversed, G can only link edges with both vertices in $\partial\mathbf{H}$. Otherwise when applying it to the vetex at infinity, it will be mapped into a vertex in $\mathbf{H}$. Now, this means that the edges linked by such glide reflection has a common vertex at infinity that is neither parabolic, semi-parabolic or improper (as in the unbounded moebius band in the example 2.3) which is a contradiction, as a canonical fundamental region can only include parabolic, semiparabolic and improper vertices at infinity. Then G is a reflection and for the same reason $G'G$ also. This shows the claim.

As G and $G'G$ are both reflections and both have to fix the same axis, namely the axis of H, we conclude that $G = G'$, which is a contradiction as we have supposed that $\langle H\rangle$ and $G\langle H\rangle G^{-1}$ are conjugate by $G' \in \Gamma^+$.

In the same lemma, Singerman showed that any group conjugate in Γ to $\langle C_{i,j-1}C_{ij}\rangle$ is conjugate in Γ^+ to $\langle C_{i,j-1}C_{ij}\rangle$. In our case, this applies to the elliptic products $\check{C}_{i,j-1}\check{C}_{ij}$, $\hat{C}_{i,j-1}\hat{C}_{ij}$ and $\tilde{C}_{i,j-1}\tilde{C}_{ij}$, too.

In the same way, any group conjugated in Γ to the group generated by the parabolic elements $\check{C}_{i1}\check{C}_{il_i}$, $\hat{C}_{i,j-1}\hat{C}_{ij}$ with $\hat{n}_{ij} = \infty$ or $\tilde{C}_{i,j-1}\tilde{C}_{ij}$ with $\tilde{n}_{ij} = \infty$, is conjugated in Γ^+ respectively to $\check{C}_{i1}\check{C}_{il_i}$, $\hat{C}_{i,j-1}\hat{C}_{ij}$ with $\hat{n}_{ij} = \infty$ or $\tilde{C}_{i,j-1}\tilde{C}_{ij}$ with $\tilde{n}_{ij} = \infty$. Indeed, if $G \in \Gamma - \Gamma^+$, then the conjugate $G\langle\check{C}_{i1}\check{C}_{il_i}\rangle G^{-1}$ in Γ of the group $\langle\check{C}_{i1}\check{C}_{il_i}\rangle$, verifies $G\langle\check{C}_{i1}\check{C}_{il_i}\rangle G^{-1} = G\langle\check{C}_{il_i}\check{C}_{i1}\check{C}_{il_i}\check{C}_{il_i}\rangle G^{-1} = G\check{C}_{il_i}\langle\check{C}_{i1}\check{C}_{il_i}\rangle(G\check{C}_{il_i})^{-1}$, so that they are also conjugated in Γ^+. The same can be done for the groups $\langle\hat{C}_{i,j-1}\hat{C}_{ij}\rangle$ with $\hat{n}_{ij} = \infty$ and $\langle\tilde{C}_{i,j-1}\tilde{C}_{ij}\rangle$ with $\tilde{n}_{ij} = \infty$ in Γ^+. The total number of parabolic conjugacy classes in Γ^+ is then $2s + l + q + \sum_{i=1}^{q}\hat{v}_i + \sum_{i=1}^{t}\tilde{v}_i$, where as before $\hat{v}_i$ is the number of parabolic products $\hat{C}_{i,j-1}\hat{C}_{ij}$ and $\tilde{v}_i$ the number of parabolic products $\tilde{C}_{i,j-1}\tilde{C}_{ij}$.

Similarly, a group conjugated to the group generated by the boundary hyperbolic elements $\tilde{C}_{i2}\tilde{C}_{it_i}$ or by the hyperbolic products $\tilde{C}_{i,j-2}\tilde{C}_{ij}$ are conjugate in Γ^+ to $\tilde{C}_{i1}\tilde{C}_{it_i}$ or $\tilde{C}_{i,j-2}\tilde{C}_{ij}$ respectively, where $j-1 \in U_i$, namely the linked side is a free-edge and there is no reflection $\tilde{C}_{i,j-1}$ in the presentation. The proof of this fact is exactly the same as the parabolic and elliptic cases. Finally, the total number of hyperbolic conjugacy classes is $2t + \sum_{i=1}^{t}|U_i|$, where $|U_i|$ is the number of boundary hyperbolic products in the i-th μ-sequence.

For calculating the genus, we take into account that if Γ is a group acting properly discontinuously on a manifold $\mathbf{H}$, then the orbit space $\mathbf{H}/\Gamma$ has the structure of an orbifold. It is also well known that if there is a cover $f : \mathbf{H}/\Gamma' \to \mathbf{H}/\Gamma$ of degree d of $\mathbf{H}/\Gamma$, then the Riemann-Hurwitz formula reads $d = \chi(\mathbf{H}/\Gamma')/\chi(\mathbf{H}/\Gamma)$, where χ is the Euler-characteristic of the corresponding orbifold. Now, as Γ^+ is a subgroup of index 2 in Γ, it can be defined an orbit space cover $f : \mathbf{H}/\Gamma^+ \to \mathbf{H}/\Gamma$ of degree 2. Therefore, $\chi(\mathbf{H}/\Gamma^+) = 2\chi(\mathbf{H}/\Gamma)$. It is well known (see [35, Section 1]) that the Euler-characteristic of an orbifold $\mathbf{H}/\Delta$ where Δ is a fuchsian group of signature $(g; m_1, ..., m_r; s; t)$ is

$$\chi(\mathbf{H}/\Delta) = 2 - 2g - \sum_{i=1}^{r}(1 - \frac{1}{m_i}) - s - t.$$

For the canonical fuchsian group Γ^+ of the NEC group Γ, we have shown that the signature is

$$sg(\Gamma^+) = (g'; m_1, m_1, m_2, m_2, ..., m_r, m_r, n_{11}, ..., n_{kk_k}, \check{n}_{12}, ..., \check{n}_{ll_l}, \hat{n}_{12} ... \hat{n}_{qq_q}, \tilde{n}_{13}, ..., \tilde{n}_{tt_t};$$
$$2s + l + q + \sum_{i=1}^{q} \hat{v}_i + \sum_{i=1}^{t} \tilde{v}_i; 2t + \sum_{i=1}^{t} |U_i|),$$

where the proper periods include only the subindices i, j where $n_{ij}, \hat{n}_{ij}, \check{n}_{ij}, \tilde{n}_{ij}$ are finite and $\hat{v}_i, \tilde{v}_i$ are the number of semi-parabolic vertices in the i-th η- and μ-sequences. Then, the Euler-characteristic of the orbit space $\mathbf{H}/\Gamma^+$ is

$$\chi(\mathbf{H}/\Gamma^+) = 2 - 2g' - 2\sum_{i=1}^{r}(1 - \frac{1}{m_i}) - \sum_{i=1}^{k}\sum_{j=1}^{k_i}(1 - \frac{1}{n_{ij}}) - \sum_{i=1}^{l}\sum_{j=1}^{l_i}(1 - \frac{1}{\check{n}_{ij}}) -$$
$$\sum_{i=1}^{q}\sum_{j=1}^{q_i}(1 - \frac{1}{\hat{n}_{ij}}) - \sum_{i=1}^{t}\sum_{j \in L_i}(1 - \frac{1}{\tilde{n}_{ij}}) - \sum_{i=1}^{t} \tilde{v}_i - 2s - 2t - \sum_{i=1}^{t} |U_i|,$$

where we have called L_i the set of $j \in \{1, ..., t_i\}$ such that $\tilde{C}_{i,j-1}\tilde{C}_{ij}$ is elliptic and we have added the parabolic term $-l$, for adding the indices $j = 1$ with $\check{n}_{i1} = \infty$ to the sum $\sum_{i=1}^{l}\sum_{j=2}^{l_i}(1 - \frac{1}{\check{n}_{ij}})$, and the parabolic terms $-q$ and $-\sum_{i=1}^{q} \hat{v}_i$, for adding to the sum $\sum_{i=1}^{q}\sum_{j=2, j \in \hat{V}_i}^{q_i}(1 - \frac{1}{\hat{n}_{ij}})$ the indices $j = 1, j \in \hat{V}_i$ with $\hat{n}_{ij} = \infty$.

