Tesis Doctoral

2014

# Soluciones Propias Aproximadas de Problemas de Optimización Vectorial

Lidia Huerga Pastor

Licenciada en Matemáticas

Doctorado en Matemática Aplicada

Director: Vicente Novo Sanjurjo

Codirector: César Gutiérrez Vaquero

*A mis padres*

## Agradecimientos

*Quiero agradecer a los profesores Dr. Vicente Novo Sanjurjo y Dr. César Gutiérrez Vaquero por su impecable trabajo de dirección, por su gran dedicación y por todas sus aportaciones y enseñanzas que van más allá del desarrollo de esta tesis. Me siento muy afortunada de tenerlos como directores.*

*También quiero expresar mi especial agradecimiento al profesor Dr. Bienvenido Jiménez Martín, por todo su apoyo y sus valiosas aportaciones para la realización de este trabajo.*

*Gracias a todos mis compañeros del Departamento de Matemática Aplicada I por su afectuosa acogida y por los buenos momentos que he pasado con ellos desde el primer día.*

*Quiero manifestar también mi agradecimiento a la profesora Dra. Christiane Tammer, de la Universidad Martin-Luther de Halle, Alemania, y al profesor Dr. Lionel Thibault, de la Universidad de Montpellier 2, Francia, por su cordial recibimiento en sus respectivos centros de trabajo, por la ayuda y por todas sus enseñanzas y aportaciones durante el tiempo que he trabajado con ellos.*

*Por último, mi más profundo agradecimiento a mi pareja, Ricardo, por su ayuda, apoyo incondicional y paciencia durante todo este tiempo, a mi hermano, José Antonio, por levantarme el ánimo siempre que lo necesito y, especialmente, a mis padres, Tomás y María Luisa, por haberme dado la educación que disfruto, y que tanto valoro, y por impulsarme siempre a alcanzar mis metas. Sirva este trabajo como reconocimiento a su esfuerzo por ayudarme a ser quien quiero.*

# Prólogo

Esta memoria se presenta para optar al grado de Doctor por la Universidad Nacional de Educación a Distancia (UNED) en el programa de doctorado en Matemática Aplicada, con mención de *Doctorado Europeo.* Siguiendo la normativa de la Universidad, la Introducción y el capítulo dedicado a las conclusiones y futuras líneas de desarrollo se han redactado en español y el resto de la memoria en inglés.

Mi interés en el estudio de los problemas de optimización surgió durante la realización del Máster en Modelización Matemática y Computación de la Universidad de Valladolid, durante los años académicos 2008–2009 y 2009–2010, tras cursar las asignaturas *Optimización Matemática* y *Optimización numérica* que en él se impartían.

En el año 2010 me fue concedida una ayuda FPI (referencia BES-2010-033742) dentro del Programa Nacional de Formación de Recursos Humanos de Investigación, en el marco del Plan Nacional de Investigación Científica, Desarrollo e Innovación Tecnológica 2008–2011, para la realización de la tesis doctoral, asociada al proyecto *Optimización de Funciones Vectoriales y de Multifunciones* (referencia MTM2009–09493), siendo el investigador principal del mismo el profesor Dr. Vicente Novo Sanjurjo, del Departamento de Matemática Aplicada I de la E.T.S. de Ingenieros Industriales de la UNED.

A partir de ese momento, entré a formar parte del grupo de investigación *Optimización vectorial* y del Departamento de Matemática Aplicada I, y se me asignó como director de tesis al Dr. Vicente Novo Sanjurjo y como codirector al profesor Dr. César Gutiérrez Vaquero, de la Universidad de Valladolid, ambos grandes expertos en el campo de la Optimización Matemática, con numerosos trabajos de investigación publicados en revistas de reconocido prestigio internacional.

Debido al desarrollo científico y tecnológico, se hace necesario resolver problemas de optimización cada vez más complejos, lo que da lugar al desarrollo de teorías de optimización cada vez más generales y a la definición de conceptos de solución que representen apropiadamente los objetivos a cumplir. Siguiendo esta línea, el trabajo desarrollado en este documento se centra en la resolución de problemas de optimización vectorial por medio de una clase de soluciones propias aproximadas que extiende y mejora varias de las más importantes de este tipo conocidas de la literatura.

Para cualquier comentario o sugerencia sobre la investigación presentada en esta memoria, el lector puede dirigirse a la siguiente dirección de correo electrónico: lhuerga@bec.uned.es.

Madrid, Julio de 2014

Lidia Huerga

# Índice

# Capítulo 1

# Introducción y preliminares

## 1.1 Introducción

La constante necesidad del ser humano de tomar la mejor decisión entre una serie de opciones o alternativas, ha propiciado el desarrollo de la Teoría de Optimización Matemática. En áreas de trabajo como la Economía y la Industria, el éxito en la ejecución de una actividad determinada suele estar estrechamente relacionado con la capacidad de abstraer correctamente los objetivos a mejorar, teniendo en cuenta las restricciones marcadas por las características del trabajo, en definitiva, con el buen planteamiento y resolución de un problema de optimización asociado a la actividad a realizar. Determinar la cantidad óptima de un producto que se debe producir en un mercado con unas características establecidas para maximizar el beneficio o planificar la producción de un artículo para, a la vez, maximizar la cantidad elaborada y minimizar los costes de fabricación, son dos ejemplos de problemas de optimización frecuentes en estos ámbitos de trabajo.

Sin embargo, los problemas de optimización son muy diversos y, en ocasiones, elegir la o las mejores opciones entre las posibles alternativas que se presentan puede convertirse en una tarea complicada. En el segundo ejemplo, maximizar la producción y minimizar los costes de fabricación son dos objetivos en conflicto, lo cual dificulta la búsqueda de una planificación de producción óptima.

Como consecuencia natural del desarrollo científico y tecnológico, se hace necesario resolver problemas de optimización cada vez más complejos, propiciando el desarrollo de teorías de optimización cada vez más generales, y la búsqueda de conjuntos de soluciones que mejor representen los objetivos a cumplir.

Un problema de optimización vectorial consiste en, dada una función entre dos espacios vectoriales, determinar los valores de la variable que minimizan (o maximizan) dicha función, sobre un conjunto definido por restricciones. Estos valores son las soluciones del problema de optimización. La función recibe el nombre de función objetivo, el espacio de definición de la misma se llama espacio inicial o de decisión, el de llegada espacio final u objetivo, el conjunto determinado por las restricciones se conoce como conjunto factible y sus elementos se llaman puntos factibles.

Si el espacio final es $\mathbb{R}$, entonces se obtiene un problema de optimización escalar, y si el espacio de decisión es $\mathbb{R}^m$ y el final es $\mathbb{R}^n$, con $n > 1$, el problema de optimización se denomina multiobjetivo, en el que se tienen $n$ objetivos correspondientes con cada una de las componentes de la función objetivo.

En problemas de optimización escalar el concepto de mínimo es claro ya que el conjunto de los números reales está dotado de una relación de orden total. Sin embargo, en problemas de optimización vectorial la noción de solución no es tan obvia, puesto que en el espacio objetivo no suele estar definida una relación de orden total. Esta dificultad ya se refleja en el caso multiobjetivo debido a que, en la práctica, cada uno de los objetivos a menudo representan criterios enfrentados entre sí, dando lugar a elementos no comparables en la imagen del conjunto factible. Este hecho hace necesaria la búsqueda de un concepto de solución del problema de optimización que refleje adecuadamente las decisiones que deben ser consideradas como óptimas.

Fue Pareto [91] quien, en 1906, introdujo la noción de solución eficiente o no dominada para un problema multiobjetivo, considerando en el espacio final el orden parcial natural por componentes. Estas soluciones se conocen como eficientes Pareto y el problema como multiobjetivo Pareto. Más tarde, Yu [118] extendió esta noción a espacios objetivo generales, en los que la relación de orden viene definida a través de un cono de orden. Si este cono es convexo, induce un preorden parcial compatible con la estructura vectorial del espacio objetivo, y si además es puntiagudo se obtiene un orden parcial (véase la Sección 1.2). Más específicamente, se definen las soluciones eficientes o no dominadas de un problema de optimización vectorial como aquellos puntos factibles para los que no existe ningún otro elemento factible que mejore el valor de la función objetivo, según la relación de orden establecida por el cono de orden en el espacio final (véase Definición 1.2.3).

Normalmente, cuando se resuelve un problema de optimización vectorial no se necesitan todas las soluciones eficientes del mismo, sino una selección de éstas. El proceso de selección se complica cuando el conjunto de soluciones eficientes es muy amplio, lo cual sucede con frecuencia en la práctica. Surgen así los conceptos de solución eficiente propia, basados todos ellos en la idea de descartar del conjunto de soluciones eficientes del problema, aquéllas que tengan un comportamiento anómalo o no deseable, siguiendo un determinado criterio.

La primera noción de eficiencia propia fue introducida por Kuhn y Tucker [66] en 1951, en el marco multiobjetivo Pareto y modificada años después por Geoffrion [34]. Las soluciones propias en el sentido de Geoffrion se interpretan como soluciones eficientes para las que una mejora en uno de los objetivos provoca un empeoramiento drástico en otro (véase Definición 1.2.5).

En 1977, Borwein [9] define una noción de eficiencia propia para problemas de optimización vectorial no necesariamente multiobjetivo Pareto, utilizando el cono tangente de Bouligand de la extensión cónica (mediante el cono de orden) de la imagen del conjunto factible (véase Definición 1.2.6) y, seguidamente, Benson [6] define un concepto similar reemplazando el cono tangente por la clausura del cono generado (véase Definición 1.2.7), que extiende la noción de Geoffrion e implica la de Borwein. Otro concepto importante de solución propia de problemas de optimización vectorial es el definido por Henig en [54] (Definición 1.2.8), más restrictivo que el dado por Benson, y que se define por medio de conos que contienen al cono de orden del espacio objetivo en su interior topológico. Bajo hipótesis de convexidad, estos tres tipos de solución coinciden y se pueden caracterizar a través de soluciones de problemas de optimización escalar asociados al problema de optimización vectorial original, lo que facilita su cálculo.

Otra cualidad destacable de las soluciones eficientes propias es que, bajo ciertas condiciones, proporcionan conjuntos densos respecto al conjunto eficiente (véase [5, 11, 25, 39]).

Las nociones de eficiencia propia anteriores son las más consolidadas. Posteriormente, se han introducido modificaciones de estos conceptos, con el fin de obtener nuevas propiedades. En [11], Borwein y Zhuang definieron una noción de solución supereficiente que puede ser considerada como una combinación entre los conceptos de eficiencia propia de Benson y Borwein y el dado por Henig, y en [121], Zheng introdujo un concepto de eficiencia propia en el sentido de Henig utilizando conos dilatadores (véase [11]).

En la práctica, los métodos de resolución de un problema de optimización se basan en técnicas de búsqueda iterativa en las que se emplean algoritmos o heurísticas que devuelven como solución puntos próximos a la solución real. Algunos ejemplos de estos métodos son los algoritmos evolutivos o genéticos (véase [1, 21]). Así pues, cobra interés el estudio de las soluciones aproximadas de un problema de optimización, dando lugar a numerosos trabajos en la literatura.

En problemas de optimización escalar, las primeras aportaciones al estudio de las soluciones aproximadas se deben a Loridan [77], Loridan y Morgan [79], Strodiot, Nguyen y Heukemes [103] y Yokoyama [117].

En el campo de la optimización vectorial, la primera y posiblemente más utilizada noción de solución aproximada fue introducida en 1979 por Kutateladze [70]. Este concepto da lugar a conjuntos eficientes aproximados demasiado grandes, en el sentido de que son posibles sucesiones de soluciones aproximadas que no tienden a soluciones eficientes exactas cuando el error tiende a cero. La causa de este comportamiento anómalo es que el error de aproximación se cuantifica en el espacio objetivo mediante un único vector del cono de orden.

Más adelante, se introdujeron nuevos e importantes conceptos de eficiencia aproximada. Merecen destacarse los dados por Vályi [113], White [115], Németh [89], Helbig [53] y Tanaka [107]. Todos ellos son nociones de solución no dominada en los que se sustituye el cono de orden del espacio objetivo por un conjunto con el que se determina el error de aproximación. Esta característica común motivó la definición de solución $(C, \varepsilon)$-eficiente para un problema de optimización vectorial introducida por Gutiérrez, Jiménez y Novo [48, 49] en 2006, donde $C$ es un conjunto del espacio objetivo y $\varepsilon$ un escalar no negativo (véase Definición 1.2.10).

La clase de soluciones $(C, \varepsilon)$-eficientes unifica y extiende los conceptos de eficiencia aproximada mencionados anteriormente, reduciéndose a cada uno de ellos mediante la elección del conjunto $C$ apropiado (véase Nota 1.2.11). De este modo, resulta de gran interés el estudio de esta noción de solución aproximada, no sólo por su carácter globalizador, sino por la posibilidad de elegir conjuntos $C$ específicos que den lugar a soluciones aproximadas con mejores propiedades, por ejemplo, con mejor comportamiento límite cuando $\varepsilon$ tiende a cero.

El cálculo de soluciones eficientes propias mediante procedimientos iterativos sugiere de nuevo estudiar conceptos de eficiencia propia aproximada. Los más conocidos y utilizados son los introducidos, respectivamente, por Li y Wang [74], Rong [97] (véase también [110]) y El Maghri [83] (véase Definiciones 1.2.16, 1.2.15 y 1.2.17). La primera de ellas está dada en el caso multiobjetivo Pareto y se basa en el concepto de eficiencia propia de Geoffrion, y las otras dos son válidas en problemas de optimización vectorial generales y están motivadas, respectivamente, por la eficiencia propia en el sentido de Benson y Henig. Estos tres tipos de solución tienen como denominador común la combinación de una noción de eficiencia propia con el concepto de eficiencia aproximada en el sentido de Kutateladze.

Recientemente, Gao, Yang y Teo [33] han introducido una clase de soluciones propias aproximadas para problemas de optimización vectorial motivada por la noción de solución $(C, \varepsilon)$-eficiente (véase Definición 1.2.19). En el presente documento se estudia en profundidad una extensión de este concepto, que puede considerarse como la versión propia en el sentido de Benson de las soluciones $(C, \varepsilon)$-eficientes, y que ha dado lugar a una nueva clase de soluciones propias aproximadas, llamadas $(C, \varepsilon)$-propias Benson.