In the case of the non-cocompact NEC group Γ, the orbifold Euler-characteristic of $\mathbf{H}/\Gamma$ can be calculated directly via the usual definition $\chi(\mathbf{H}/\Gamma) = F - E + V$, where F, E, V are respectively the number of faces, edges and vertices of a triangulation of $\mathbf{H}/\Gamma$, where each vertex and edge weighs $1/k$, with k the order of its stabilizer, see [58, Def 13.3.3, Prop. 13.3.4 and Examples]. We use the canonical form of the fundamental region and the equivalence of the pairing edges and related vertices for calculating it. First of all, we have necessarily only one face, so $F = 1$. The number of edges of the fundamental region is one per paired ones and half for mirror edges. The free-sides do not belong to the fundamental region (non-compact part) and actually count as 0. We have then

$$E = E_1 + E_2,$$

with

$$E_1 = s + r + \eta g + k + l + q + t,$$

where $\eta = 1, 2$ depending on the sign of the non-cocompact NEC signature and for mirror edges

$$E_2 = \frac{1}{2}\sum_{i=1}^{k}(k_i+1) + \frac{1}{2}\sum_{i=1}^{l} l_i + \frac{1}{2}\sum_{i=1}^{q} q_i + \frac{1}{2}\sum_{i=1}^{t}(t_i - |U_i|).$$

As mentioned above, if o is the order of the stabilizer of a vertex, then $\frac{1}{o}$ is the weight at counting them. The parabolic, semiparabolic and improper vertices are counted as 0. Then, the total number of weighted vertices is

$$V = 1 + \sum_{i=1}^{r}\frac{1}{m_i} + \sum_{i=1}^{k}\sum_{j=1}^{k_i}\frac{1}{2}\frac{1}{n_{ij}} + \sum_{i=1}^{l}\sum_{j=2}^{l_i}\frac{1}{2}\frac{1}{\check{n}_{ij}} + \sum_{i=1}^{q}\sum_{j=2}^{q_i}\frac{1}{2}\frac{1}{\hat{n}_{ij}} + \sum_{i=1}^{t}\sum_{j\in L_i}\frac{1}{2}\frac{1}{\tilde{n}_{ij}} + \frac{1}{2}k,$$

where as before we call L_i the set of $j \in \{1, ..., t_i\}$ such that $\tilde{C}_{i,j-1}\tilde{C}_{i,j}$ is elliptic. The conic vertices count $\frac{1}{m_i}$ and the corner vertices count $\frac{1}{2}\frac{1}{n_{ij}}$, $\frac{1}{2}\frac{1}{\check{n}_{ij}}$, $\frac{1}{2}\frac{1}{\hat{n}_{ij}}$ and $\frac{1}{2}\frac{1}{\tilde{n}_{ij}}$. The vertices of the fundamental region paired respectively by the hyperbolic transformations E_i are of order 2 and count $\frac{1}{2}k$ and all other vertices are conjugated and count 1. We can then conclude

$$\begin{aligned}
\chi(\mathbf{H}/\Gamma) &= F - E + V = 1 - s - r - \eta g - k - l - q - t - \frac{1}{2}\sum_{i=1}^{k} k_i - \frac{1}{2}k - \frac{1}{2}\sum_{i=1}^{l} l_i - \\
&- \frac{1}{2}\sum_{i=1}^{q} q_i - \frac{1}{2}\sum_{i=1}^{t}(t_i - |U_i|) + 1 + \sum_{i=1}^{r}\frac{1}{m_i} + \sum_{i=1}^{k}\sum_{j=1}^{k_i}\frac{1}{2}\frac{1}{n_{ij}} + \sum_{i=1}^{l}\sum_{j=2}^{l_i}\frac{1}{2}\frac{1}{\check{n}_{ij}} + \\
&+ \sum_{i=1}^{q}\sum_{j=1}^{q_i}\frac{1}{2}\frac{1}{\hat{n}_{ij}} + \sum_{i=1}^{r}(\frac{1}{m_i} - 1) + \sum_{i=1}^{t}\sum_{j\in L_i}\frac{1}{2}\frac{1}{\tilde{n}_{ij}} + \frac{1}{2}k = \\
&= 2 - s - \eta g - k - l - q - t + \sum_{i=1}^{k}\sum_{j=1}^{k_i}\frac{1}{2}(\frac{1}{n_{ij}} - 1) + \sum_{i=1}^{l}\sum_{j=1}^{l_i}\frac{1}{2}(\frac{1}{\check{n}_{ij}} - 1) + \sum_{i=1}^{q}\sum_{j=1}^{q_i}\frac{1}{2}(\frac{1}{\hat{n}_{ij}} - 1) + \\
&+ \sum_{i=1}^{t}\sum_{j\in L_i}\frac{1}{2}(\frac{1}{\tilde{n}_{ij}} - 1) - \frac{1}{2}\sum_{i=1}^{t}|U_i| - \frac{1}{2}\sum_{i=1}^{t}\tilde{v}_i,
\end{aligned}$$

where we have used the fact that $t_i = 2|U_i| + |L_i| + \tilde{v}_i$, i.e. the total number of edges in the i-th μ-sequence is the same as the total number of vertices (considering the first and last vertices the same as they are paired). In turn the total number of vertices is the number of improper vertices ($2|U_i|$) plus the number of proper vertices ($\tilde{v}_i$) plus the number of vertices in $\mathbf{H}$ ($|E_i|$).

Using the Riemann-Hurwitz formula $\chi(\mathbf{H}/\Gamma^+) = 2\chi(\mathbf{H}/\Gamma)$, we get

$$2 - 2g' - 2\sum_{i=1}^{r}(1-\frac{1}{m_i}) - \sum_{i=1}^{k}\sum_{j=1}^{k_i}(1-\frac{1}{n_{ij}}) - \sum_{i=1}^{l}\sum_{j=1}^{l_i}(1-\frac{1}{\check{n}_{ij}}) - \sum_{i=1}^{q}\sum_{j=1}^{q_i}(1-\frac{1}{\hat{n}_{ij}})$$
$$-\sum_{i=1}^{t}\sum_{j\in E_i}(1-\frac{1}{\tilde{n}_{ij}}) - 2s - 2t - \sum_{i=1}^{t}|U_i| - \sum_{i=1}^{t}\tilde{v}_i =$$
$$= 4 - 2s - 2\eta g - 2k - 2l - 2q - 2t + 2\sum_{i=1}^{r}(\frac{1}{m_i}-1) + \sum_{i=1}^{k}\sum_{j=1}^{k_i}(\frac{1}{n_{ij}}-1) +$$
$$+\sum_{i=1}^{l}\sum_{j=1}^{l_i}(\frac{1}{\check{n}_{ij}}-1) + \sum_{i=1}^{q}\sum_{j=1}^{q_i}(\frac{1}{\hat{n}_{ij}}-1) + \sum_{i=1}^{t}\sum_{j\in\tilde{U}_i}(\frac{1}{\tilde{n}_{ij}}-1) - \sum_{i=1}^{t}|U_i| - \sum_{i=1}^{t}\tilde{v}_i.$$

Rearranging the terms and simplifying, we finally obtain
$2g' = -2 + 2\eta g + 2k + 2l + 2q$ and so $g' = \eta g + k + l + q - 1$. □

3.5 Topological classification of the orbit space

The main goal of this section is to classify topologically the (non-compact) orbit space from the signature of the group. In the case of cocompact fuchsian groups, given a signature $(g; m_1, m_2, ..., m_r)$, the related orientable orbit space is classified up to homeomorphism by the genus g. Similarly, the orbit space of a cocompact NEC group is identified up to homeomorphism by the invariants $g, \pm, k$, i.e. genus, orientability and the number of boundary components of the space. As a surface with a closed disc removed is topologically equivalent to a surface with a point removed, the orientable orbit space of a non-cocompact fuchsian group of signature $(g; m_1, m_2, ..., m_r; s; t)$, where s is the number of punctures and t the number of funnels, is then defined by the invariants $g, s+t$.

In order to obtain the invariants classifying topologically the orbit space corresponding to a non-cocompact NEC group in terms of its signature, we introduce the basic classification of non-cocompact surfaces following the classical results of Richards [50], Brown and Messer [7] and Konya [34].

Let $P_1 \supset P_2 \supset ...$ be a nested sequence of unbounded connected regions of a surface S such that:

(1) the boundary of P_n in S is compact for all n,

(2) for any bounded subset A of S, $P_n \cap A = \varnothing$ for n sufficiently large.

We say that two sequences $P_1 \supset P_2 \supset ...$ and $Q_1 \supset Q_2 \supset ...$ are equivalent if, for any n there is an integer N such that $P_n \subset Q_N$ and vice versa. We denote by p^* the equivalence class of sequences containing $p = P_1 \supset P_2 \supset ...$ that we call an end of S. The set $B(S)$ of all ends of S is a topological space having as elements the ends of S and endowed with the following topology: for any set U in S whose boundary in S is compact, we define U^* to be the set of all ends p^* represented by some $p = P_1 \supset P_2 \supset ...$, such that $P_n \supset U$ for n sufficiently large. A bordered surface is said to be planar if every compact subsurface in it is of genus zero. A surface without borders is planar if and only if every Jordan curve separates it. We say that the end p^* is planar and/or orientable if the sets P_n are planar and/or orientable for all sufficiently large n.

A surface S with non-empty boundary is of infinite genus and/or infinitely nonorientable if there is no bounded subset A of S such that $S - A$ has genus 0 and/or is orientable. A non-compact surface S can be classified attending to its orientability in four types, namely orientable, infinitely nonorientable, nonorientable with an odd number of "cross cups" or with an even number of "cross cups". The spaces of ends $B(S)$ is then defined as a nested triple of the sets $B(S) \supset B'(S) \supset B''(S)$ where $B'(S)$, $B''(S)$ are the parts of $B(S)$ which are not planar and infinitely nonorientable respectively. Richards proved in [50] that two surfaces without boundaries of the same genus and orientability class are homeomorphic if and only if their spaces of ends considered as triple of spaces are topologically equivalent.

In case of surfaces with boundaries we have to consider additionally the ends contained on the boundaries: we define similarly the triple of spaces $C(S), C'(S), C''(S)$ corresponding to all the ends on the boundary, and the subsets of non planar ends on the boundary and infinitely nonorientable ends on the boundary. Additionally, we say that two ends are adjacent if there exists a boundary component for which they are the end of it. Two boundary ends are said to be equivalent if they belong to the same sequence of adjacent ends. We finally define the set D as the quotient of C by the equivalence relation above. Equipped with the boundary space of ends and D, Prishlyak and Mischenko [49] proved that two surfaces S_1 and S_2 with the same genus, orientability class and same number of compact boundaries are homeomorphic if and only if there is a homeomorphism which maps $B(S_1)$ onto $B(S_2)$, $B'(S_1)$ onto $B'(S_2)$, $B''(S_1)$ onto $B''(S_2)$, $C(S_1)$ onto $C(S_2)$, $C'(S_1)$ onto $C'(S_2)$, $C''(S_1)$ onto $C''(S_2)$ and $D(S_1)$ onto $D(S_2)$.

For a surface S that is an orbit space generated by a finitely generated non-cocompact NEC group, we know that S has finite genus, and therefore by definition $B'(S) = C'(S) = \varnothing$, and S has finite orientability class, also by definition $B''(S) = C''(S) = \varnothing$. So we have proved:

Lemma 3.20. *Let Γ be a finitely generated non-cocompact NEC group and $S = \boldsymbol{H}/\Gamma$, then $B'(S) = B''(S) = C'(S) = C''(S) = \varnothing$.*

Let Γ be an NEC group of signature:

$$sg = (g; \pm; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_q)\}; \{\tilde{C}_1, ..., \tilde{C}_t\}).$$

For classifying the compact core of the orbit space we have the genus g, orientability $\pm$ and the number k of compact borders.

Now, for the non-cocompact part, the space of ends $B(S)$ corresponds to points and closed disks removed on the surface, i.e. punctures and simple funnels. As a surface with a puncture removed is topologically (but not conformally) equivalent to a surface with a simple funnel, their total number is an invariant, that we write $s + n^0$, where n^0 is the total number of μ-cycles that are simple funnels. In other words n^0 is the number of funnels with 0 cuts.

Similarly, the space of ends on the boundary $C(S)$ corresponds to boundaries on which we remove points and closed segments, so that each border with only one semi-puncture (total number l) is topologically equivalent to a funnel with one (non-compact) cut (total number n^1). Let us call n_k, $k \geqslant 2$, the number of η-cycles $\hat{C}_i$, such that $|\hat{V}_i| = \hat{v}_i = k$, i.e. boundary components with exactly k semi-punctures. And we call n^k the number of μ-cycles $\tilde{C}_i$ with a total number of k cuts. As before, semi-punctures (points removed on a border) and cuts on the funnels (closed segments removed on the border) are topologically, but not conformally equivalent.

Let us define $N_k = n_k + n^k$, $k \geqslant 2$, $N_0 = s + n^0$ and $N_1 = l + n^1$. Let us define the number

$$N = \prod_{k=0}^{M} p_{k+1}^{N_k},$$

with M the maximum k such that $N_k \neq 0$ and p_k the k-th prime number. We call N the *diagram invariant* (motivated by the definition of diagram and the homeomorphism theorem of non-cocompact 2-manifolds [7, Theorem 2.2]). It identifies from a topological point of view the space of ends $B(S)$ (N_0) and $C(S)$ (N_k, $k \geqslant 1$) of the surface $S = \mathbf{H}/\Gamma$. Finally, we have proved the following theorem:

Theorem 3.21. *The orbit space $\boldsymbol{H}/\Gamma$ of a finitely generated non-cocompact NEC group Γ with signature:*

$$sg = (g; \pm; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\check{C}_1, ..., \check{C}_l\}; \{\hat{C}_1, ..., \hat{C}_q)\}; \{\tilde{C}_1, ..., \tilde{C}_t\})$$

is a non-compact surface topologically classified by the invariants $(g, \pm, k, N)$, where g is the genus of $\boldsymbol{H}/\Gamma$, "$\pm$" stands for its orientability, k is the number of compact boundary components and N is the diagram invariant defined above.