La noción de solución $(C, \varepsilon)$-propia Benson se define con la finalidad de obtener un conjunto de soluciones aproximadas que represente al conjunto eficiente con un pequeño error, es decir, tal que, en el espacio final, los límites cuando el error tiende hacia cero de estas soluciones aproximadas se encuentren en la clausura del conjunto eficiente. Esta propiedad esencial no es común en las nociones de eficiencia aproximada. En otras palabras, con frecuencia, los conceptos de eficiencia aproximada pueden generar sucesiones de soluciones aproximadas que se alejan del conjunto eficiente tanto como se quiera. Por ejemplo, como se explicó más arriba, la eficiencia aproximada en el sentido de Kutateladze presenta este mal comportamiento límite, por dar lugar a conjuntos de soluciones aproximadas demasiado grandes. En consecuencia, las clases de solución dadas por Li y Wang, Rong, Tuan y El Maghri, respectivamente, no son apropiadas para aproximar el conjunto eficiente. Sin embargo, para conjuntos $C$ adecuados, se demuestra que las soluciones $(C, \varepsilon)$-propias Benson sí lo son, y este hecho supone que la extensión del concepto introducido por Gao, Yang y Teo es significativa, ya que las hipótesis que garantizan el buen comportamiento límite no se cumplen para esta última clase de soluciones.

Otra cualidad importante de las soluciones $(C, \varepsilon)$-propias Benson es su caracterización por medio de escalarización en problemas convexos, es decir, a través de soluciones aproximadas de problemas de optimización escalares asociados al problema de optimización vectorial, suponiendo determinadas condiciones de convexidad generalizada.

Esta memoria se vertebra en torno al estudio de las soluciones $(C, \varepsilon)$-propias Benson de un problema de optimización vectorial. Los objetivos fundamentales son estudiar en profundidad las propiedades de esta clase de soluciones, caracterizarlas mediante escalarizaciones lineales asumiendo condiciones de convexidad generalizada y utilizarlas para introducir y estudiar problemas duales aproximados y $\varepsilon$-subdiferenciales de funciones vectoriales. La tesis se estructura del siguiente modo.

En la Sección 1.2 se fija el marco de trabajo y se detallan las notaciones utilizadas y los conceptos previos necesarios. En particular, se define una nueva noción de convexidad generalizada de naturaleza aproximada (Definición 1.2.21) que generaliza las más importantes conocidas de la literatura. La mayor parte de los resultados presentados en la memoria se obtienen bajo esta condición.

Al comienzo de los Capítulos 2, 3, 4, 5 y 6 se presenta de forma breve el estado del arte sobre los contenidos que se tratan en cada uno de ellos.

En el Capítulo 2 se introduce el concepto de solución $(C, \varepsilon)$-propia Benson, se compara con los principales conceptos de eficiencia propia aproximada de la literatura y se estudian sus propiedades. En particular, se caracteriza este tipo de soluciones a través de escalarización lineal, bajo hipótesis de convexidad generalizada. Los resultados obtenidos mejoran notablemente los presentados por Gao, Yang y Teo en [33] (véanse las Notas 2.2.7 y 2.2.9).

También, se analiza el comportamiento límite de las soluciones $(C, \varepsilon)$-propias Benson y se obtienen condiciones basadas en la elección apropiada del conjunto $C$, bajo las que el límite superior en el sentido de Painlevé-Kuratowski del conjunto de soluciones $(C, \varepsilon)$-propias Benson está incluído en la clausura del conjunto eficiente.

La importancia de los teoremas de punto de silla radica en su estrecha relación con las teorías de dualidad y con otras herramientas usuales en Optimización Matemática, como las condiciones de optimalidad de tipo Kuhn-Tucker. Por ello, se considera de interés el estudio de estos resultados también para soluciones $(C, \varepsilon)$-propias Benson, dedicando el Capítulo 3 a este fin. En él, se introduce una multifunción Lagrangiana que extiende las más importantes vectoriales de la literatura y un nuevo concepto de punto de silla aproximado mediante soluciones $(C, \varepsilon)$-propias Benson del problema Lagrangiano asociado. Se determinan teoremas de punto de silla para las soluciones $(C, \varepsilon)$-propias Benson del problema de optimización vectorial original, que se reducen a teoremas conocidos en el caso exacto, en ocasiones con hipótesis más débiles.

Es remarcable que la condición de holgura complementaria puede ser acotada si se elige $C$ adecuadamente, lo cual no suele suceder en la mayoría de los teoremas de punto de silla de la literatura para soluciones aproximadas de problemas de optimización vectorial (véase [29, 41, 98, 114]).

Las teorías de dualidad surgen con la finalidad de proporcionar problemas alternativos, llamados duales, que permitan resolver de forma más fácil el problema de optimización original, llamado primal. Con esta premisa, en el Capítulo 4 se introducen dos problemas duales de tipo Lagrangiano y se obtienen teoremas de dualidad débil y fuerte para las soluciones $(C, \varepsilon)$-propias Benson del problema primal, bajo hipótesis de estabilidad y de convexidad generalizada. El primero de ellos se define por medio de una Lagrangiana escalar aproximada y se reduce al conocido problema dual introducido por Jahn [63, 64] en el caso exacto. El segundo problema se formula mediante una función dual definida a través de soluciones $(C, \varepsilon)$-propias Benson del problema de optimización Lagrangiano asociado a la multifunción Lagrangiana introducida en el Capítulo 3, y está motivado por el problema dual definido por Li [72] en el caso exacto.

Los Capítulos 5 y 6 se han dedicado al estudio de un nuevo concepto de $\varepsilon$-subdiferencial propia para funciones vectoriales. La $\varepsilon$-subdiferencial de una función escalar es una herramienta fundamental para determinar el conjunto de soluciones aproximadas de un problema de optimización convexo no necesariamente diferenciable, mediante la correspondiente regla de Fermat. Por ello, tiene gran interés extender este concepto a funciones vectoriales. En el Capítulo 5 se define la $(C, \varepsilon)$-subdiferencial Benson de una función vectorial, mediante soluciones $(C, \varepsilon)$-propias Benson de problemas de optimización vectorial no restringidos y se analizan sus principales propiedades. En particular, sus elementos se caracterizan mediante $\varepsilon$-subgradientes de funciones escalares asociadas asumiendo hipótesis de convexidad generalizada, y se demuestra que para conjuntos $C$ adecuados el conjunto de soluciones eficientes exactas de un problema de optimización vectorial se puede aproximar a través de sucesiones minimizantes. En este sentido, la $(C, \varepsilon)$-subdiferencial Benson mejora las $\varepsilon$-subdiferenciales propias aproximadas más importantes de la literatura porque no satisfacen esta propiedad.

Al final del Capítulo 5 se presentan fórmulas de tipo Moreau-Rockafellar para calcular la $(C, \varepsilon)$-subdiferencial Benson de la suma de dos funciones vectoriales, y en el Capítulo 6 se obtienen reglas de la cadena. Estas reglas de cálculo permiten aplicar los resultados teóricos a problemas prácticos y por este motivo resultan de gran interés. Para su desarrollo, se ha introducido y estudiado una $\varepsilon$-subdiferencial vectorial fuerte relacionada con un nuevo concepto de solución aproximada fuerte de problemas de optimización vectorial, y también una nueva condición de regularidad que extiende la conocida condición de regularidad dada por Raffin [95] a las soluciones $(C, \varepsilon)$-propias Benson.

Finalmente, el Capítulo 7 recoge las conclusiones y nuevas líneas futuras de investigación que han surgido durante la elaboración de esta memoria.

## 1.2 Preliminaries

In this document, $X$ and $Y$ are Hausdorff locally convex topological vector spaces and $Y$ is ordered through an ordering cone $D$ as usual, i.e.,

$$y_1, y_2 \in Y, y_1 \leq_D y_2 \Longleftrightarrow y_2 - y_1 \in D,$$

where $D$ is assumed to be nontrivial ($D \neq \{0\}$), convex, closed and pointed ($D \cap (-D) = \{0\}$). In this way, the relation $\leq_D$ defines a partial order, compatible with the linear structure in $Y$, that is, given $y_1, y_2, y_3 \in Y$ and $\alpha \geq 0$,

$$y_1 \leq_D y_2 \Longrightarrow y_1 + y_3 \leq_D y_2 + y_3, \quad \alpha y_1 \leq_D \alpha y_2.$$

The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$ and $\mathbb{R}_+ := \mathbb{R}^1_+$. In particular, if $Y = \mathbb{R}^n$ then the order defined through the cone $D = \mathbb{R}^n_+$ is called Pareto order.

The topological dual space of $X$ (respectively, $Y$) is denoted by $X^*$ (respectively, $Y^*$), and the duality pairing in $X$ (respectively, in $Y$) is denoted by $\langle x^*, x \rangle$, $x^* \in X^*$, $x \in X$ (respectively, $\langle y^*, y \rangle$, $y^* \in Y^*$, $y \in Y$). Moreover, we consider an arbitrary locally convex topology $\mathcal{T}$ on $Y^*$ compatible with the dual pair, i.e., $(Y^*, \mathcal{T})^* = Y$ (see [38, 64, 92, 93]).

The set of continuous linear mappings from $X$ to $Y$ will be denoted by $\mathcal{L}(X, Y)$.

We define the element $+\infty_Y$ as the supremum in $Y$ with respect to the ordering $\leq_D$ and we denote $\overline{Y} = Y \cup \{+\infty\}$. In other words, it holds that $y \leq_D +\infty_Y$, for all $y \in \overline{Y}$, and if there exists $y \in \overline{Y}$ such that $+\infty_Y \leq_D y$, then $y = +\infty_Y$.

The algebraic operations in $Y$ are extended as follows:

$$+\infty_Y + y = y + (+\infty_Y) = +\infty_Y,\ \alpha \cdot (+\infty_Y) = +\infty_Y,\ \forall\, y \in Y,\ \forall\, \alpha > 0.$$

We also suppose that $0 \cdot (+\infty_Y) = 0$. In particular, if $Y = \mathbb{R}$ and $D = \mathbb{R}_+$ then $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, where $+\infty := +\infty_{\mathbb{R}}$.

Given a set $F \subset Y$, we denote by $\operatorname{int} F$, $\operatorname{bd} F$, $\operatorname{cl} F$, $\operatorname{co} F$, $F^c$ and $\operatorname{cone} F$ the topological interior, the boundary, the closure, the convex hull, the complement and the cone generated by $F$, respectively. We say that $F$ is solid if $\operatorname{int} F \neq \emptyset$ and $F$ is coradiant if $\alpha F \subset F$, $\forall\, \alpha \geq 1$. The coradiant set generated by $F$ is denoted by $\operatorname{shw} F$ ("shadow" of $F$, see [119]), i.e.,

$$\operatorname{shw} F = \bigcup_{\alpha \geq 1} \alpha F.$$

The cone $D$ is said to be based if there exists a nonempty convex set $B \subset D$ such that $0 \notin \operatorname{cl} B$ and $\operatorname{cone} B = D$ (see [38, Definiton 2.2.14(i)]). The set $B$ is called a base of $D$.

The positive polar cone and the strict positive polar cone of $D$ are denoted by $D^+$ and $D^{s+}$, respectively, i.e.,

$$D^+ = \{\mu \in Y^* : \langle \mu, d \rangle \geq 0, \forall\, d \in D\},$$
$$D^{s+} = \{\mu \in Y^* : \langle \mu, d \rangle > 0, \forall\, d \in D\backslash\{0\}\}.$$

It is clear that $\operatorname{int} D^+ \subset D^{s+}$. Given $\mu \in D^{s+}$, we denote $B_\mu = \{d \in D : \langle \mu, d \rangle = 1\}$ and $C_\mu = B_\mu + D\backslash\{0\}$. The set $B_\mu$ is a base of $D$ (see [64, Lemma 1.28($a$)]).

Given a mapping $f : X \to \overline{Y}$, we denote the effective domain, the image, the graph and the epigraph of $f$ by $\operatorname{dom} f$, $\operatorname{Im} f$, $\operatorname{gr} f$ and $\operatorname{epi} f$, respectively, i.e.,

$$\operatorname{dom} f = \{x \in X : f(x) \in Y\},$$
$$\operatorname{Im} f = \{f(x) : x \in \operatorname{dom} f\},$$
$$\operatorname{gr} f = \{(x, y) \in X \times Y : y = f(x)\},$$
$$\operatorname{epi} f = \{(x, y) \in X \times Y : f(x) \leq_D y\}.$$

We say that $f$ is proper if $\operatorname{dom} f \neq \emptyset$. For each $y^* \in Y^*$ we assume that $\langle y^*, f(x) \rangle = +\infty$, $\forall\, x \in X, x \notin \operatorname{dom} f$. Thus, $y^* \circ f : X \to \overline{\mathbb{R}}$ and $\operatorname{dom}(y^* \circ f) = \operatorname{dom} f$.

In this document, we study the following vector optimization problem:

$$\text{Minimize } f(x) \text{ subject to } x \in S, \tag{$\mathcal{P}_S$}$$

where the objective mapping $f$ is proper and the feasible set $S \subset X$ verifies that $S_0 := S \cap \operatorname{dom} f \neq \emptyset$ in order to avoid trivial problems. The solutions of problem $(\mathcal{P}_S)$ are determined in terms of the preference relation given by $\leq_D$.

If $S = X$, then the problem is unconstrained and it will be denoted by $(\mathcal{P})$, i.e.,

$$\text{Minimize } f(x) \text{ subject to } x \in X. \tag{$\mathcal{P}$}$$

Frequently, the feasible set is defined by a cone constraint, i.e., in the following way:

$$S = \{x \in X : g(x) \in -K\}, \tag{1.1}$$

where $g : X \to Z$, $Z$ is an ordered Hausdorff locally convex topological vector space and $K \subset Z$ is the ordering cone. The topological dual space of $Z$ is denoted by $Z^*$. In the particular case $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$, $n > 1$, $D = \mathbb{R}^n_+$, $Z = \mathbb{R}^p$ and $K = \mathbb{R}^p_+$, problem $(\mathcal{P}_S)$ with $S$ defined as in (1.1) reduces to the well-known Pareto multiobjective optimization problem with inequality constraints in which we have $n$ objectives, corresponding with the components $f_i : \mathbb{R}^m \to \overline{\mathbb{R}}$, $i \in \{1, 2, \ldots, n\}$ of $f$, and the feasible set is formulated in terms of $p$ inequality constraints defined by means of the components $g_1, g_2, \ldots, g_p$ of $g$. This problem reduces to an ordinary scalar optimization problem with inequality constraints when $n = 1$.

Along this work, when we are referring to problem $(\mathcal{P}_S)$ with the feasible set given as in (1.1), we assume that the cone $K$ is solid, convex and $K \neq Z$. Moreover, we say that problem $(\mathcal{P}_S)$ satisfies the Slater constraint qualification if there exists $x \in X$ such that $g(x) \in -\operatorname{int} K$.