Example 3.22. *Let us consider the surface of the example 3.5. According to the Theorem 3.21, we have the following invariants that classify topologically (up to homeomorphism) the surface:*

- *Invariants of the compact core: the genus $g = 3$, orientability "$+$" and number $k = 1$ of compact borders,*
- *N_0: one puncture, $s = 1$ and one simple funnel $n^0 = 1$, $N_0 = 2$,*
- *N_1: one $\nu-$cycle, $l = 1$ and no funnel with only one cut, $n^1 = 0$, such that $N_1 = 1$,*
- *N_2: one $\eta-$cycle with $n_2 = 1$ and two μ-cycles with two cuts, $N_2 = 3$,*

Finally, $M = 2$ and the diagram invariant $N = 2^{N_0}3^{N_1}5^{N_2} = 2^2 3^1 5^3 = 1500$ and therefore the list of invariants is $(3, +, 1, 1500)$.

CHAPTER 4

Other results on NEC groups

In this chapter, we study additional properties of not necessarily non-cocompact NEC groups, namely the elementary NEC groups and the limit set. In Section 4.1 we describe all the elementary NEC groups, including those with orientation reversing isometries. In Section 4.2, we prove that an NEC group and its canonical fuchsian subgroup have the same limit set, and then introduce the classification of NEC groups of first and second kind, similarly to the classification of fuchsian groups.

4.1 Elementary NEC groups

In this section we describe a class of subgroups of isometries of the hyperbolic plane called elementary groups, which have a particularly simple structure.

Definition 4.1. An NEC group Γ is elementary if there is a finite Γ-orbit in $\partial\mathbf{H} \cup \mathbf{H}$.

The elementary fuchsian groups are of one of the following types (see Katok [32, Theorem 2.4.3]):

1. $\Gamma \simeq C_m$, a finite cyclic group generated by an elliptic element of order m,

2. $\Gamma \simeq C_\infty$ is an infinite cyclic group generated by either a parabolic or a hyperbolic element,

3. $\Gamma \simeq C_\infty \oplus C_2$, conjugated in $Isom(\mathbf{H})$ to $\langle S, T\rangle$ where $S(z) = kz, k > 1$ and $T(z) = \frac{-1}{z}$.

For proper elementary NEC groups, we have the following classification.

Theorem 4.2. *A proper elementary NEC group* Γ *is a group of one of the following types:*

1. $\Gamma \simeq C_2$, *generated by a reflection,*

2. $\Gamma \simeq C_\infty$, *generated by a glide reflection,*

3. $\Gamma \simeq D_m$, *dihedral generated by an elliptic element of order* m *and a reflection fixing the elliptic fixed point,*

4. $\Gamma \simeq D_\infty$, *generated by a parabolic element and a reflection fixing the parabolic fixed point, or by a reflection fixing* a *and* b *in* $\partial\boldsymbol{H}$ *and an elliptic involution swapping* a *with* b*, or by a glide reflection and an elliptic involution swapping the fixed points of the glide reflection, or by a hyperbolic element and a reflection swapping the fixed points of the hyperbolic element,*

5. $\Gamma \simeq C_2 \oplus C_2$, *generated by either a reflection fixing* $a, b \in \partial\boldsymbol{H}$*, or a reflection swapping* a *and* b*, or a glide reflection fixing* a, b*, and an elliptic element swapping* a *with* b*,*

6. $\Gamma \simeq C_\infty \oplus C_2$, *generated by a hyperbolic element and either a reflection or a glide reflection both fixing the same points in* $\partial\boldsymbol{H}$*,*

7. $\Gamma \simeq D_\infty \oplus C_2$, *generated by a hyperbolic element fixing* a *and* b*, an elliptic involution swapping* a *with* b *and a reflection preserving* $\{a, b\}$*,*

8. $\Gamma \simeq D_\infty \rtimes C_2$, *generated by a hyperbolic element fixing* a *and* b*, an elliptic involution swapping* a *with* b *and a reflection swapping* a *with* b*,*

9. $\Gamma \simeq C_\infty \oplus C_2$ *or* $(C_\infty \oplus C_2) \rtimes C_2$, *generated by a hyperbolic element fixing* a *and* b*, an elliptic involution swapping* a *with* b *and a glide reflection fixing* a *and* b*.*

Proof. Attending to the number of points of the finite orbit and whether this orbit is in **H** or ∂**H** we have the following possibilities:

Case 1: Γ has an orbit consisting of one point $a \in \mathbf{H}$. We can write then $\Gamma = stab\{a\}$. As no glide reflection, hyperbolic or parabolic element fixes any point in **H**, then the group can only contain reflections and elliptic elements. We deduce then Γ is

(1.1) a group generated by a unique reflection R as generator, $\Gamma = \langle R \rangle \simeq C_2$.

(1.2) a group generated by an elliptic element E and a reflection R both fixing a, $\Gamma = \langle E, R \rangle \simeq D_m$.

Observe that we have $\Gamma^+ = \{Id\}$ or $\langle elliptic \rangle$.

Case 2: Γ has an orbit consisting of one point $a \in \partial\mathbf{H}$. Again, we write $\Gamma = stab\{a\}$. We have the following cases:

(2.1) a group generated by a unique element S, $\Gamma = \langle S \rangle$. We have the following possibilities:

(2.1.1) $S = R$ is a reflection, $\Gamma = \langle R \rangle \simeq C_2$, or

(2.1.2) $S = D$ is a glide reflection, $\Gamma = \langle D \rangle \simeq C_\infty$.

We have $\Gamma^+ = \langle Id \rangle$ or $\langle hyperbolic \rangle$.

(2.2) A group generated by a parabolic element P and an orientation reversing element S, both fixing $a \in \partial\mathbf{H}$, $\Gamma = \langle P, S \rangle$, so that we get:

(2.2.1) $\Gamma = \langle P, R\, reflection\ fixing\ a \in \partial\mathbf{H} \rangle = \langle R, PR | R^2 = (PR)^2 = 1 \rangle \simeq C_2 * C_2 \simeq D_\infty$.

In this case, S cannot be a glide reflection fixing a since otherwise Γ^+ would contain a parabolic element and the hyperbolic element S^2, which is impossible as a fuchsian group cannot contain a parabolic element and a hyperbolic element fixing the same point, see for example see Katok [32, proof of Theorem 2.4.3]. Additionally, we have $\Gamma^+ = \langle Id \rangle$ or $\langle Parabolic \rangle$.

(2.3) A group generated by a hyperbolic element H and an orientation reversing element S, both fixing $a \in \partial\mathbf{H}$, $\Gamma = \langle H, S \rangle$. Let $b \in \partial\mathbf{H}$ be the other fixed point of H. Then, we have two possibilities:

(2.3.1) $\Gamma = \langle H, R\, reflection\ fixing\ a \in \partial\mathbf{H} \rangle$.

Claim $R(b) = b$.

Proof. Suppose $R(b) \neq b$. Then, $R \neq HRH^{-1}$ since otherwise $R(b) = HRH^{-1}(b) = HR(b) \Rightarrow R(b) \in Fix(H) = \{a, b\}$ which is a contradiction. So $RHRH^{-1}$ is parabolic fixing a, which is impossible as mentioned above. So we deduce $R(b) = b$, this shows our claim.

Choosing $\{a, b\} = \{0, \infty\}$ we conclude then $\Gamma = \langle z \mapsto \lambda z, R \mapsto -\bar{z}\rangle = C_\infty \oplus C_2$.

(2.3.2) $\Gamma = \langle H, D\ glide\ reflection\ fixing\ a \in \partial\mathbf{H}\rangle$.

Claim $D(b) = b$.

Proof. Suppose $D(b) \neq b$. Then, $D^2(b) \neq b$, because the $\langle D\rangle$–orbit of b is infinite. Therefore, Γ^+ contains the hyperbolic elements H and D^2 with different fixed point sets. This is impossible, see [32, proof of Theorem 2.4.3]. So, $D(b) = b$, showing our claim.