Given $\mu \in D^{s+}$, we consider the following scalar optimization problem related to problem $(\mathcal{P}_S)$:

$$\text{Minimize } (\mu \circ f)(x) \text{ subject to } x \in S. \tag{$\mathcal{P}_\mu$}$$

In the next definition we state the well-known normality and stability concepts associated with the scalar optimization problem $(\mathcal{P}_\mu)$ and its classical Lagrange dual problem when $S$ is defined by the cone constraint $g(x) \in -K$ (see [30, 64]).

**Definition 1.2.1.** Let $\mu \in D^{s+}$. It is said that problem $(\mathcal{P}_\mu)$ is normal if

$$-\infty < \inf_{x\in S}\{(\mu \circ f)(x)\} = \sup_{\lambda\in K^+} \inf_{x\in X}\{(\mu \circ f + \lambda \circ g)(x)\}.$$

Moreover, $(\mathcal{P}_\mu)$ is said to be stable if it is normal and

$$\sup_{\lambda\in K^+} \inf_{x\in X}\{(\mu \circ f + \lambda \circ g)(x)\}$$

is attained.

The following definition introduces the well-known concepts of minimal and maximal efficient points of a nonempty set $F \subset Y$. Both concepts are determined by the preference relation $\leq_D$.

**Definition 1.2.2.** It is said that $y_0 \in F$ is an efficient minimal (respectively, maximal) point of $F$ if there is not a point $y \in F$, with $y \neq y_0$ such that $y \leq_D y_0$ (respectively, $y_0 \leq_D y$).

The set of efficient minimal (respectively, maximal) points of $F$ is denoted by $\mathrm{Min}(F)$ (respectively, $\mathrm{Max}(F)$). It is clear that $y_0 \in \mathrm{Min}(F)$ (respectively, $y_0 \in \mathrm{Max}(F)$) if and only if $y_0 \in F$ and

$$(F - y_0) \cap (-D\backslash\{0\}) = \emptyset \quad \text{(respectively, } (F - y_0) \cap (D\backslash\{0\}) = \emptyset).$$

Next, we recall some classical solution concepts of problem $(\mathcal{P}_S)$ (see, for instance, [20, 35, 38, 64, 81, 88, 101]).

**Definition 1.2.3.** A point $x_0 \in S_0$ is an efficient solution of problem $(\mathcal{P}_S)$, if there is not a feasible point $x \in S_0$ such that $f(x) \leq_D f(x_0)$, $f(x) \neq f(x_0)$.

We denote the set of efficient solutions of $(\mathcal{P}_S)$ by $\mathrm{E}(f, S)$. Observe that

$$\mathrm{E}(f, S) = f^{-1}(\mathrm{Min}(f(S_0))) \cap S_0$$

and so $x_0 \in \mathrm{E}(f, S)$ if and only if

$$(f(S_0) - f(x_0)) \cap (-D\backslash\{0\}) = \emptyset. \tag{1.2}$$

If $D$ is solid and we replace $D\backslash\{0\}$ by $\operatorname{int} D$ in (1.2) then we obtain the well-known concept of weakly efficient solution of $(\mathcal{P}_S)$.

**Definition 1.2.4.** A point $x_0 \in S_0$ is a strong solution of problem $(\mathcal{P}_S)$ if $f(x_0) \leq_D f(x)$, for all $x \in S$. The set of strong solutions of $(\mathcal{P}_S)$ is denoted by $\mathrm{SE}(f, S)$.

Usually, the set $\mathrm{E}(f, S)$ is too big, and it is necessary to select an appropriate subset of efficient solutions from it, in such a way that efficient solutions with bad properties in a certain sense are not included. The concepts of proper efficiency arise in order to carry out this choice.

The following proper efficiency notion was introduced by Geoffrion in [34] and it works for Pareto multiobjective optimization problems.

**Definition 1.2.5.** Suppose that $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$ and $D = \mathbb{R}^n_+$. A point $x_0 \in S_0$ is a Geoffrion proper solution of $(\mathcal{P}_S)$, denoted by $x_0 \in \mathrm{Ge}(f, S)$, if it is efficient and there exists $k > 0$ such that for all $i \in \{1, 2, \dots, n\}$ and for all $x \in S_0$ satisfying that $f_i(x) < f_i(x_0)$ there exists $j \in \{1, 2, \dots, n\}$ such that $f_j(x_0) < f_j(x)$ and

$$\frac{f_i(x_0) - f_i(x)}{f_j(x) - f_j(x_0)} \leq k.$$

From the definition above, we see that the set $\mathrm{Ge}(f, S)$ is formed by the efficient solutions for which an improvement in one of the objectives implies a considerable worsening in another objective.

The following notions of proper efficiency work for (non necessarily multi-objective) vector optimization problems. The common idea of these concepts, introduced by Borwein [9], Benson [6] and Henig [54], respectively, is to replace one of the sets that takes part in (1.2) by another set bigger than first one, in order to obtain a more restrictive notion of efficiency and avoid, in this way, anomalous efficient solutions of $(\mathcal{P}_S)$.

In the sequel, $T(F, y_0)$ denotes the Bouligand tangent cone of a nonempty set $F \subset Y$ at $y_0 \in F$, i.e.,

$$T(F, y_0) = \{y \in Y : \exists t_n \to 0^+, \exists y_n \in F \text{ such that } \lim_{n\to+\infty} t_n^{-1}(y_n - y_0) = y\}.$$

**Definition 1.2.6.** A point $x_0 \in S_0$ is a proper efficient solution of $(\mathcal{P}_S)$ in the sense of Borwein if

$$T(f(S_0) + D, f(x_0)) \cap (-D) = \{0\}.$$

The set of proper efficient solutions of $(\mathcal{P}_S)$ in the sense of Borwein is denoted by $\mathrm{Bor}(f, S)$.

**Definition 1.2.7.** A point $x_0 \in S_0$ is a proper efficient solution of problem $(\mathcal{P}_S)$ in the sense of Benson, if

$$\mathrm{cl}\,\mathrm{cone}(f(S_0) + D - f(x_0)) \cap (-D) = \{0\}.$$

We denote the set of proper efficient solutions of $(\mathcal{P}_S)$ in the sense of Benson by $\mathrm{Be}(f, S)$.

**Definition 1.2.8.** A point $x_0 \in S_0$ is a proper efficient solution of problem $(\mathcal{P}_S)$ in the sense of Henig, if there exists a convex cone $D' \subset Y$, $D' \neq Y$, such that $D\backslash\{0\} \subset \mathrm{int}\, D'$ and

$$(f(S_0) - f(x_0)) \cap (-\,\mathrm{int}\, D') = \emptyset.$$

We denote the set of proper efficient solutions of $(\mathcal{P}_S)$ in the sense of Henig by $\mathrm{He}(f, S)$. It is clear that $x_0 \in \mathrm{He}(f, S)$ whenever $x_0$ is a weakly efficient solution of $(\mathcal{P}_S)$ with respect to an ordering cone $D'$ such that $D' \neq Y$ and $D\backslash\{0\} \subset \mathrm{int}\, D'$.

**Remark 1.2.9.** It follows that

$$\mathrm{SE}(f, S) \subset \mathrm{He}(f, S) \subset \mathrm{Be}(f, S) \subset \mathrm{Bor}(f, S) \subset \mathrm{E}(f, S).$$

For the first inclusion above, we assume that there exists a convex cone $D' \subset Y$, $D' \neq Y$, such that $D\backslash\{0\} \subset \mathrm{int}\, D'$. This assumption is satisfied, for example, if $D^+$ is solid.

Moreover, when $(\mathcal{P}_S)$ is a Pareto multiobjective optimization problem, we have that $\mathrm{Ge}(f,S) = \mathrm{Be}(f,S)$ (see [6, Theorem 3.2]). Furthermore, under generalized convexity assumptions it follows that $\mathrm{He}(f,S) = \mathrm{Be}(f,S) = \mathrm{Bor}(f,S)$ (see, for instance, [101, Theorems 3.1.1 and 3.1.2], [35, Theorems 6.4.2 and 6.4.5] and Corollary 2.2.16).

In the practice, the methods for solving an optimization problem use iterative algorithms that converge to the theoretical solution. Thus, these methods usually obtain as solution feasible points close to the real solution. For this reason, the study of approximate solutions of an optimization problem gains importance. In vector optimization, the notions of approximate efficiency are known as $\varepsilon$-efficiency concepts and their solutions are called $\varepsilon$-efficient.

In [48, 49], Gutiérrez, Jiménez and Novo introduced a new notion of approximate solution of problem $(\mathcal{P}_S)$ that extends and unifies the most important concepts of approximate efficiency given in the literature, as it is shown in Remark 1.2.11.

For each nonempty set $C \subset Y$, we define the set-valued mappings $C, C_0 : \mathbb{R}_+ \to 2^Y$ as follows:

$$C(\varepsilon) = \begin{cases} \varepsilon C & \text{if} \quad \varepsilon > 0 \\ \mathrm{cone} C & \text{if} \quad \varepsilon = 0, \end{cases}$$

and $C_0(\varepsilon) := C(\varepsilon)$ if $0 \notin C(\varepsilon)$, $C_0(\varepsilon) := C(\varepsilon)\backslash\{0\}$ otherwise.

**Definition 1.2.10.** Let $C \subset Y$ be nonempty and $\varepsilon \geq 0$. It is said that $x_0 \in S_0$ is a $(C,\varepsilon)$-efficient solution of problem $(\mathcal{P}_S)$, denoted by $x_0 \in \mathrm{AE}(f,S,C,\varepsilon)$, if

$$(f(S_0) - f(x_0)) \cap (-C_0(\varepsilon)) = \emptyset. \tag{1.3}$$

In the same way as for the exact case, if $C$ is solid and we replace $C$ by $\mathrm{int}\, C$ in (1.3) we obtain the concept of weakly $(C,\varepsilon)$-efficient solution of $(\mathcal{P}_S)$ (see [48, 49]). It is obvious that if $C = D$ or $\varepsilon = 0$ and $\mathrm{cone}\, C = D$, then $\mathrm{AE}(f,S,C,\varepsilon) = \mathrm{E}(f,S)$. Moreover, if $D$ is solid, the set of weakly $(D,0)$-efficient solutions of $(\mathcal{P}_S)$ coincides with the set of weakly efficient solutions.

**Remark 1.2.11.** Definition 1.2.10 reduces to the following well-known approximate efficiency concepts considering specific sets $C$ (see [48, 49, 52]).

- If $C = q + D$, $q \in D\backslash\{0\}$, then we obtain the $\varepsilon$-efficiency concept given by Kutateladze [70]. In the sequel we denote $C_q := q + D$.
- If $C = H + D$, where $H \subset D\backslash\{0\}$, then Definition 1.2.10 reduces to the approximate efficiency notion due to Németh [89].
- If $C = [\mu > 1]$, where $\mu \in D^+\backslash\{0\}$ and $[\mu > 1] = \{y \in Y : \langle \mu, y\rangle > 1\}$, then Definition 1.2.10 reduces to the $\varepsilon$-efficiency notion given by Vályi [113, 114].
- If $C = D \cap [\mu > 1]$, $\mu \in D^+\backslash\{0\}$, then we obtain the $\varepsilon$-efficiency concept introduced by Helbig [53].
- Suppose that $D$ is solid. If $C = D \cap (q - D)^c$, $q \in D\backslash\{0\}$, then we obtain the $\varepsilon$-efficiency concept introduced by White [115].
- Assume that $Y$ is normed. If we consider the set $C = D \cap \mathcal{B}^c$, where $\mathcal{B} \subset Y$ denotes the unit open ball, then we recover the approximate efficiency notion given by Tanaka [107].

Observe that Definition 1.2.10 reduces to the notions stated in Remark 1.2.11 by considering sets $C \subset D$ and also via the set $[\mu > 1]$, which is not included in $D$. This is a meaningful property, because for appropriate sets $C$ not included in $D$ the set of $(C, \varepsilon)$-efficient solutions of $(\mathcal{P}_S)$ has a good limit behaviour when the error $\varepsilon$ tends to zero, in the sense that the limits of $(C, \varepsilon)$-efficient solutions are included in the efficient set, under certain additional hypotheses, as it is shown in Lemma 2.3.2.

In particular, the set $C$ should verify the following natural restriction:

$$C \cap (-D\backslash\{0\}) = \emptyset. \tag{1.4}$$

Indeed, if we consider a set $C$ such that $C \cap (-D\backslash\{0\}) \neq \emptyset$ and $x_0 \in S_0$ is a $(C, \varepsilon)$-efficient solution of $(\mathcal{P}_S)$, then we have in particular that

$$(f(S_0) - f(x_0)) \cap ((-C(\varepsilon)) \cap (D\backslash\{0\})) = \emptyset. \tag{1.5}$$

This means that for this set $C$, we are discarding as possible approximate efficient solutions of $(\mathcal{P}_S)$ the feasible points $x_0 \in S$ for which there exists at least one point $\bar{x} \in S_0$ satisfying that $f(\bar{x}) - f(x_0) \in (-C(\varepsilon)) \cap (D\backslash\{0\}) \subset D\backslash\{0\}$. This latter inclusion implies in particular that $f(x_0) \leq_D f(\bar{x})$, so by choosing a set $C \subset Y$ that does not verify (1.4), we are following a pointless procedure to obtain suitable approximate efficient solutions of $(\mathcal{P}_S)$. This situation is illustrated in the following example.

**Example 1.2.12.** Consider $X = Y = \mathbb{R}^2$, $D = \mathbb{R}^2_+$ and $C \subset \mathbb{R}^2$ such that $C \cap (-\mathbb{R}^2_+\backslash\{0\}) \neq \emptyset$. Denote by $\mathcal{B}$ the Euclidean unit closed ball in $\mathbb{R}^2$ and suppose that $S = \mathcal{B} + \mathbb{R}^2_+$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the identity mapping. It is easy to see that

$$\mathrm{E}(f, S) = \mathcal{S} \cap (-\mathbb{R}^2_+),$$

where $\mathcal{S}$ denotes the Euclidean unit sphere in $\mathbb{R}^2$.

Let $\bar{d} \in C \cap (-\mathbb{R}^2_+\backslash\{0\})$. If $\varepsilon > 0$ and $x_0 \in \mathrm{AE}(f, S, C, \varepsilon)$, by statement (1.5) we have that $f(x_0) - \varepsilon\bar{d} \notin \mathcal{B} + \mathbb{R}^2_+$, which is a contradiction. Then,

$$\mathrm{AE}(f, S, C, \varepsilon) = \emptyset, \ \forall \varepsilon > 0,$$

and the corresponding $(C, \varepsilon)$-efficient concept is worthless.

In a natural way, the $(C, \varepsilon)$-efficient solution notion of $(\mathcal{P}_S)$ leads to notions of approximate minimal and maximal point of a nonempty set $F \subset Y$.

**Definition 1.2.13.** A point $y_0 \in F$ is a $(C, \varepsilon)$-minimal (respectively, $(C, \varepsilon)$-maximal) point of $F$, denoted by $y_0 \in \mathrm{Min}(F, C, \varepsilon)$ (respectively, $y_0 \in \mathrm{Max}(F, C, \varepsilon)$), if $y_0 \in \mathrm{AE}(i, F, C, \varepsilon)$ (respectively, $y_0 \in \mathrm{AE}(-i, F, C, \varepsilon)$), where $X = Y$, $i(y) := y$ and $-i(y) := -y$, for all $y \in Y$.

It follows that

$$\mathrm{AE}(f, S, C, \varepsilon) = f^{-1}(\mathrm{Min}(f(S_0), C, \varepsilon)) \cap S_0.$$

Clearly, if $C = D$ or $\varepsilon = 0$ and $\operatorname{cone} C = D$, then $\mathrm{Min}(F, C, \varepsilon) = \mathrm{Min}(F)$ and $\mathrm{Max}(F, C, \varepsilon) = \mathrm{Max}(F)$.

Next, we recall an approximate strong solution notion of $(\mathcal{P}_S)$ introduced by Kutateladze (see [70]) that generalizes the concept of strong solution from an approximate point of view.

**Definition 1.2.14.** Let $q \in D\backslash\{0\}$ and $\varepsilon \geq 0$. We say that $x_0 \in S_0$ is a strong $\varepsilon$-efficient solution of $(\mathcal{P}_S)$ with respect to $q$, denoted by $\mathrm{SE}(f, S, q, \varepsilon)$, if

$$f(x_0) - \varepsilon q \leq_D f(x), \quad \forall\, x \in S.$$

With respect to the approximate proper efficiency, several concepts have been introduced in the literature. Next we recall some of them. The first one was introduced by Rong [97] for vector optimization problems and it is based on the Benson proper efficiency concept (see Definition 1.2.7), the second one is due to Li and Wang [74] and it is motivated by the notion of proper efficiency in the sense of Geoffrion (see Definition 1.2.5), and the third one was given by El Maghri [83] and it is motivated by the proper efficiency due to Henig (see Definition 1.2.8). The three notions can be considered as proper versions of the $\varepsilon$-solution notion due to Kutateladze [70].

**Definition 1.2.15.** Consider $q \in D\backslash\{0\}$ and $\varepsilon \geq 0$. A point $x_0 \in S_0$ is a Benson $\varepsilon$-proper solution of problem $(\mathcal{P}_S)$ with respect to $q$, denoted by $x_0 \in \mathrm{Be}(f, S, q, \varepsilon)$, if

$$\mathrm{cl}(\mathrm{cone}(f(S_0) + \varepsilon q + D - f(x_0))) \cap (-D) = \{0\}.$$

**Definition 1.2.16.** Assume that $Y = \mathbb{R}^n$ and $D = \mathbb{R}^n_+$ and consider $q \in \mathbb{R}^n_+\backslash\{0\}$ and $\varepsilon \geq 0$. A point $x_0 \in S_0$ is a Geoffrion $\varepsilon$-proper solution of $(\mathcal{P}_S)$ with respect to $q$, denoted by $x_0 \in \mathrm{Ge}(f, S, q, \varepsilon)$, if $x_0 \in \mathrm{AE}(f, S, \varepsilon q + \mathbb{R}^n_+, 1)$ and there exists $k > 0$ such that for each $x \in S_0$ and $i \in \{1, 2, \ldots, n\}$ with $f_i(x_0) > f_i(x) + \varepsilon q_i$ there exists $j \in \{1, 2, \ldots, n\}$ such that $f_j(x_0) < f_j(x) + \varepsilon q_j$ and

$$\frac{f_i(x_0) - f_i(x) - \varepsilon q_i}{f_j(x) - f_j(x_0) + \varepsilon q_j} \leq k.$$

**Definition 1.2.17.** Consider $q \in D\backslash\{0\}$ and $\varepsilon \geq 0$. A point $x_0 \in S_0$ is a Henig $\varepsilon$-proper solution of problem $(\mathcal{P}_S)$ with respect to $q$, denoted by $x_0 \in \mathrm{He}(f, S, q, \varepsilon)$, if there exists a convex cone $D' \subset Y$, $D' \neq Y$, such that $D\backslash\{0\} \subset \mathrm{int}\, D'$ and $(f(S_0) + \varepsilon q - f(x_0)) \cap (-D'\backslash\{0\}) = \emptyset$.

**Remark 1.2.18.** Let $q \in D\backslash\{0\}$.

($a$) If $\varepsilon = 0$, then

$$\mathrm{SE}(f, S, q, 0) = \mathrm{SE}(f, S),\ \mathrm{Be}(f, S, q, 0) = \mathrm{Be}(f, S),\ \mathrm{He}(f, S, q, 0) = \mathrm{He}(f, S),$$

$$\mathrm{Ge}(f,S,q,0)=\mathrm{Ge}(f,S).$$

$(b)$ It follows that

$$\mathrm{SE}(f,S,q,\varepsilon)\subset \mathrm{He}(f,S,q,\varepsilon)\subset \mathrm{Be}(f,S,q,\varepsilon)\subset \mathrm{AE}(f,S,C_q,\varepsilon).$$

For the first inclusion above, we assume that there exists a convex cone $D'\subset Y$, $D'\neq Y$, such that $D\backslash\{0\}\subset \operatorname{int} D'$. Moreover, in Corollary 2.2.16 it is proved under cone convexity assumptions that $\mathrm{He}(f,S,q,\varepsilon)=\mathrm{Be}(f,S,q,\varepsilon)$, and $\mathrm{Ge}(f,S,q,\varepsilon)=\mathrm{Be}(f,S,q,\varepsilon)$, whenever $Y=\mathbb{R}^n$ and $D=\mathbb{R}^n_+$.

In [33], Gao, Yang and Teo introduced the following concept of approximate proper efficiency, which is based on the $(C,\varepsilon)$-efficiency notion stated in Definition 1.2.10 (see [33, Definition 3.3 and Remark 2]).

**Definition 1.2.19.** Let $C\subset Y$ be a nonempty, pointed and coradiant set and $\varepsilon\geq 0$. It is said that a point $x_0\in S_0$ is a properly $\varepsilon$-efficient solution of $(\mathcal{P}_S)$ with respect to $C$ if

$$\operatorname{cl}\operatorname{cone}(f(S_0)+C(\varepsilon)-f(x_0))\cap(-C(0))=\{0\}. \tag{1.6}$$

Next, we recall some classical notions of cone convexity and generalized convexity that will appear in this work.

**Definition 1.2.20.** Let $M\subset X$, with $M_0:=M\cap\operatorname{dom} f\neq\emptyset$. It is said that the mapping $f$ is

- $D$-convex on $M$ if $M$ is convex and

$$f(\alpha x_1+(1-\alpha)x_2)\leq_D \alpha f(x_1)+(1-\alpha)f(x_2),\quad \forall\, x_1,x_2\in M,\ \forall\,\alpha\in[0,1].$$

- $D$-convexlike on $M$ if $f(M_0)+D$ is convex.
- $D$-subconvexlike on $M$ if $f(M_0)+\operatorname{int} D$ is convex ($D$ is assumed to be solid).
- nearly $D$-subconvexlike on $M$ if $\operatorname{cl}\operatorname{cone}(f(M_0)+D)$ is convex.

In particular, if $f$ is $D$-convex on $M$ then $M_0$ is convex.

Moreover, it is not hard to verify that $f$ is $D$-convex on $M$ if and only if $M$ is convex and $\mu \circ f$ is convex on $M$, for all $\mu \in D^+$.

It follows that

$$\begin{aligned} f \text{ } D\text{-convex on } M &\implies f \text{ } D\text{-convexlike on } M \\ &\implies f \text{ } D\text{-subconvexlike on } M \\ &\implies f \text{ nearly } D\text{-subconvexlike on } M. \end{aligned}$$

In this document, the following new notion of generalized convexity for a vector-valued mapping will be considered. Roughly speaking, it is an "approximate" version of the notion of nearly subconvexlikeness, due to Yang, Li and Wang [116].

**Definition 1.2.21.** Consider $\varepsilon \geq 0$ and two nonempty sets $M \subset X$, with $M_0 \neq \emptyset$ and $C \subset Y$. The mapping $f$ is said to be nearly $(C, \varepsilon)$-subconvexlike on $M$ if $\operatorname{cl} \operatorname{cone}(f(M_0) + C(\varepsilon))$ is convex.

The notion above reduces to the concept of nearly cone subconvexlikeness (see [116]) when $C$ is a cone or $\varepsilon = 0$. In both cases we say that $f$ is nearly $C(0)$-subconvexlike on $M$.

The next result is very useful in order to check if the mapping $f$ is nearly $(C, \varepsilon)$-subconvexlike on $M$.

**Theorem 1.2.22.** Consider two nonempty sets $M \subset X$ and $C \subset Y$, a proper mapping $f : X \to \overline{Y}$ such that $M_0 \neq \emptyset$ and $\varepsilon \geq 0$.

($a$) If $C$ is convex and $f$ is $E$-convexlike on $M$, where $E \subset Y$ is a cone such that $C + E = C$, then $f$ is nearly $(C, \varepsilon)$-subconvexlike on $M$.

($b$) If $C$ is convex and $f$ is $E$-subconvexlike on $M$, where $E \subset Y$ is a convex cone such that $C + E = C$, then $f$ is nearly $(C, \varepsilon)$-subconvexlike on $M$.

($c$) If $C$ is convex and coradiant, and $f$ is $\operatorname{cone} C$-convexlike on $M$, then $f$ is nearly $(C, \varepsilon)$-subconvexlike on $M$.

*Proof.* $(a)$ Suppose that $\varepsilon > 0$. From the hypotheses we deduce that $f(M_0) + E$ and $C(\varepsilon)$ are convex sets, and so $f(M_0) + E + C(\varepsilon)$ is convex. Moreover, it is clear that

$$E + C(\varepsilon) = \varepsilon(E + C) = C(\varepsilon).$$

Therefore, $f(M_0)+C(\varepsilon)$ is convex and it follows that $f$ is nearly $(C, \varepsilon)$-subconvexlike on $M$.

Now consider the case $\varepsilon = 0$. From the hypotheses we deduce that $f(M_0)+E$ and $\mathrm{cl}(C(0))$ are convex sets, and so $f(M_0) + E + \mathrm{cl}(C(0))$ is convex. On the other hand, it follows that $E \subset \mathrm{cl}(C(0))$. Indeed, let $\bar{c} \in C$ be arbitrary. For each $y \in E$ we have that

$$y + \alpha\bar{c} = \alpha((1/\alpha)y + \bar{c}) \in \mathrm{cone}(E + C) = C(0), \quad \forall\, \alpha > 0,$$

and by taking the limit when $\alpha \to 0$ we see that $y \in \mathrm{cl}(C(0))$. Therefore, as $\mathrm{cl}(C(0))$ is a convex cone, we deduce that

$$E + \mathrm{cl}(C(0)) = \mathrm{cl}(C(0))$$

and then $f(M_0) + \mathrm{cl}(C(0))$ is convex. Moreover, it is clear that

$$\mathrm{cl}\,\mathrm{cone}(f(M_0) + C(0)) = \mathrm{cl}\,\mathrm{cone}(f(M_0) + \mathrm{cl}(C(0)))$$

and it follows that $f$ is nearly $(C, 0)$-subconvexlike on $M$.

$(b)$ The proof follows the same reasoning as in part $(a)$, taking into account that

$$\mathrm{cl}\,\mathrm{cone}(f(M_0) + \mathrm{int}\, E + C(\varepsilon)) = \mathrm{cl}\,\mathrm{cone}(f(M_0) + E + C(\varepsilon)),$$
$$\mathrm{cl}\,\mathrm{cone}(f(M_0) + \mathrm{int}\, E + \mathrm{cl}\, C(0)) = \mathrm{cl}\,\mathrm{cone}(f(M_0) + E + \mathrm{cl}\, C(0)),$$

since $\mathrm{cl}\, E = \mathrm{cl}(\mathrm{int}\, E)$.

$(c)$ By applying [48, Lemma 3.1(v)] we deduce that $C + \mathrm{cone}\, C = C$, and the result is a consequence of part $(a)$. □

**Remark 1.2.23.** If $f$ is $D$-convex on $S$, $A \in \mathcal{L}(X, Y)$ and $y \in Y$, then it follows that the mapping $f + A + y : X \to \overline{Y}$, $(f + A + y)(x) = f(x) + Ax + y$ for all

$x \in X$, is $D$-convex on $S$, and by Theorem 1.2.22($a$) we deduce that it is nearly $(C, \varepsilon)$-subconvexlike on $S$, for all $\varepsilon \geq 0$ and for all nonempty convex set $C \subset Y$ such that $C + D = C$.

Another mathematical tool used in this document is the well-known Brøndsted-Rockafellar $\varepsilon$-subdifferential (see [17]), that we recall in the following definition.

**Definition 1.2.24.** Let $h : X \to \overline{\mathbb{R}}$ be a proper mapping and $\varepsilon \geq 0$. The $\varepsilon$-subdifferential of $h$ at a point $x_0 \in \operatorname{dom} h$ is defined as follows:

$$\partial_\varepsilon h(x_0) = \{x^* \in X^* : h(x) \geq h(x_0) - \varepsilon + \langle x^*, x - x_0 \rangle, \forall\, x \in X\}.$$

The elements of $\partial_\varepsilon h(x_0)$ are called $\varepsilon$-subgradients of $h$ at $x_0$.

Given a proper mapping $h : X \to \overline{\mathbb{R}}$, a nonempty set $M \subset X$ such that $M \cap \operatorname{dom} h \neq \emptyset$ and $\varepsilon \geq 0$, we denote

$$\varepsilon\text{-argmin}_M h = \{x \in M : h(x) - \varepsilon \leq h(z), \forall\, z \in M\} \subset \operatorname{dom} h$$

and $\operatorname{argmin}_M h := 0\text{-argmin}_M h$.