Again, choosing $\{a, b\} = \{0, \infty\}$ we conclude $\Gamma = \langle H : z \mapsto \lambda z, D : z \mapsto -\mu\bar{z}\ (\mu > 1)\rangle$. Since $\langle H, D^2\rangle$ is cyclic, see [32, Theorem 2.3.5], we may write $\langle H, D^2\rangle = \langle hyperbolic\ F\rangle$. So $\Gamma = \langle F : z \mapsto \alpha z, D : z \mapsto -\mu\bar{z}\ (\mu > 1)\rangle = \langle F, D | FD = DF, D^2 = F^m\ for\ some\ m\rangle \simeq$

$$\simeq \begin{cases} \langle DF^k\rangle = C_\infty & \text{if } m = 2k+1, \\ \langle F\rangle \oplus \langle DF^{-k}\rangle = C_\infty \oplus C_2 & \text{if } m = 2k. \end{cases}$$

Case 3: Γ has an orbit consisting of two points $a, b \in \partial\mathbf{H}$. Again, we write $\Gamma = stab\{a, b\}$. In this case, Γ^+ cannot contain parabolic elements, since the orbit of a point under a parabolic element is infinite, unless the point if the fixed point of the parabolic element. We have the following cases:

(3.1) a group generated by a unique element S, $\Gamma = \langle S\ preserving\ \{a, b\} \subset \partial\mathbf{H}\rangle$. We have the following possibilities, where we may assume $\{a, b\} = \{0, \infty\}$:

(3.1.1) $\Gamma = \langle R\ reflection\ fixing\ a, b \in \partial\mathbf{H}\rangle = \langle z \mapsto -\bar{z}\rangle \simeq C_2$,

(3.1.2) $\Gamma = \langle R\ reflection\ swapping\ a\ with\ b\rangle = \langle z \mapsto \dfrac{\mu}{\bar{z}},\ \mu > 0\rangle \simeq C_2$,

(3.1.3) $\Gamma = \langle D\ glide\ reflection\ fixing\ a, b\rangle = \langle z \mapsto -\lambda\bar{z},\ \lambda > 0\rangle \simeq C_\infty$.

(3.2) A group generated by two elements E and S, with E an elliptic involution swapping a with b in $\partial\mathbf{H}$, and S an orientation reversing isometry preserving $\{a, b\}$, $\Gamma = \langle E, S\rangle$. We have the following possibilities, where again $\{a, b\} = \{0, \infty\}$:

(3.2.1) $\Gamma = \langle E, R\ reflection\ z \mapsto -\bar{z}\ fixing\ a, b\rangle = \langle z \mapsto -\dfrac{1}{z}, z \mapsto -\bar{z}\rangle \simeq C_2 \oplus C_2$,

(3.2.2) $\Gamma = \langle E, R\ \textit{reflection}\ z \mapsto \dfrac{\mu}{\bar{z}} \textit{swapping a with } b\rangle \simeq$
$$\simeq \begin{cases} C_2 \oplus C_2 & \text{if } \mu = 1,\ \textit{this is the case } (3.2.1), \\ C_2 * C_2 = D_\infty & \text{otherwise.} \end{cases}$$
Observe that ER is such $z \mapsto -\dfrac{\bar{z}}{\mu}$ and so ER is a glide reflection fixing a, b.

(3.2.3) $\Gamma = \langle E, D\ \textit{glide reflection fixing}\ a, b \in \partial\mathbf{H}\rangle$. This is the case (3.2.2) with the product ER a glide reflection.

(3.3) A group generated by two elements H and S, with H a hyperbolic element $H(z) = \lambda z$ ($\lambda > 1$) fixing $a = 0$ and $b = \infty$, and S an orientation reversing element preserving $\{a, b\} = \{0, \infty\}$, $\Gamma = \langle H, S\rangle$. We have the following possibilities:

(3.3.1) $\Gamma = \langle H, R\ \textit{reflection}\ z \mapsto -\bar{z}\ \textit{fixing}\ a, b\rangle \simeq C_\infty \oplus C_2$, which is the case (2.3.1) above.

(3.3.2) $\Gamma = \langle H, R\ \textit{reflection}\ z \mapsto \dfrac{\mu}{\bar{z}} \textit{swapping a with } b\rangle = \langle RH, R\rangle \simeq C_2 * C_2 = D_\infty$, because $RH : z \mapsto \dfrac{\mu\lambda}{\bar{z}}$, is a reflection.

(3.3.3) $\Gamma = \langle H, D\ \textit{glide reflection fixing}\ a, b\rangle \simeq C_\infty$ or $C_\infty \oplus C_2$. This is the case (2.3.2).

(3.4) A group generated by three elements H, E and S, with H a hyperbolic element $H(z) = \lambda z$ fixing $a = 0$ and $b = \infty$, E an elliptic involution $E(z) = -\dfrac{1}{z}$ swapping 0 and ∞ and S an orientation reversing element preserving $\{0, \infty\}$. In this case, we can write $\Gamma^+ \supsetneqq \langle EH, E\rangle = C_2 * C_2 = D_\infty$, as $EH(z) = -\dfrac{1}{\lambda z}$ has order 2.
We have the following possibilities:

(3.4.1) $\Gamma = \langle H, E, R\ \textit{reflection}\ z \mapsto -\bar{z}\ \textit{fixing}\ a, b \in \partial\mathbf{H}\rangle \simeq \Gamma^+ \oplus C_2 \simeq (C_2 * C_2) \oplus C_2 = D_\infty \oplus C_2$.

(3.4.2) $\Gamma = \langle H, E, R\ \textit{reflection}\ z \mapsto \dfrac{\mu}{\bar{z}}\ \textit{swapping a with } b\rangle \simeq$
$$\simeq \begin{cases} D_\infty \oplus C_2 & \text{if } \mu = 1,\ \textit{this is the case } 3.4.1, \\ D_\infty \rtimes C_2 = & \text{otherwise.} \end{cases}$$
For the case $\mu \neq 1$, we have considered first that $(RE)^2$ is an orientation preserving element fixing two points in $\partial\mathbf{H}$ and therefore is hyperbolic and so $\langle H, (RE)^2\rangle$ is a cyclic group as both are hyperbolic groups fixing the same points. Then there exists a hyperbolic element F such that $(RE)^2 = F^n$ for some n. We can then write $\langle H, E, R\rangle = \langle F, E, R | (EF)^2 = E^2 = R^2 = 1, RER = EF^n\rangle$. Now, it is clear that

$\Gamma^+ = \langle F, E | (EF)^2 = E^2 = 1\rangle \simeq C_2 * C_2 \simeq D_\infty$ is normal in Γ, $\Gamma^+ \cap \langle R | R^2 = 1\rangle = \varnothing$ and any element of Γ can be written in the form TR with $T \in \Gamma^+$, i.e. Γ is the semidirect product of Γ^+ and $\langle R | R^2 = 1\rangle$.

(3.4.3) $\Gamma = \langle H, E, D\ glide\ reflection\ z \mapsto -\mu\bar{z}\ fixing\ a, b\rangle \simeq \langle H, D\rangle \rtimes C_2$.
In this case, the semidirect product of $\langle H, D\rangle$ and $\langle E | E^2 = 1\rangle$ is straightforward considering that $EHE = H^{-1}$ and $EDE = D^{-1}$ and as seen in (2.3.2), $\langle H, D\rangle \simeq \begin{cases} C_\infty \\ or \\ C_\infty \oplus C_2. \end{cases}$

Case 4: Γ has an orbit consisting of $k \geqslant 2$ points in $\mathbf{H}$ or $k \geqslant 3$ points in $\partial\mathbf{H}$. First of all, Γ^+ cannot contain elements of infinite order since the parabolic and hyperbolic elements can have only either fixed points at infinity or infinite orbits (see [32, pg. 39]). In fact, $\Gamma^+ = \{id\}$ or $\langle elliptic\ of\ order\ n\rangle$. We have the following cases:

(4.1) $\Gamma^+ = \{Id\}$ and so $\Gamma = \langle reflection\rangle \simeq C_2$,

(4.2) $\Gamma^+ = \langle E\ elliptic\ order\ m\ fixing\ p\rangle$ and so $\Gamma = \langle E, R\ reflection; fixing\ p\rangle \simeq D_m$.