Observe that the elements of $\varepsilon\text{-argmin}_M h$ are the suboptimal solutions with error $\varepsilon$ (exact solutions if $\varepsilon = 0$) of the following implicitly constrained scalar optimization problem:

$$\text{Minimize } h(x) \text{ subject to } x \in M.$$

For $\varepsilon \geq 0$ and $x_0 \in \operatorname{dom} h$, it is easy to check that $x^* \in \partial_\varepsilon h(x_0)$ if and only if $x_0 \in \varepsilon\text{-argmin}_X(h - x^*)$. In particular, given $x_0 \in \operatorname{dom} f$ and two arbitrary mappings $y^* \in Y^*$ and $A \in \mathcal{L}(X, Y)$, it follows that

$$x_0 \in \varepsilon\text{-argmin}_X(y^* \circ (f - A)) \iff y^* \circ A \in \partial_\varepsilon (y^* \circ f)(x_0). \tag{1.7}$$

Next, we recall two proper $\varepsilon$-subdifferentials for vector mappings due to Tuan (see [110] and El Maghri [83], respectively, that will be quoted in this document. The first one was introduced for set-valued mappings (in this document, we only consider the single-valued case), and it is defined in terms of Benson $\varepsilon$-proper solutions of $(\mathcal{P})$ with respect to a vector $q$, and the second one is defined by means of Henig $\varepsilon$-proper solutions of $(\mathcal{P})$ with respect to a vector $q$.

**Definition 1.2.25.** Let $\varepsilon \geq 0$, $q \in D\backslash\{0\}$ and $x_0 \in \operatorname{dom} f$. The $\varepsilon q$-Benson proper subdifferential of $f$ at $x_0$ is defined as

$$\partial^B_{\varepsilon q} f(x_0) := \{T \in \mathcal{L}(X, Y) : x_0 \in \operatorname{Be}(f - T, X, q, \varepsilon)\}.$$

The elements of $\partial^B_{\varepsilon q} f(x_0)$ are called $\varepsilon q$-Benson proper subgradients of $f$ at $x_0$.

**Definition 1.2.26.** Let $\varepsilon \geq 0$, $q \in D\backslash\{0\}$ and $x_0 \in \operatorname{dom} f$. The $\varepsilon q$-Henig proper subdifferential of $f$ at $x_0$ is defined as

$$\partial^p_{\varepsilon q} f(x_0) := \{T \in \mathcal{L}(X, Y) : x_0 \in \operatorname{He}(f - T, X, q, \varepsilon)\}.$$

The elements of $\partial^p_{\varepsilon q} f(x_0)$ are called $\varepsilon q$-proper subgradients of $f$ at $x_0$ in the sense of Henig.

Finally, for a nonempty set $C \subset Y$, we define the mapping $\tau_C : Y^* \to \mathbb{R} \cup \{-\infty\}$ as follows:

$$\tau_C(y^*) = \inf_{y \in C}\{\langle y^*, y\rangle\}, \quad \forall\, y^* \in Y^*.$$

Observe that $\tau_C = -\sigma_{-C}$, where $\sigma_{-C} : Y^* \to \mathbb{R} \cup \{+\infty\}$ denotes the support function of $-C$ (see, for instance [120]), i.e.,

$$\sigma_{-C}(y^*) = \sup_{y \in -C}\{\langle y^*, y\rangle\}, \quad \forall\, y^* \in Y^*. \tag{1.8}$$

Moreover, we denote

$$C^{\tau+} = \{y^* \in Y^* : \tau_C(y^*) \geq 0\}.$$

It follows that $C^{\tau+} = (\operatorname{cone} C)^+$ and for each two nonempty sets $C_1, C_2 \subset Y$ and $\alpha_1, \alpha_2 \in \mathbb{R}_+$, it is clear that (we assume $0 \cdot (-\infty_{\mathbb{R}}) = 0$)

$$\begin{aligned} \tau_{\alpha_1 C_1 + \alpha_2 C_2}(y^*) &= \alpha_1 \tau_{C_1}(y^*) + \alpha_2 \tau_{C_2}(y^*), \quad \forall\, y^* \in Y^*, \\ \tau_{C_1(\alpha_1) + C_2(\alpha_2)}(y^*) &= \alpha_1 \tau_{C_1}(y^*) + \alpha_2 \tau_{C_2}(y^*), \quad \forall\, y^* \in C_1^{\tau+} \cap C_2^{\tau+}. \end{aligned} \tag{1.9}$$

# Capítulo 2

# Approximate proper efficiency

## 2.1 Introduction

In Section 1.2 we collected the most known concepts of approximate proper efficiency of the literature. They have been studied for stating scalarization results and multiplier rules, and for defining new concepts of $\varepsilon$-subdifferential of cone convex vector mappings.

To be precise, Li and Wang [74] and Liu [76] obtained, respectively, nonlinear and linear scalarization results for Geoffrion $\varepsilon$-proper solutions of $(\mathcal{P}_S)$ with respect to a vector $q$ (see Definition 1.2.16). Moreover, Liu stated multiplier rules for these solutions in convex and nondifferentiable Pareto multiobjective optimization problems with inequality and equality constraints.

Analogously, Rong [97] obtained linear scalarization results for Benson $\varepsilon$-proper solutions of problem $(\mathcal{P}_S)$ with respect to a vector $q$ (see Definition 1.2.15), and Tuan [110] characterized these type of solutions through linear scalarizations in vector optimization problems with set-valued mappings. Also, Tuan defined an $\varepsilon$-subdifferential for set-valued mappings through these solutions.

El Maghri [83] followed the same line than Tuan but considering Henig $\varepsilon$-proper solutions and single-valued mappings (see Definition 1.2.17).

On the other hand, Gao, Yang and Teo [33] obtained linear and nonlinear scalarization results for the properly $\varepsilon$-efficient solutions of $(\mathcal{P}_S)$ (see Definition 1.2.19). This concept generalized the previous ones.

In this chapter we introduce a new concept of approximate proper solution in the sense of Benson of problem $(\mathcal{P}_S)$, which is a meaningful generalization of the notion of properly $\varepsilon$-efficiency given by Gao, Yang and Teo in [33], and it is based on the concept of approximate efficiency introduced by Gutiérrez, Jiménez and Novo in [48, 49] (see Definition 1.2.10). These solutions, called Benson $(C, \varepsilon)$-proper solutions, stand out because they can be characterized through approximate solutions of the related scalar optimization problems $(\mathcal{P}_\mu)$ under generalized convexity assumptions and they have a good limit behaviour when $\varepsilon \to 0$, in the sense that the limits of Benson $(C, \varepsilon)$-proper solutions are in the efficient set, whenever the set $C$ satisfies suitable properties.

In Section 2.2 we define the notion of Benson $(C, \varepsilon)$-proper efficiency, we study its basic properties and we compare it with the properly $\varepsilon$-efficiency concept given by Gao, Yang and Teo in [33]. Under nearly subconvexlikeness hypotheses, we characterize the set of Benson $(C, \varepsilon)$-proper solutions of $(\mathcal{P}_S)$ through approximate solutions of the scalar optimization problems $(\mathcal{P}_\mu)$. This characterization is a significant result, since it facilitates the calculus of the set of Benson $(C, \varepsilon)$-proper solutions in convex problems.

When we apply the results obtained in this subsection to the notion of properly $\varepsilon$-efficiency introduced by Gao, Yang and Teo, we improve the results obtained in [33]. In particular, under subconvexlikeness assumptions, Gao, Yang and Teo stated necessary and sufficient conditions for properly $\varepsilon$-efficient solutions of $(\mathcal{P}_S)$ through approximate solutions of the problems $(\mathcal{P}_\mu)$, but they obtained different approximation error in both conditions.

In Section 2.3, we study in deep the sets of Benson $(C, \varepsilon)$-proper efficient solutions of $(\mathcal{P}_S)$ when the error $\varepsilon$ tends to zero. We show that they are useful in order to obtain outer approximations for the efficient set in non necessarily convex vector optimization problems.

The reason of this good limit behaviour lies in the selection of a suitable set $C$, which lets us obtain a set of Benson $(C, \varepsilon)$-proper solutions of $(\mathcal{P}_S)$ that approximates the efficient set in a convenient way. For example, in normed spaces, this property facilitates to search a set of Benson $(C, \varepsilon)$-proper solutions in such a way that all these solutions are close to the efficient set. Observe that the well-known notions of approximate proper efficiency stated in Definitions 1.2.16, 1.2.15 and 1.2.17 quantify the error by considering a unique vector and, because of this, the obtained sets of proper $\varepsilon$-efficient solutions are usually too big and do not approximate well the efficient set.

The results stated in this chapter are based on [44, Section 3] and [42, Section 3].

## 2.2 Benson $(C, \varepsilon)$-proper solutions

In this section we introduce a new concept of approximate proper efficiency for problem $(\mathcal{P}_S)$ which is a proper version in the sense of Benson (see Definition 1.2.7) of the $(C, \varepsilon)$-efficiency notion due to Gutiérrez, Jiménez and Novo (see Definition 1.2.10) and generalizes the properly $\varepsilon$-efficiency concept given by Gao, Yang and Teo in [33] (see Definition 1.2.19). Next, we study its basic properties and we characterize this type of solutions through approximate solutions of the associated scalar problems $(\mathcal{P}_\mu)$. We show that the obtained results improve those given in [33] when we reduce them to the notion of Definition 1.2.19.

We denote

$$\mathcal{H}_Y := \{C \subset Y : C \neq \emptyset, \mathrm{cl}(C(0)) \cap (-D \backslash \{0\}) = \emptyset\}.$$

**Definition 2.2.1.** Consider $\varepsilon \geq 0$ and $C \in \mathcal{H}_Y$. A point $x_0 \in S_0$ is a Benson $(C, \varepsilon)$-proper solution of $(\mathcal{P}_S)$, denoted by $x_0 \in \mathrm{Be}(f, S, C, \varepsilon)$, if

$$\mathrm{cl\,cone}(f(S_0) + C(\varepsilon) - f(x_0)) \cap (-D) = \{0\}. \tag{2.1}$$

**Remark 2.2.2.** $(a)$ Taking into account (2.1), assumption $C \in \mathcal{H}_Y$ is needed in order to obtain a non superfluous concept.

$(b)$ Consider two nonempty sets $F_1, F_2 \subset Y$. It is obvious that

$$\mathrm{cl\,cone}(F_1 + F_2) = \mathrm{cl\,cone}(F_1 + \mathrm{cl}\, F_2). \tag{2.2}$$

Thus, if $\mathrm{cl\,cone}\, C = D$ then $\mathrm{Be}(f, S, C, 0) = \mathrm{Be}(f, S)$. In particular, this happens if we consider $q \in D$ and $C_q := q + D$ or $C_q^0 := q + D \backslash \{0\}$.

($c$) It is clear that Definition 2.2.1 reduces to Definition 1.2.15 when $C = C_q$ or $C = C_q^0$, i.e., it follows that

$$\mathrm{Be}(f, S, q, \varepsilon) = \mathrm{Be}(f, S, C_q, \varepsilon) = \mathrm{Be}(f, S, C_q^0, \varepsilon), \quad \forall\, q \in D, \quad \forall\, \varepsilon \geq 0.$$

($d$) Suppose that $C(0)$ is closed. Then, statement (2.1) reduces to (1.6) by taking the ordering cone $D = C(0)$, and hence Definition 2.2.1 reduces to Definition 1.2.19.

On the other hand, statement (1.6) only reduces to (2.1) when the cone generated by $C$ is $D$, and so $C$ must be included in $D$ necessarily.

As we will see in the next section, the Benson $(C, \varepsilon)$-proper solutions of $(\mathcal{P}_S)$ satisfy important properties concerning to its limit behaviour when $C$ is not included in $D$. Thus, the results obtained in Section 2.3 for the Benson $(C, \varepsilon)$-proper efficiency, in particular Theorem 2.3.3, cannot be applied for the properly $\varepsilon$-efficiency in the sense of Gao, Yang and Teo. Hence, in this sense, our Definition 2.2.1 is more general than Definition 1.2.19.

The Benson $(C, \varepsilon)$-proper solutions of $(\mathcal{P}_S)$ satisfy the next basic properties.

**Proposition 2.2.3.** Let $\varepsilon \geq 0$ and $C \in \mathcal{H}_Y$. It follows that

($a$) $\mathrm{Be}(f, S, C', \delta) \subset \mathrm{Be}(f, S, C, \varepsilon) = \mathrm{Be}(f, S, \mathrm{cl}\, C, \varepsilon)$, for all $C' \in \mathcal{H}_Y$ and $\delta \geq 0$ such that $C(\varepsilon) \subset \mathrm{cl}(C'(\delta))$.

($b$) $\mathrm{Be}(f, S, C + C', \varepsilon) = \mathrm{Be}(f, S, \mathrm{cl}\, C + C', \varepsilon)$, for all $C' \subset Y$ such that $C + C' \in \mathcal{H}_Y$.

($c$) If $C \subset D$ then $\mathrm{Be}(f, S) \subset \mathrm{Be}(f, S, C, \varepsilon)$.

($d$) If $C$ is coradiant, then $\mathrm{Be}(f, S, C, \delta) \subset \mathrm{Be}(f, S, C, \varepsilon)$, for all $0 \leq \delta \leq \varepsilon$. If additionally $C$ is convex, then $\mathrm{Be}(f, S, C, \varepsilon) = \mathrm{Be}(f, S, C + C(0), \varepsilon)$.

($e$) If $D$ has a compact base, then $\mathrm{Be}(f, S, C, \varepsilon) = \mathrm{Be}(f, S, C + D, \varepsilon)$.

*Proof.* Given a nonempty set $F \subset Y$, $C' \in \mathcal{H}_Y$ and $\delta \geq 0$ such that $C(\varepsilon) \subset \mathrm{cl}(C'(\delta))$, in view of (2.2) it is clear that

$$\begin{aligned} &\mathrm{cl\,cone}(F + \mathrm{cl}\, C(\varepsilon)) = \mathrm{cl\,cone}(F + C(\varepsilon)) \\ \subset\ &\mathrm{cl}(\mathrm{cone}(F + \mathrm{cl}(C'(\delta)))) = \mathrm{cl}(\mathrm{cone}(F + C'(\delta))). \end{aligned}$$

Analogously, for each nonempty set $C' \subset Y$ such that $C + C' \in \mathcal{H}_Y$ we have

$$\operatorname{cl}\operatorname{cone}(F + (C + C')(\varepsilon)) = \operatorname{cl}\operatorname{cone}(F + (\operatorname{cl} C + C')(\varepsilon)).$$

Therefore, parts $(a)$ and $(b)$ follow from the definition by considering $F = f(S_0) - f(x_0)$.

Part $(c)$ is a consequence of applying part $(a)$ to $C' = D$ and $\delta = 0$.

On the other hand, if $C$ is coradiant, then $C(\varepsilon) \subset C(\delta)$ for all $0 \leq \delta \leq \varepsilon$ and so the first part of $(d)$ follows by applying part $(a)$ to $C' = C$. If additionally $C$ is convex, then by [48, Lemma 3.1(v)] we have that $C + C(0) = C$, which finishes the proof of part $(d)$.

In order to prove part $(e)$, let us observe that

$$\operatorname{Be}(f, S, C + D, \varepsilon) \subset \operatorname{Be}(f, S, C, \varepsilon),$$

since $C \subset C + D$. Reciprocally, if $x_0 \in \operatorname{Be}(f, S, C, \varepsilon)$ then $x_0 \in S_0$ and

$$\operatorname{cl}\operatorname{cone}(f(S_0) + C(\varepsilon) - f(x_0)) \cap (-D) = \{0\}. \tag{2.3}$$

Suppose on the contrary that $x_0 \notin \operatorname{Be}(f, S, C + D, \varepsilon)$. As $(C + D)(\varepsilon) = C(\varepsilon) + D$ there exists $w \neq 0$ such that

$$w \in \operatorname{cl}\operatorname{cone}(f(S_0) + C(\varepsilon) + D - f(x_0)) \cap (-D).$$

Thus, there exist nets $(\alpha_i) \subset \mathbb{R}_+$, $(y_i) \subset f(S_0)$, $(q_i) \subset C(\varepsilon)$ and $(d_i) \subset D$ such that

$$\alpha_i(y_i + q_i - f(x_0)) + \alpha_i d_i \to w \in -D \backslash \{0\}. \tag{2.4}$$

By (2.3) and (2.4) we deduce that $(\alpha_i d_i)$ has not any null subnet and so we can suppose that $\bar{d}_i := \alpha_i d_i \in D \backslash \{0\}$, $\forall\, i$. As $D$ has a compact base $B$, for each $i$ there exist $\beta_i > 0$ and $b_i \in B$ such that $\bar{d}_i = \beta_i b_i$. Since $B$ is compact, we can assume without loss of generality that $b_i \to b$, $b \in B$. We consider the two following possibilities:

$(i)$ $(\beta_i)$ is bounded. Then, we can suppose that $(\beta_i)$ is convergent. Let $\beta := \lim_i \beta_i$. It follows that $\beta_i b_i \to \beta b \in D$. Hence, from (2.4) we have that

$$\alpha_i(y_i + q_i - f(x_0)) \to w - \beta b.$$

Since $w \in -D\backslash\{0\}$ and $D$ is pointed, $w - \beta b \in -D\backslash\{0\}$ in contradiction with (2.3).

$(ii)$ $(\beta_i)$ is not bounded. Thus, we can assume that $\beta_i \to +\infty$. Therefore, from (2.4) we deduce that

$$\frac{\alpha_i}{\beta_i}(y_i + q_i - f(x_0)) + \frac{\alpha_i}{\beta_i} d_i = \frac{\alpha_i}{\beta_i}(y_i + q_i - f(x_0)) + b_i \to 0.$$

As $b_i \to b$ it follows that

$$\frac{\alpha_i}{\beta_i}(y_i + q_i - f(x_0)) \to -b \in D\backslash\{0\},$$

which is a contradiction with (2.3). Thus, $\mathrm{Be}(f, S, C, \varepsilon) \subset \mathrm{Be}(f, S, C + D, \varepsilon)$, and the proof is complete. □

As a consequence of Proposition 2.2.3$(c)$ we have that

$$\mathrm{Be}(f, S) \subset \cap_{\varepsilon > 0} \mathrm{Be}(f, S, C, \varepsilon), \tag{2.5}$$

whenever $C \subset D$. The reciprocal inclusion is not true in general, as it will be shown in Example 2.2.13.

In order to characterize the Benson $(C, \varepsilon)$-proper solutions of $(\mathcal{P}_S)$ through scalarization in convex problems, we need the following lemma.

**Lemma 2.2.4.** Let $x_0 \in S_0$, $\varepsilon \geq 0$ and $C \in \mathcal{H}_Y$. Suppose that $\mathrm{int}\, D^+ \neq \emptyset$ and $f - f(x_0)$ is nearly $(C, \varepsilon)$-subconvexlike on $S$. If $x_0 \in \mathrm{Be}(f, S, C, \varepsilon)$, then there exists $\mu \in D^{s+} \cap C^{\tau+}$ such that

$$\mathrm{cl\, cone}(f(S_0) + C(\varepsilon) - f(x_0)) \cap (-H) = \emptyset,$$

where $H$ is the open half space defined by $H = \{y \in Y : \langle \mu, y \rangle > 0\}$.

*Proof.* As $x_0 \in \mathrm{Be}(f, S, C, \varepsilon)$ it follows that

$$\mathrm{cl\, cone}(f(S_0) + C(\varepsilon) - f(x_0)) \cap (-D) = \{0\}. \tag{2.6}$$

Since $f - f(x_0)$ is nearly $(C, \varepsilon)$-subconvexlike on $S$, we have that $\mathrm{cl\, cone}(f(S_0) + C(\varepsilon) - f(x_0))$ is a closed convex cone. Then, by applying [64, Theorem 3.22] to statement (2.6) we deduce that there exists a functional $\mu \in Y^*\backslash\{0\}$ such that

$$\langle \mu, y \rangle \geq 0, \quad \forall\, y \in \mathrm{cl\, cone}(f(S_0) + C(\varepsilon) - f(x_0)),$$
$$\langle \mu, d \rangle > 0, \quad \forall\, d \in D\backslash\{0\}.$$

From the first inequality we deduce in particular that $\langle \mu, y\rangle \geq 0$, for all $y \in C$, and by the second inequality $\mu \in D^{s+}$. Thus, $\mu \in D^{s+} \cap C^{\tau+}$, and the proof is complete by taking the open half space $H$ associated to $\mu$. □

From Lemma 2.2.4 we deduce that in problems where $\operatorname{int} D^+ \neq \emptyset$ and $f - f(x)$ is nearly $(C,\varepsilon)$-subconvexlike on $S$ for all $x \in S_0$, the existence of Benson $(C,\varepsilon)$-proper solutions implies $D^{s+} \cap C^{\tau+} \neq \emptyset$. Moreover, it is clear that

$$D^{s+} \cap C^{\tau+} \neq \emptyset \Rightarrow \operatorname{cl}(C(0)) \cap (-D\backslash\{0\}) = \emptyset. \tag{2.7}$$

This fact motivates the following notation for referring to sets $C$ such that Definition 2.2.1 is not superfluous when $\operatorname{int} D^+ \neq \emptyset$ and $f - f(x)$ is nearly $(C,\varepsilon)$-subconvexlike on $S$ for all $x \in S_0$:

$$\mathcal{F}_Y := \{C \in \mathcal{H}_Y : C \neq \emptyset, D^{s+} \cap C^{\tau+} \neq \emptyset\}.$$

**Example 2.2.5.** Let $q \in Y\backslash(-D\backslash\{0\})$, $C_q = q + D$ and assume that $\operatorname{int} D^+ \neq \emptyset$. It follows that $C_q \in \mathcal{F}_Y$. Indeed, by [64, Lemma 3.21($d$)] we see that $D^{s+} = \operatorname{int} D^+$ and so $D^{s+} \neq \emptyset$.

If $q = 0$ then $C_q^{\tau+} = D^+$ and $D^{s+} \cap C_q^{\tau+} = D^{s+}$. Thus $C_q \in \mathcal{F}_Y$.

If $q \neq 0$ then $q \notin -D$ and by applying a separation theorem (see, for instance, [64, Theorem 3.18]) we deduce that there exists $\mu_1 \in D^+\backslash\{0\}$ satisfying $\langle \mu_1, q\rangle > 0$. By taking $\mu_2 \in D^{s+}$ and $\alpha > 0$ such that $\langle \mu_1, q\rangle + \alpha\langle \mu_2, q\rangle > 0$ we obtain that $\mu := \mu_1 + \alpha\mu_2 \in D^{s+} \cap C_q^{\tau+}$, and so $C_q \in \mathcal{F}_Y$.

On the other hand, if $C$ is convex and $\operatorname{int} D^+ \neq \emptyset$, by [9, Proposition 2] we deduce that the reciprocal implication in statement (2.7) is also true. Therefore, under these assumptions we have the following "primal" characterization:

$$\emptyset \neq C \subset Y, \quad C \in \mathcal{F}_Y \iff \operatorname{cl}(C(0)) \cap (-D\backslash\{0\}) = \emptyset,$$

i.e., $\mathcal{F}_Y = \mathcal{H}_Y$.

The following theorem is a necessary condition for Benson $(C,\varepsilon)$-proper solution of $(\mathcal{P}_S)$ through linear scalarization under nearly $(C,\varepsilon)$-subconvexlikeness assumptions.

**Theorem 2.2.6.** Let $x_0 \in S_0$, $\varepsilon \geq 0$ and $C \in \mathcal{F}_Y$. Suppose that $\operatorname{int} D^+ \neq \emptyset$ and $f - f(x_0)$ is nearly $(C, \varepsilon)$-subconvexlike on $S$. If $x_0 \in \operatorname{Be}(f, S, C, \varepsilon)$ then there exists $\mu \in D^{s+} \cap C^{\tau+}$ such that $x_0 \in \varepsilon\tau_C(\mu)\text{-argmin}_S(\mu \circ f)$.

*Proof.* Let $x_0 \in \operatorname{Be}(f, S, C, \varepsilon)$. By Lemma 2.2.4 there exists $\mu \in D^{s+} \cap C^{\tau+}$ such that

$$\operatorname{cl} \operatorname{cone}(f(S_0) + C(\varepsilon) - f(x_0)) \cap (-H) = \emptyset,$$

where $H$ is the open half space defined by $H = \{y \in Y : \langle \mu, y \rangle > 0\}$. Hence, it is clear that

$$\langle \mu, f(x) \rangle + \langle \mu, d \rangle - \langle \mu, f(x_0) \rangle \geq 0, \quad \forall\, x \in S_0, \forall\, d \in C(\varepsilon). \tag{2.8}$$

In particular, we have that

$$(\mu \circ f)(x) \geq (\mu \circ f)(x_0) - \inf_{d \in C(\varepsilon)} \{\langle \mu, d \rangle\} = (\mu \circ f)(x_0) - \varepsilon\tau_C(\mu), \quad \forall\, x \in S_0.$$

Therefore, $x_0 \in \varepsilon\tau_C(\mu)\text{-argmin}(\mu \circ f)$, and the proof is complete. □

**Remark 2.2.7.** $(a)$ If $Y$ is normed then we can suppose in Theorem 2.2.6 that $\|\mu\| = 1$ since one can divide in statement (2.8) by $\|\mu\|$.

$(b)$ Suppose that $C(0) = D$. In [33, Theorem 5.4], the authors obtained the following necessary condition:

$$\operatorname{Be}(f, S, C, \varepsilon) \subset \bigcup_{\mu \in D^{s+}, \|\mu\| = 1} \varepsilon\beta\text{-argmin}_S(\mu \circ f), \tag{2.9}$$

where $\beta = \inf\{\|d\| : d \in C\}$. To obtain this result they assume that $Y$ is a normed space, $C$ is solid and coradiant, $D$ is locally compact or $D^+$ is solid and $f$ is $D$-subconvexlike on $S$.

Theorem 2.2.6 improves [33, Theorem 5.4] since $Y$ is a Hausdorff locally convex space, $C$ does not need to be coradiant neither solid (and so $D$ can be not solid) and the mapping $f - f(x_0)$ is assumed to be nearly $(C, \varepsilon)$-subconvexlike on $S$, which can be a weaker generalized convexity condition than the subconvexlikeness of $f$ (see [99]) and Theorem 1.2.22$(b)$. Moreover, if $Y$ is normed, let us observe that the error $\varepsilon\beta$ in statement (2.9) is greater than the error $\varepsilon\tau_C(\mu)$ in Theorem 2.2.6, since

$$\tau_C(\mu) = \inf\{\langle \mu, d \rangle : d \in C\} \leq \inf\{\|d\| : d \in C\} = \beta, \quad \forall\, \mu \in D^{s+}, \|\mu\| = 1.$$

Furthermore, the inequality can be strict as it is observed in the following example: Let $Y = (\mathbb{R}^2, \|\cdot\|_2)$, $D = \mathbb{R}^2_+$, $C = \{(y_1, y_2) \in \mathbb{R}^2_+ : y_1 + y_2 \geq 1\}$ and $\mu = (1/\sqrt{5}, 2/\sqrt{5})$. It is clear that $\beta = 1/\sqrt{2}$ and $\tau_C(\mu) = 1/\sqrt{5}$.

$(c)$ Theorem 2.2.6 reduces to the single-valued version of [110, Theorem 4.4(i)] by considering $C = C_q$, $q \in D$.

Next result is a sufficient condition for Benson $(C,\varepsilon)$-proper solutions of $(\mathcal{P}_S)$ through linear scalarization.