□

Remark 4.3. *From the theorem is clear that the elementary NEC groups are finitely generated. Additionally, an elementary group is non-cocompact.*

4.2 Limit orbits and the classification of NEC groups

Let $z \in \mathbf{H}$ and $\{T_n\}$ a sequence of distinct elements of an NEC group Γ. If the sequence $\{T_n z\}$ has a limit, then this is a limit point of Γ.

Definition 4.4. The limit set of an NEC group Γ is the set of all limit points of Γ-orbits Γz, $z \in \mathbf{H}$ and is denoted by $\Lambda(\Gamma)$.

For fuchsian groups the limit set can be one of the following (see Katok [32, Theorem 3.4.6]):

1. a set consisting of $0, 1, 2$ points in $\partial\mathbf{H}$,

2. a perfect nowhere dense subset of $\partial\mathbf{H}$,

3. $\partial\mathbf{H}$.

We have the following results:

Theorem 4.5. *Let* Γ *be an NEC group and let* $\Lambda(\Gamma)$, $\Lambda(\Gamma^+)$ *be the limit sets of the NEC group and of its canonical fuchsian subgroup. Then* $\Lambda(\Gamma) = \Lambda(\Gamma^+)$.

Proof. First of all, it is clear that $\Lambda(\Gamma^+) \subseteq \Lambda(\Gamma)$. Conversely, for each $a \in \Lambda(\Gamma)$, there exists a sequence $\{T_n\}$ in Γ with $T_n z \to a$. We can ensure that there exists an infinite subsequence of $\{T_n\}$ such that either all the elements belong to Γ^+ or to $\Gamma - \Gamma^+$. If the elements belong to Γ^+, then $a \in \Lambda(\Gamma^+)$. If they belong to $\Gamma - \Gamma^+$, then as Γ^+ is a subgroup of index two of Γ, we can decompose the group into a union of two disjoint sets $\Gamma^+ \cup T\Gamma^+$ with $T \in \Gamma - \Gamma^+$ and therefore we have a sequence $S_n = T^{-1}T_n$ in Γ^+ such that $S_n z \to T^{-1}a \in \Lambda(\Gamma^+)$. But $TS_n z = TS_n T^{-1}Tz \to a$, so we have a sequence of elements of $R_n = TS_nT^{-1} \in \Gamma^+$ and a $w = Tz \in \mathbf{H}$ such that $R_n w \to a$ and therefore $a \in \Lambda(\Gamma^+)$. □

The following corollary is immediate:

Corollary 4.6. *The limit set of an NEC group is one of the following:*

1. *a set consisting of* $0, 1, 2$ *points in* $\partial\boldsymbol{H}$,
2. *a perfect nowhere dense subset of* $\partial\boldsymbol{H}$,
3. $\partial\boldsymbol{H}$.

A fuchsian group Λ is said to be of the first kind if its limit set is $\partial\mathbf{H}$, otherwise is of the second kind. Specifically, elementary fuchsian groups are of the second kind. Fuchsian groups of the first kind are finitely generated and have finite covolume. Based on the Theorem 4.5 we can introduce the same classification of NEC groups and obtain similar results:

Definition 4.7. An NEC group Γ is called of the first kind if its limit set is $\partial\mathbf{H}$. Otherwise it is called of the second kind.

Corollary 4.8. *Let* Γ *be an NEC group and* Γ^+ *its fuchsian canonical subgroup. Then* Γ *is of the first kind if and only if* Γ^+ *is of the first kind.*

Proof. The proof is a direct consequence of $\Lambda(\Gamma) = \Lambda(\Gamma^+)$. □

Corollary 4.9. *Let* Γ *be a finitely generated non elementary NEC group. Then* Γ *is of the first kind if and only if it has a fundamental region of finite area.*

Proof. Γ is of the first kind if and only if Γ^+ is of the first kind and this happens in case of non-elementary finitely generated fuchsian groups if and only if there is a fundamental region of Γ^+ of finite area (for example applying Beardon [5, Theorem 10.1.2]). Now, as Γ^+ is a subgroup of index two of Γ, there is a fundamental region of Γ whose area is half the area of a fundamental region of Γ^+ and therefore finite. $\square$

Finally we can easily derive the following corollaries:

Corollary 4.10. *Let* Γ *be a finitely generated non elementary NEC group of first kind. Then the signature of* Γ *does not include any* $\tilde{C}$ *cycles*

$$sg = (g; \pm; s; [m_1, ..., m_r]; \{C_1, ..., C_k\}; \{\hat{C}_1, ..., \hat{C}_l)\}; \{\check{C}_1, ..., \check{C}_q)\}; \{-\}).$$

Proof. By Theorem 3.19, the signature of the canonical fuchsian group Γ^+ is

$$sg(\Gamma^+) = (\eta g + k + l + q - 1; m_1, m_1, m_2, m_2, ..., m_r, m_r, n_{11}, ..., n_{kk_k}, \check{n}_{12}, ..., \check{n}_{ll_l},$$
$$\hat{n}_{1,2}...\hat{n}_{q,q_q}, \tilde{n}_{1,3}...\tilde{n}_{t,t_t}; 2s + l + \sum_{i=1}^{q} \hat{v}_i + \sum_{i=1}^{t}\sum_{j=1}^{t_i} \tilde{n}_{i,j}; 2t + \sum_{i=1}^{t} |U_i|),$$

where $\eta = 2$ for sign " + " and $\eta = 1$ for sign " − ". As the fuchsian group is finitely generated of first kind, then $2t + \sum_{i=1}^{t} |U_i| = 0$. This means that $t = 0$ and therefore the signature of the canonical group is reduced to

$$sg(\Gamma^+) = (\eta g + k + l + q - 1; m_1, m_1, ..., m_r, m_r, n_{11}, ..., n_{kk_k}, \check{n}_{12}, ..., \check{n}_{ll_l},$$
$$\hat{n}_{1,2}...\hat{n}_{q,q_q}; 2s + l + \sum_{i=1}^{q} \hat{v}_i; 0),$$

in other words, there is no μ-sequence in the fundamental region, i.e. the related data in the signature is empty. $\square$

Corollary 4.11. *Let* Γ *be a finitely generated non elementary NEC group of first kind of* *signature as in corollary 4.10. Then, the measure of any fundamental region is*

$$\mu(\Gamma) = 2\pi[\eta g + k + l + q - 2 + s + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) +$$
$$+ \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k_i}(1 - \frac{1}{n_{ij}}) + \frac{1}{2}\sum_{i=1}^{l}\sum_{j=1}^{l_i}(1 - \frac{1}{\check{n}_{ij}}) + \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1}^{q_i}(1 - \frac{1}{\hat{n}_{ij}})],$$

where $\eta = 2$ for sign " $+$ " and $\eta = 1$ for sign " $-$ ".

Proof. The area of a non elementary finitely generated fuchsian group Λ of signature $(g; m_1, ..., m_r; s; 0)$ is

$$\mu(\Lambda) = 2\pi[2g - 2 + s + \sum_{i=1}^{r}(1 - \frac{1}{m_i})].$$

By corollary 4.10, the signature of the canonical fuchsian group Γ^+ is

$$sg(\Gamma^+) = (\eta g + k + l + q - 1; m_1, m_1, m_2, m_2, ..., m_r, m_r, n_{11}, ..., n_{kk_k}, \check{n}_{12}, ..., \check{n}_{ll_l},$$
$$\hat{n}_{1,2}, ..., \hat{n}_{q,q_q}; 2s + l + \sum_{i=1}^{q}\hat{v}_i; 0),$$

where $\eta = 2$ for sign " $+$ " and $\eta = 1$ for sign " $-$ ". As Γ^+ has index two in Γ, we have $\mu(\Gamma^+) = 2\mu(\Gamma)$. So we get

$$\mu(\Gamma) = 2\pi[\eta g + k + l + q - 2 + s + \frac{1}{2}l + \frac{1}{2}\sum_{i=1}^{q}\hat{v}_i + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) +$$
$$+ \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k_i}(1 - \frac{1}{n_{ij}}) + \frac{1}{2}\sum_{i=1}^{l}\sum_{j=2}^{l_i}(1 - \frac{1}{\check{n}_{ij}}) + \frac{1}{2}\sum_{i=1}^{q}\sum_{j\in\hat{V}_i}(1 - \frac{1}{\hat{n}_{ij}})] =$$
$$= 2\pi[\eta g + k + l + q - 2 + s + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) +$$
$$+ \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k_i}(1 - \frac{1}{n_{ij}}) + \frac{1}{2}\sum_{i=1}^{l}\sum_{j=1}^{l_i}(1 - \frac{1}{\check{n}_{ij}}) + \frac{1}{2}\sum_{i=1}^{q}\sum_{j=1}^{q_i}(1 - \frac{1}{\hat{n}_{ij}})].$$

□

CHAPTER 5

Conclusions

The focus of this thesis has been oriented to understand the structure and classification, from an algebraic, geometrical and topological point of view, of finitely generated non-cocompact NEC groups.

The main original results obtained are summarized below:

1. We have defined a canonical fundamental polygon of an NEC group Γ, providing details of the structure of its fundamental region, specifically studying the properties of edges and vertices at infinity for which the notions of ν-, η- and μ-sequences have been introduced.

2. We have obtained the presentation of these groups via generators and relations closely linked to the geometry of the canonical fundamental region, including orientation preserving, reversing isometries and hyperbolic boundary and parabolic elements.

3. We have defined a signature for non-cocompact groups that collects the algebraical (presentation), geometrical (marked polygon) and topological (identification of the orbit space $\mathbf{H}/\Gamma$) data of the groups.

4. We have described the non-compact orbit space (non compact Klein surfaces) and defined topological invariants that classify them up to homeomorphism.

5. We have identified the necessary and sufficient conditions for the existence of type-preserving isomorphisms between two groups given their signatures.

6. We have obtained the signature of the (non-cocompact) canonical fuchsian subgroup of a proper NEC group given its signature.

7. We have presented the form of all elementary NEC groups and the structure of the limit set of an NEC group Γ. Due to the fact that the canonical fuchsian group is a normal subgroup of index two, we also proved that the limit sets of Γ and Γ^+ are the same.

8. Using the fact that the limit sets $\Lambda(\Gamma)$ and $\Lambda(\Gamma^+)$ are the same, we have applied directly results and concepts of the theory of fuchsian groups for studying properties of NEC groups (e.g. classification of NEC groups in first and second kind, signature of finitely generated NEC groups of first kind, finite measure of first kind NEC groups finitely generated and their value in terms of the signature).

5.1 Future work

The introduction in this thesis of basic results of non-cocompact finitely generated NEC groups as stated above can be immediately applied in three directions:

1. **Study of algebraic properties of NEC groups**, including for example the study of the signatures of their (normal) subgroups, finitely maximal NEC groups, the rank of finitely generated NEC groups, etc. Linked to the concepts of measure and Euler-characteristic of a fundamental region is the problem of showing that given an abstract signature, there exists an NEC group with this signature if and only if

$$\eta g + k + l + q + t + s - 2 + \sum_{i=1}^{r}(1-\frac{1}{m_i}) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k_i}(1-\frac{1}{n_{ij}}) + \frac{1}{2}\sum_{i=1}^{l}\sum_{j=2}^{l_i}(1-\frac{1}{\check{n}_{ij}}) +$$
$$+\frac{1}{2}\sum_{i=1}^{q}\sum_{j=2}^{q_i}(1-\frac{1}{\hat{n}_{ij}}) + \frac{1}{2}\sum_{i=1}^{t}(\tilde{v}_i + |U_i| + \sum_{j\in E_i}(1-\frac{1}{\tilde{n}_{ij}})) > 0,$$

where $\eta = 2$, if $sign(s) =$ " + " and $\eta = 1$.

2. **Study of non-compact Klein surfaces:** Prove a uniformization theorem for non-compact Klein surfaces, namely under which conditions a non-compact Klein surface of

finite genus is an orbit space of the form $\mathbf{H}/\Gamma$, with Γ a finitely generated NEC group. In general, the study of the properties of the symetries and moduli spaces linked to non-compact Klein surfaces.

3. **Properties of the groups of automorphisms of non-compact Klein surfaces:** The study of the automorphism groups of non-compact Klein surfaces is a topic about which not much is known and where the results of this thesis might contribute. For instance, the automorphism groups of non-compact Riemann surfaces with a finitely generated fundamental group are finite, as shown by Greenberg in [25] and this work might help to understand under which conditions a Klein surface which is the orbit space of a finitely generated non-cocompact NEC group has a finite group of automorphisms. Also the study of the structure of the automorphism groups for different non-compact Klein surfaces (surfaces with punctures, funnels with or without cuts, etc.) or the existence of surfaces with automorphism groups in a prescribed class are topics for a future development of the results in this thesis.

Bibliography

[1] Anderson, J. W., *Hyperbolic Geometry*, Springer, (2005). (ISBN:1852339349).

[2] Apanasov, B. N., *Conformal Geometry of Discrete Groups and Manifolds*, Walter de Gruyter, (2011). (ISBN:3110808056).

[3] Beardon, A. F., *Convex fundamental domains of Fuchsians groups*, Proc. Camb. Phil. Soc. 76, (1973), 511-513.

[4] Beardon, A. F. , *Hyperbolic Polygons and Fuchsian Groups*, Journal of the London Mathematical Society, (1979), s2-20: 247-254.

[5] Beardon, A. F., *The Geometry of Discrete Groups (Graduate texts in mathematics, 91)*,Springer-Verlag, (1983). (ISBN: 0387907882).

[6] Borthwick, D., *Spectral Theory of Infinite-Area Hyperbolic Surfaces*,Birkäuser, Boston (2007).

[7] Brown E. M., Messer R., *The classification of two-dimensional manifolds*, Transactions of the American Mathrmatical Society (1979), vol. 255.

[8] Bujalance, E., *Normal subgrops of N.E.C. Groups*, Math. Z. 178, (1981), 331-341.

[9] Bujalance, E., *Normal N.E.C. signatures*, Illinois Journal of Mathematics vol. 26, Number 3, (1982).

[10] Bujalance, E., *Automorphisms groups of compact Klein surfaces*, Manuscripta mathematica 56, (1986), 105-124

[11] Bujalance E., Etayo, J., Gamboa, J. M. *Teoría elemental de grupos*,UNED ediciones, (2002). (ISBN: 8436244362).

[12] Bujalance, E., Cirre, F. J., Gamboa, J. M., Gromadzki, G., *Symmetries of Compact Riemann Surfaces (Lecture Notes in Mathematics)*, Springer-Verlag, (2007). (ISBN:9783642148279).