**Theorem 2.2.8.** Let $\varepsilon \geq 0$ and $C \in \mathcal{F}_Y$. It follows that

$$\bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \varepsilon\tau_C(\mu)\text{-argmin}_S(\mu \circ f) \subset \mathrm{Be}(f, S, C, \varepsilon).$$

*Proof.* Let us consider $\mu \in D^{s+} \cap C^{\tau+}$ and

$$x_0 \in \varepsilon\tau_C(\mu)\text{-argmin}_S(\mu \circ f). \tag{2.10}$$

Suppose on the contrary that $x_0 \notin \mathrm{Be}(f, S, C, \varepsilon)$. Then, there exist $v \in D\backslash\{0\}$ and nets $(\alpha_i) \subset \mathbb{R}_+$, $(x_i) \subset S_0$ and $(d_i) \in C(\varepsilon)$ such that $\alpha_i(f(x_i) + d_i - f(x_0)) \to -v$. Since $\mu \in D^{s+}$ and $v \in D\backslash\{0\}$, we deduce that

$$\alpha_i(\langle\mu, f(x_i)\rangle + \langle\mu, d_i\rangle - \langle\mu, f(x_0)\rangle) \to -\langle\mu, v\rangle < 0.$$

Therefore, we can suppose that there exists $i_0$ such that $\alpha_{i_0}(\langle\mu, f(x_{i_0})\rangle + \langle\mu, d_{i_0}\rangle - \langle\mu, f(x_0)\rangle) < 0$. In particular,

$$\langle\mu, f(x_{i_0})\rangle + \langle\mu, d_{i_0}\rangle - \langle\mu, f(x_0)\rangle < 0,$$

and as $d_{i_0} \in C(\varepsilon)$ we deduce that

$$(\mu \circ f)(x_{i_0}) + \varepsilon\tau_C(\mu) - (\mu \circ f)(x_0) \leq (\mu \circ f)(x_{i_0}) + \langle\mu, d_{i_0}\rangle - (\mu \circ f)(x_0) < 0,$$

which is a contradiction with (2.10). □

**Remark 2.2.9.** $(a)$ Suppose that $C(0) = D$. In [33, Theorem 5.5], the authors obtained the following sufficient optimality condition for Benson $(C,\varepsilon)$-proper solutions of $(\mathcal{P}_S)$ by assuming that $Y$ is a Banach space, $C$ is coradiant, $0 \notin \operatorname{cl} C$ and $D^+$ is solid:

$$\bigcup_{\mu \in D^{s+}} \varepsilon\theta\text{-argmin}_S(\mu \circ f) \subset \text{Be}(f, S, C, \varepsilon),$$

where $\theta = \inf\{\|d\| : d \in C\} \inf\{\|\xi - \mu\| : \xi \in Y^* \backslash D^+\}$. By [8, Lemma 2.7] we see that

$$\inf_{d \in D, \|d\|=1} \{\langle \mu, d \rangle\} \geq \inf\{\|\xi - \mu\| : \xi \in Y^* \backslash D^+\}, \quad \forall\, \mu \in D^{s+}.$$

From this statement is clear that

$$\langle \mu, d \rangle \geq \|d\| \inf\{\|\xi - \mu\| : \xi \in Y^* \backslash D^+\}, \quad \forall\, d \in D, \forall\, \mu \in D^{s+}. \tag{2.11}$$

By applying this inequality to elements of $C$ we deduce that

$$\begin{aligned}\tau_C(\mu) &= \inf\{\langle \mu, d \rangle : d \in C\} \\ &\geq \inf\{\|d\| : d \in C\} \inf\{\|\xi - \mu\| : \xi \in Y^* \backslash D^+\} = \theta,\ \forall \mu \in D^{s+}.\end{aligned}$$

Therefore,

$$\bigcup_{\mu \in D^{s+}} \varepsilon\theta\text{-argmin}_S(\mu \circ f) \subset \bigcup_{\mu \in D^{s+}} \varepsilon\tau_C(\mu)\text{-argmin}_S(\mu \circ f)$$

and so [33, Theorem 5.5] is a particular case of Theorem 2.2.8, since $C^{\tau+} = D^+$ and so $D^{s+} \cap C^{\tau+} = D^{s+}$. Moreover, let us underline that the assumptions of Theorem 2.2.8 are more general than the assumptions of [33, Theorem 5.5].

($b$) Theorem 2.2.8 reduces to the single-valued version of [110, Theorem 4.4(ii)] by considering $C = C_q$, $q \in D$.

The next corollary is an immediate consequence of Theorems 2.2.6 and 2.2.8.

**Corollary 2.2.10.** Let $\varepsilon \geq 0$ and $C \in \mathcal{F}_Y$. Suppose that $\operatorname{int} D^+ \neq \emptyset$ and $f - f(x)$ is nearly $(C, \varepsilon)$-subconvexlike on $S$ for all $x \in S_0$. Then,

$$\text{Be}(f, S, C, \varepsilon) = \bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \varepsilon\tau_C(\mu)\text{-argmin}_S(\mu \circ f).$$

**Remark 2.2.11.** $(a)$ Corollary 2.2.10 shows that there is not gap between the approximation errors obtained in the necessary condition of Theorem 2.2.6 and in the sufficient condition in Theorem 2.2.8.

Moreover, if $Y$ is normed, $C(0)=D$ and $\mu\in D^{s+}$, $\|\mu\|=1$, it is easy to check that (see Remarks 2.2.7$(b)$ and 2.2.9$(a)$)

$$\theta\le\tau_C(\mu)\le\beta,$$

and so the errors of [33, Theorems 5.4 and 5.5] are not the same.

For example, consider $Y=(\mathbb{R}^2,\|\cdot\|_2)$, $D=\mathbb{R}^2_+$, $C=\{(y_1,y_2)\in\mathbb{R}^2_+ : y_1+y_2\ge 1\}$ and $\mu=(1/\sqrt{5},2/\sqrt{5})$. It is clear that $\theta=1/\sqrt{10}$, $\tau_C(\mu)=1/\sqrt{5}$ and $\beta=1/\sqrt{2}$.

$(b)$ If $C=C_q$, $q\in D$, then Corollary 2.2.10 reduces to [97, Theorem 4], where the order cone $D$ is assumed to be solid and a convexity assumption stronger than nearly $(C,\varepsilon)$-subconvexlikeness is considered.

$(c)$ When $\varepsilon=0$ and $C=D$, Corollary 2.2.10 reduces to well-known characterizations of (exact) Benson proper solutions of a vector optimization problem. In particular, it reduces to [4, Corollary 4.1], [6, Theorem 4.2], [19, Theorem 4.1], [73, Theorem 3.1(ii)], the equivalence (i) $\Longleftrightarrow$ (iii) of [93, Theorem 3.2] and the single-valued version of [116, Theorem 6.2]. Moreover, [6, Theorem 4.2] and [73, Theorem 3.1(ii)] are referred to a finite dimensional objective space, [6, Theorem 4.2], [19, Theorem 4.1] and [73, Theorem 3.1(ii)] use stronger convexity assumptions and [4, Corollary 4.1], [19, Theorem 4.1] and [73, Theorem 3.1(ii)] require a solid order cone.

Next we show another important direct consequence of Theorem 2.2.8. Given a set $C\in\mathcal{F}_Y$, $\varepsilon\ge 0$ and $\mu\in D^{s+}\cap C^{\tau+}$, we denote

$$\mathcal{F}_{C(\varepsilon)}(\mu) \;:=\; \{(C',\delta)\in\mathcal{F}_Y\times\mathbb{R}_+ : \tau_{C'}(\mu)\ge 0,\ \delta\tau_{C'}(\mu)\ge\varepsilon\tau_C(\mu)\}.$$

**Corollary 2.2.12.** Let $C\in\mathcal{F}_Y$, $\varepsilon\ge 0$ and $\mu\in D^{s+}\cap C^{\tau+}$. Then

$$\varepsilon\tau_C(\mu)\text{-argmin}_S(\mu\circ f)\subset\bigcap_{(C',\delta)\in\mathcal{F}_{C(\varepsilon)}(\mu)}\mathrm{Be}(f,S,C',\delta).$$

Now we can show by an example, that the reciprocal inclusion of statement (2.5) is not true in general.

**Example 2.2.13.** Let us consider problem $(\mathcal{P}_S)$ with the following data: $X = Y = \mathbb{R}^2$, $f(y_1, y_2) = (y_1, y_2)$, for all $(y_1, y_2) \in \mathbb{R}^2$, $S' = \{(y_1, y_2) \in -\mathbb{R}^2_+ : y_1^2 + y_2^2 = 1\}$, $S = S' + \mathbb{R}^2_+$, $D = \mathbb{R}^2_+$ and $C = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 0,\ y_2 \geq 0,\ y_1 + y_2 \geq 1\}$.

It is clear that $\mathrm{Be}(f, S) = S' \backslash \{(-1, 0), (0, -1)\}$. Moreover,

$$(-1, 0) \in \bigcap_{\varepsilon > 0} \mathrm{Be}(f, S, C, \varepsilon).$$

Indeed, by applying Remark 1.2.23 we see that $f - f(y)$ is nearly $(C, \varepsilon)$-subconvexlike on $S$, for all $y \in S$ and for all $\varepsilon \geq 0$, and by Corollary 2.2.10 it follows that

$$\mathrm{Be}(f, S, C, \varepsilon) = \bigcup_{\mu \in \operatorname{int} \mathbb{R}^2_+} \varepsilon \tau_C(\mu)\text{-}\mathrm{argmin}_S(\mu \circ f), \tag{2.12}$$

since $D^{s+} \cap C^{\tau+} = \operatorname{int} \mathbb{R}^2_+$. It is easy to check that

$$\mathrm{argmin}_S(\mu \circ f) = \{-\mu / \|\mu\|\}, \quad \forall\, \mu \in \operatorname{int} \mathbb{R}^2_+,$$

where $\|\cdot\|$ denotes the Euclidean norm, and

$$\tau_C(\mu) = \mu_2, \quad \forall\, \mu = (\mu_1, \mu_2) \in \operatorname{int} \mathbb{R}^2_+,\ \mu_1 \geq \mu_2.$$

Then, fixed an arbitrary $\varepsilon > 0$ and by defining $\mu_n = (1, 1/n) \in \operatorname{int} \mathbb{R}^2_+$ for all $n \in \mathbb{N}$, $n \geq 1$, there exists $n_\varepsilon \in \mathbb{N}$ large enough such that

$$\langle \mu_{n_\varepsilon}, (-1, 0) \rangle \leq -\|\mu_{n_\varepsilon}\| + \varepsilon / n_\varepsilon,$$

i.e., $(-1, 0) \in \varepsilon \tau_C(\mu_{n_\varepsilon})\text{-}\mathrm{argmin}_S(\mu_{n_\varepsilon} \circ f)$, and by (2.12) we conclude that $(-1, 0) \in \mathrm{Be}(f, S, C, \varepsilon)$, for all $\varepsilon > 0$.

Under generalized convexity assumptions, the sets of Benson $(C, \varepsilon)$-proper solutions of problem $(\mathcal{P}_S)$ satisfy additional properties, as it is showed in the following proposition.

**Proposition 2.2.14.** Consider $\varepsilon \geq 0$, $C \in \mathcal{F}_Y$, $x_0 \in \operatorname{dom} f$ and suppose that $\operatorname{int} D^+ \neq \emptyset$ and $f - f(x_0)$ is nearly $(C, \varepsilon)$-subconvexlike on $S$. Then,

$(a)$ $x_0 \in \mathrm{Be}(f,S,C,\varepsilon) \Rightarrow x_0 \in \bigcap_{\delta\geq\varepsilon} \mathrm{Be}(f,S,C,\delta)$.

$(b)$ $x_0 \in \mathrm{Be}(f,S,C,\varepsilon) \iff x_0 \in \mathrm{Be}(f,S,\operatorname{co} C,\varepsilon) \iff x_0 \in \mathrm{Be}(f,S,\operatorname{shw} C,\varepsilon)$.

$(c)$ Let $C' \subset D$. Then $x_0 \in \mathrm{Be}(f,S,C,\varepsilon) \Rightarrow x_0 \in \mathrm{Be}(f,S,C+C',\varepsilon)$.

If additionally $0 \in \operatorname{cl} C'$, then $x_0 \in \mathrm{Be}(f,S,C,\varepsilon) \iff x_0 \in \mathrm{Be}(f,S,C{+}C',\varepsilon)$.

*Proof.* $(a)$ Assume that $x_0 \in \mathrm{Be}(f,S,C,\varepsilon)$ and let $\delta \geq \varepsilon$. By Theorem 2.2.6 there exists $\mu \in D^{s+} \cap C^{\tau+}$ such that $x_0 \in \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_S(\mu\circ f)$. As $(C,\delta) \in \mathcal{F}_{C(\varepsilon)}(\mu)$, by Corollary 2.2.12 we deduce that $x_0 \in \mathrm{Be}(f,S,C,\delta)$.

$(b)$ By Proposition 2.2.3$(a)$ we have that

$$x_0 \in \mathrm{Be}(f,S,\operatorname{co} C,\varepsilon) \cup \mathrm{Be}(f,S,\operatorname{shw} C,\varepsilon) \Rightarrow x_0 \in \mathrm{Be}(f,S,C,\varepsilon).$$

On the other hand, consider $x_0 \in \mathrm{Be}(f,S,C,\varepsilon)$. Then, by Theorem 2.2.6 there exists $\mu \in D^{s+} \cap C^{\tau+}$ such that

$$x_0 \in \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_S(\mu\circ f). \tag{2.13}$$

For each $y^* \in Y^*$, recall that $\tau_C(y^*) = -\sigma_{-C}(y^*)$ (see (1.8)). By using well-known properties of the support function (see, for instance, [120]) we have that

$$\tau_C(y^*) = -\sigma_{-C}(y^*) = -\sigma_{-\operatorname{co} C}(y^*) = \tau_{\operatorname{co} C}(y^*), \quad \forall\, y^* \in Y^*,$$

and

$$\begin{aligned}\tau_C(y^*) \geq \tau_{\operatorname{shw} C}(y^*) &= -\sigma_{-\operatorname{shw} C}(y^*) \geq -\sigma_{-(C+\operatorname{cone} C)}(y^*)\\ &= -\sigma_{-C}(y^*) - \sigma_{-\operatorname{cone} C}(y^*) = \tau_C(y^*), \quad \forall\, y^* \in C^{\tau+}.\end{aligned}$$

Hence, $(\operatorname{co} C,\varepsilon), (\operatorname{shw} C,\varepsilon) \in \mathcal{F}_{C(\varepsilon)}(\mu)$ and from (2.13), by applying Corollary 2.2.12, we deduce that

$$x_0 \in \mathrm{Be}(f,S,C,\varepsilon) \Rightarrow x_0 \in \mathrm{Be}(f,S,\operatorname{co} C,\varepsilon) \cap \mathrm{Be}(f,S,\operatorname{shw} C,\varepsilon),$$

and the proof of part $(b)$ is complete.

$(c)$ The first statement of this part follows by reasoning as in part $(a)$, since $\tau_C(\mu) \leq \tau_{C+C'}(\mu)$, for all $\mu \in D^{s+}$ and for all $C' \subset D$.

Moreover, if $0 \in \operatorname{cl} C'$ then $C(\varepsilon) \subset (C + \operatorname{cl} C')(\varepsilon)$ and by applying Proposition 2.2.3($a$) and ($b$) we have that

$$\operatorname{Be}(f, S, C + C', \varepsilon) = \operatorname{Be}(f, S, C + \operatorname{cl} C', \varepsilon) \subset \operatorname{Be}(f, S, C, \varepsilon),$$

which proves the second statement and the proof of part ($c$) finishes. □

**Remark 2.2.15.** When the hypotheses of Proposition 2.2.14 are true for all $x \in S_0$, by part ($c$) we deduce in particular that

$$\operatorname{Be}(f, S, C, \varepsilon) = \operatorname{Be}(f, S, C + D, \varepsilon)$$

and so we can consider the set $C' := C + D$ instead of $C$, which usually has better properties than $C$. The same conclusion follows by Proposition 2.2.3($e$) when $D$ has a compact base.

We finish the section with the following corollary, in which we prove that Geoffrion, Benson and Henig $\varepsilon$-proper solution notions of problem $(\mathcal{P}_S)$ with respect to $q$ (see Definitions 1.2.16, 1.2.15 and 1.2.17) coincide when $\operatorname{int} D^+ \neq \emptyset$, $S$ is a convex set and $f$ is $D$-convex on $S$.