[13] Bujalance, E., Gromadyki, G., *On automorphisms of Klein surfaces with invariant subsets*, Osake J. Math. 50, (2013), 251-269.

[14] Bujalance, E., Cirre, F.J. Conder, M.D.E., *Riemann surfaces with maximal real symmetry*, Journal of Algebra 443, (2015).

[15] Bujalance, E., Costa A. F., Izquierdo, M., *On Riemann surfaces of genus g with 4g automorphisms*, Topology and its Applications, Vol. 18 (2017), Elsevier, 1-18.

[16] Cirre, F. J., Monerri A. J., *Finitely generated non-cocompact NEC groups*, To appear in Contemp. Math., Amer. Math. Soc., Providence, RI, 2021.

[17] Cirre, F. J., Monerri A. J., *On the structure and classification of finitely generated non-cocompact NEC groups*. In preparation.

[18] Cooper, D., Hodgson C. D., Kerckhoff S. P., *Three-dimensional Orbifolds and Conemanifolds*, Mathematical Society of Japan Memoirs, Volume 5 (2000).

[19] Costa, A. F., Martínez, E., *Planar hyperelliptic Klein surfaces and fundamental regions of NEC groups*, London Math. Soc. Lect. Note Series 173 (1992).

[20] Etayo, J. J., Martínez, E., *Hyperbolic polygons and NEC groups*, Proc. Camb. Soc. 104 261 (1988).

[21] Fenchel, W., Nielsen, J., *Discontinuous groups of isometries in the hyperbolic plane*, Walter de Gruyter, (2003). (ISBN:3110175266).

[22] Fricke, R., Klein, F., *Vorlesungen über die Theorie der autonorphen Funktionen*, 2 Bde. Leipzieg: Teubner (1926).

[23] Jones, G. A., Singerman S., *Complex functions*, Cambridge University Press (2002). (ISBN:052131366X).

[24] Gilman, J., *Two-generator discrete subgroups of* $\mathrm{PSL}(2,\mathbb{R})$, Mem. Amer. Math. Soc. **117** (1995), no. 561, x+204 pp.

[25] Greenberg, L., *Conformal transformations of Riemann surfaces*, Amer. J. Math. **82**, 1960, 749–760.

[26] Greenberg, L., *Maximal Fuchsian groups*, Bulletin of the American Mathematical Society, American Mathematical Society, (1963) 569-573.

[27] Greenberg, L., *Finitness theorems for Fuchsian and Klenian groups*, Proceedings of the Conference, Cambridge, 1975, Academic Press, (1977) 199-257.

[28] Hoare, A. H. M., Karrass, A., Solitar, D., *Subgroups of NEC groups*, Comm. Pure Appl. Math. 26 (1973), 731-744.

[29] Iversen, B., *Hyperbolic Geometry*, London Mathematical Society. 25 (1992). (ISBN:9780521435284).

[30] R. C. James: *Combinatorial topology of surfaces*, Math. Mag. **29**, (1955), 1–39.

[31] Jones, A. G., Singerman, D., *Complex Functions*, Cambridge University Press, (1987). (ISBN:052131366).

[32] Katok, S., *Fuchsian Groups*, Chicago Lectures in Mathematics, (1992). (ISBN:0226425835).

[33] Kaufmann, R., Zieschang, H., *On the rank of NEC groups*, London Math. Soc. Lect. Note Series 173 (1992).

[34] Kónya, E., *On the fundamental theorem of compact and noncompact surfaces*, Annales Mathematicae et Informaticae 32 (2005), 211-224.

[35] Liebeck M. W., Shalev A., *Fuchsian groups, finite simple groups and representation varieties*, Inventiones Mathematicae 159 (2005), 317-367.

[36] Lyndon, R. C., Schupp, P.E., *Combinatorial Group Theory*, Springer, (2001). (ISBN:9783642618963).

[37] Lyndon, R. C., *Non Euclidean chystallographic groups (Lecture Notes in Mathematics 372)*, Springer-Verlag, (1974). (ISBN:9783540068457).

[38] Macbeath, A. M., *On a theorem of Hurwitz* Proc. Glasgow Math. Assoc. 5 (1961), 90-96

[39] Macbeath, A. M., *Groups of homeomorphisms of a simply connected space*, Ann. of Math. 79 (1964), 473-488.

[40] Macbeath, A. M., *The classification of non-Euclidean plane crystallographic groups*, Canad. J. Math. 19 (1967), 1192-1205.

[41] Macbeath, A. M., Hoare A. H. M., *Groups of hyperbolic crystallography*, Math. Proc. Camb. Phil. Soc. 79 (1976), 235-249.

[42] Maclachlan, C., *Maximal normal Fuchsian groups*, Illinois J. Math. 15 (1971), no. 1, 104-113. http://projecteuclid.org/euclid.ijm/1256052824.

[43] Martinez, E., *Convex fundamental regions for NEC groups*, Arch. Math. 47 (1986), 457-464.

[44] May, C. L., *Large automorphism groups of compact Klein surfaces with boundary*,I. Glasgow Math. J. 18 (1977), 1-10.

[45] Munkres, J. R., *Topología*, Prentice Hall, (2002). (ISBN:9788420531809).

[46] Patterson, S. J., *The limit set of a Fuchsian group*, Acta Math. 136 (1976), 241 - 273.

[47] Poincare, H., *Theorie des groupes fuchsiens*, Acta Mathematica 1, 1-62 (1882).

[48] Preston, R., *Projective structures and fundamental domains on compact Klein surfaces*,Ph. D. Thesis, University of Texas, (1975).

[49] Prishlyak, A. O., Mischenko, K. I., *Classification of noncompact surfaces with boundary*, Methods Funct. Anal. Topology 13 (2007), no. 1, 62-66.

[50] Richards, I., *On the classification of noncompact surfaces*, Transactions of the American Mathematical Society, Vol. 106, No. 2 (Feb., 1963), pp. 259-269.

[51] Seppala, M., *Geometry of Riemann Surfaces and Teichmuller Spaces*, Elsevier, (1992). (ISBN:0444558438).

[52] Singerman, D., *Automorphisms of compact non-orientable Riemann surfaces*, Glasgow Math J. 12 (1971), 50-59.

[53] Singerman, D., *Finitely Maximal Fuchsian Groups*, Journal of The London Mathematical Society-second Series 6. (1972), 29-38.

[54] Singerman, D., *On the structure of non-Euclidean crystallographic groups*, Proc. Camb. Soc. 76 233; (1974).

[55] Singerman, D., *Universal tesellations*, Revista Matemática de la Universidad Complutense de Madrid Vol. 1, números 1, 2, 3; (1988).

[56] Singerman, D., *Superficies de Riemann y Cristalografía*, Disertaciones del Seminario de Matemáticas Fundamentales, nº25, 1-28, (2000).

[57] Swan, R. G., *Generators and Relations for certain Special Linear Groups*, Advances in Mathematics 6, 1-77 233; (1971).

[58] Thurston, W. P., *The Geometry and Topology of Three-Manifolds*, Electronic version 1.1, http://library.msri.org/books/gt3m/, Princenton University (2002).

[59] Wilkie, H. C., *On non-Euclidean crystallographic groups*, Math. Zeitschr. 91 (1966), 87-102.

[60] Zieschang, H., Vogt, E., Coldewey, H.-D. *Surfaces and Planar Discontinuous Groups (Lecture Notes in Mathematics 835)*,Springer-Verlag, (1980). (ISBN:0387100245).

[61] Zieschang, H., *Finite Groups of Mapping Classes of Surfaces (Lecture Notes in Mathematics 875)*, Springer-Verlag, (2006). (ISBN:03540387714).