**Corollary 2.2.16.** Let $\varepsilon \geq 0$ and $q \in D \backslash \{0\}$. Suppose that $\operatorname{int} D^+ \neq \emptyset$, $S$ is convex and $f$ is $D$-convex on $S$. Then,

($a$) $\operatorname{Be}(f, S, q, \varepsilon) = \operatorname{He}(f, S, q, \varepsilon)$.

($b$) If $Y = \mathbb{R}^p$ and $D = \mathbb{R}^p_+$, $\operatorname{Ge}(f, S, q, \varepsilon) = \operatorname{Be}(f, S, q, \varepsilon)$.

*Proof.* ($a$) As $f$ is $D$-convex on $S$, by Remark 1.2.23 it follows that $f - f(x)$ is nearly $(C_q, \varepsilon)$-subconvexlike on $S$ for all $x \in S_0$. Then, by Corollary 2.2.10 and [83, Theorem 3.1, statement (3)] we deduce that

$$\operatorname{Be}(f, S, q, \varepsilon) = \operatorname{Be}(f, S, C_q, \varepsilon) = \bigcup_{\mu \in D^{s+}} \varepsilon\langle \mu, q \rangle\text{-}\operatorname{argmin}_S(\mu \circ f) = \operatorname{He}(f, S, q, \varepsilon). \tag{2.14}$$

($b$) By [76, Theorem 3] we see that

$$\operatorname{Ge}(f, S, q, \varepsilon) = \bigcup_{\mu \in \operatorname{int} \mathbb{R}^p_+} \varepsilon\langle \mu, q \rangle\text{-}\operatorname{argmin}_S(\mu \circ f)$$

and by (2.14) the proof is finished. □

## 2.3 Limit behaviour

In this section we study the limit behaviour of Benson $(C,\varepsilon)$-proper solutions of problem $(\mathcal{P}_S)$ when $\varepsilon$ tends to zero. The main result (see Theorem 2.3.3) is a direct consequence of the limit behaviour when $\varepsilon$ tends to zero of certain outer approximations of the sets $\mathrm{Be}(f,S,C,\varepsilon)$ based on $(C,\varepsilon)$-efficient solutions. Next we justify why this kind of results are necessary.

Let us denote the upper limit of a set-valued mapping $\mathcal{E}:\mathbb{R}_+\to 2^X$ at zero by $\limsup_{\varepsilon\to 0}\mathcal{E}(\varepsilon)$ (see, for instance, [38]), i.e.,

$$x\in\limsup_{\varepsilon\to 0}\mathcal{E}(\varepsilon) \iff \text{there exist nets } \mathbb{R}_+\backslash\{0\}\supset(\varepsilon_i)\to 0 \text{ and}$$
$$X\supset(x_i)\to x \text{ with } x_i\in\mathcal{E}(\varepsilon_i),\ \forall\, i.$$

Let problem $(\mathcal{P}_S)$ be such that $X=Y$ and $f(y)=y$, for all $y\in S$. From the density theorems of Arrow-Barankin-Blackwell's type (see [5, 25, 39] and the references therein) and under certain compactness assumptions on $S$, it follows that

$$\mathrm{E}(f,S)\subset\mathrm{cl}\bigcup_{\mu\in D^{s+}}\mathrm{argmin}_S(\mu\circ f).$$

Thus, by applying Theorem 2.2.8 to $C=D$ and since $\mathrm{Be}(f,S)\subset\mathrm{E}(f,S)$ we deduce that

$$\mathrm{Be}(f,S)\subset\mathrm{E}(f,S)\subset\mathrm{cl}\,\mathrm{Be}(f,S).$$

In other words and roughly speaking, in vector optimization problems, the Benson proper efficient solutions approximate the efficient set. From this point of view, it is natural to ask for sets $C\in\mathcal{H}_Y$ such that the limits of Benson $(C,\varepsilon)$-proper solutions when $\varepsilon$ tends to zero are in the closure of the efficient set. When a set $C$ does not satisfy this limit behaviour, the associated Benson $(C,\varepsilon)$-proper solution concept is not suitable at all. This fact motivates the following notion.

**Definition 2.3.1.** We say that $C\in\mathcal{H}_Y$ is admissible to approximate efficient solutions of problem $(\mathcal{P}_S)$ (admissible for $(\mathcal{P}_S)$ in short form) if

$$\limsup_{\varepsilon\to 0}\mathrm{Be}(f,S,C,\varepsilon)\subset\mathrm{cl}\,\mathrm{E}(f,S).$$

In Theorem 2.3.3$(c)$ we provide admissible sets for any problem $(\mathcal{P}_S)$ whose objective mapping is continuous and $S_0$ is closed. Moreover, in Example 2.3.6 we show that Benson $(C_q, \varepsilon)$-proper solutions of problem $(\mathcal{P}_S)$ are not suitable to approximate efficient solutions.

**Lemma 2.3.2.** Let $C \subset Y$ be nonempty.

$(a)$ If $D \subset \operatorname{cone} C$ then

$$\bigcap_{\varepsilon>0} \mathrm{AE}(f, S, C, \varepsilon) \subset \mathrm{E}(f, S).$$

$(b)$ If $f : S_0 \subset X \to Y$ is continuous, $S_0$ is closed, $C \neq Y$ is solid and coradiant and $D \subset \operatorname{cone}(\operatorname{int} C)$ then

$$\limsup_{\varepsilon\to 0} \mathrm{AE}(f, S, C, \varepsilon) \subset \mathrm{E}(f, S).$$

*Proof.* $(a)$ By the hypothesis it is clear that

$$D\backslash\{0\} \subset \operatorname{cone} C\backslash\{0\} = \bigcup_{\varepsilon>0} C_0(\varepsilon), \tag{2.15}$$

and for each $x_0 \in \bigcap_{\varepsilon>0} \mathrm{AE}(f, S, C, \varepsilon)$ we deduce that

$$(f(S_0) - f(x_0)) \cap (-D\backslash\{0\}) \subset \bigcup_{\varepsilon>0}((f(S_0) - f(x_0)) \cap (-C_0(\varepsilon))) = \emptyset,$$

i.e., $x_0 \in \mathrm{E}(f, S)$.

$(b)$ Let $x_0 \in \limsup_{\varepsilon\to 0} \mathrm{AE}(f, S, C, \varepsilon)$ and suppose that $x_0 \notin \mathrm{E}(f, S)$. As $S_0$ is closed, from the definitions we deduce that there exist two nets $(\varepsilon_i) \subset \mathbb{R}_+\backslash\{0\}$ and $(x_i)$, $x_i \in \mathrm{AE}(f, S, C, \varepsilon_i)$ for all $i$, $\varepsilon_i \to 0$ and $x_i \to x_0$, and a point $x \in S_0$ such that $f(x) - f(x_0) \in -D\backslash\{0\}$.

Since $x_i \to x_0$ and $f$ is continuous, by statement (2.15) with $\operatorname{int} C$ instead of $C$ we can suppose without loss of generality that there exists $\varepsilon_0 > 0$ such that $f(x) - f(x_i) \in -C_0(\varepsilon_0)$ for all $i$, and then $x_i \notin \mathrm{AE}(f, S, C, \varepsilon_0)$ for all $i$.

As $\varepsilon_i \to 0$ there exists $i_0$ such that $\varepsilon_{i_0} < \varepsilon_0$, and a contradiction is obtained, since $x_{i_0} \in \mathrm{AE}(f, S, C, \varepsilon_{i_0})$ and by [49, Theorem 3.4(ii)] we have that $\mathrm{AE}(f, S, C, \varepsilon_{i_0}) \subset \mathrm{AE}(f, S, C, \varepsilon_0)$. ◻

**Theorem 2.3.3.** Consider $C \in \mathcal{H}_Y$. We have that

$(a)$ $\text{Be}(f, S, C, \varepsilon) \subset \text{AE}(f, S, C + D\backslash\{0\}, \varepsilon)$, for all $\varepsilon \geq 0$.

$(b)$ If $D \subset \text{cone}(C + D\backslash\{0\})$ then $\bigcap_{\varepsilon>0} \text{Be}(f, S, C, \varepsilon) \subset \text{E}(f, S)$.

$(c)$ If $f : S_0 \subset X \to Y$ is continuous, $S_0$ is closed, $C+D\backslash\{0\}$ is solid and coradiant and $D \subset \text{cone}(\text{int}(C + D\backslash\{0\}))$ then

$$\limsup_{\varepsilon \to 0} \text{Be}(f, S, C, \varepsilon) \subset \text{E}(f, S)$$

and so $C$ is admissible for problem $(\mathcal{P}_S)$.

*Proof.* $(a)$ For each $\varepsilon \geq 0$ and $x_0 \in \text{Be}(f, S, C, \varepsilon)$ it is obvious that

$$\begin{aligned}&(f(S_0) + C(\varepsilon) - f(x_0)) \cap (-D\backslash\{0\})\\ &\subset \text{cl cone}(f(S_0) - f(x_0) + C(\varepsilon)) \cap (-D\backslash\{0\}) = \emptyset.\end{aligned}$$

Therefore,

$$(f(S_0) - f(x_0)) \cap (-(C(\varepsilon) + D\backslash\{0\})) = \emptyset,$$

and since $(C + D\backslash\{0\})_0(\varepsilon) \subset C(\varepsilon) + D\backslash\{0\}$ we deduce that $x_0 \in \text{AE}(f, S, C + D\backslash\{0\}, \varepsilon)$.

Parts $(b)$ and $(c)$ follow by part $(a)$ and Lemma 2.3.2$(a)$,$(b)$. For part $(c)$, let us observe that $C \in \mathcal{H}_Y$ implies $0 \notin C + D\backslash\{0\}$ and so $C + D\backslash\{0\} \neq Y$. □

**Remark 2.3.4.** By Theorem 2.3.3$(c)$ and roughly speaking, one can obtain efficient solutions of problem $(\mathcal{P}_S)$ through limits of Benson $(C, \varepsilon)$-proper solutions when there exists a set $C \in \mathcal{H}_Y$ such that $C + D\backslash\{0\}$ is solid and coradiant, and $D \subset \text{cone}(\text{int}(C+D\backslash\{0\}))$. This happens, for example, when the objective space $Y$ is normed and the order cone $D$ is based.

Indeed, let $B \subset D\backslash\{0\}$ a base of $D$ and let us denote the unit open ball of $Y$ by $\mathcal{B}$ and the distance from zero to $B$ by $d_B(0)$, i.e.,

$$d_B(0) = \inf_{b \in B} \|b\| > 0.$$

For each $\delta \in (0, d_B(0))$, the nonempty set $C_\delta := \operatorname{shw}(B + \delta\mathcal{B}) \subset Y$ is convex, open, coradiant, $D \subset \operatorname{cone} C_\delta$ and $C_\delta + D\backslash\{0\} = C_\delta$. Moreover, $0 \notin \operatorname{cl} C_\delta$ and by applying the separation theorem (see [38, Theorem 2.2.8]) we deduce that $C_\delta \in \mathcal{F}_Y \subset \mathcal{H}_Y$. Therefore, these sets satisfy the assumptions of Theorem 2.3.3($c$) and so they can be used to approximate efficient solutions of problem $(\mathcal{P}_S)$ via Benson $(C_\delta, \varepsilon)$-proper solutions when $\varepsilon$ tends to zero.

As an interesting application, in the next example we obtain optimality conditions for efficient solutions of unconstrained convex Pareto bi-objective optimization problems.

**Example 2.3.5.** Consider problem $(\mathcal{P}_S)$ with the following data: $X$ is an arbitrary Hausdorff locally convex topological vector space, $S = X$, $f_1, f_2 : X \to \mathbb{R}$ are two convex mappings such that $f_1$ or $f_2$ is continuous at an arbitrary point $x_0 \in X$, $Y = \mathbb{R}^2$ and $D = \mathbb{R}^2_+$.

This problem is usually solved by iterative algorithms based on the well-known Weighting Method (see [88, Part II, Section 3.1]). However, from a numerical point of view, this method does not ensure that the objective values of the attained solutions are near to the efficient objective values. To be precise, by applying this kind of iterative algorithms one generates $m$ solutions $(x_n)_{n=1}^m \subset X$, where $x_n \in \varepsilon\text{-}\operatorname{argmin}_X(\mu_1^n f_1 + \mu_2^n f_2)$, $\mu_1^n, \mu_2^n > 0$, and the values $f(x_n)$ could be too far to the set $f(\mathrm{E}(f, X))$.

For example, if $X = \mathbb{R}^2$, $f_1(z_1, z_2) = z_1^2$, $f_2(z_1, z_2) = z_2^2$ and $\varepsilon > 0$, it is easy to check that $\mathrm{E}(f, X) = \{(0, 0)\}$, $x_n = (\sqrt{n}, 0) \in \varepsilon\text{-}\operatorname{argmin}_X((\varepsilon/(2n))f_1 + (1 - \varepsilon/(2n))f_2)$ and $\|f(x_n) - f(0, 0)\| \to +\infty$ if $n \to +\infty$.

In order to overcome this drawback, next we apply Theorem 2.3.3($c$) by considering the sets $C_\delta$ defined in Remark 2.3.4. Let

$$B = \{(y_1, y_2) \in \mathbb{R}^2_+ : y_1 + y_2 = 1\}$$

and let $\mathcal{B}_1$ be the open unit ball of $\mathbb{R}^2$ given by the $\ell_1$ norm. It is clear that $d_B(0) = 1$ and for all $\delta \in (0, 1)$,

$$C_\delta = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 > 1 - \delta, y_1 + \delta y_2 > 0, \delta y_1 + y_2 > 0\}.$$

Moreover,

$$C_\delta^{\tau+} = \{(\mu_1, \mu_2) \in \mathbb{R}^2_+ : \min\{\mu_1 - \delta\mu_2, -\delta\mu_1 + \mu_2\} \geq 0\} = \operatorname{cone} B'$$

where

$$B' = \{(\mu_1, \mu_2) \in B : \mu_1 \in [\delta/(1+\delta), 1/(1+\delta)]\},$$

and for each $(\mu_1, \mu_2) \in C_\delta^{\tau+}$

$$\tau_{C_\delta}(\mu) = \begin{cases} -\delta\mu_1 + \mu_2 & \text{if} \quad \mu_1 \geq \mu_2 \\ \mu_1 - \delta\mu_2 & \text{if} \quad \mu_1 < \mu_2. \end{cases}$$

Then, for each $(\mu_1, \mu_2) \in B'$ we have

$$\tau_{C_\delta}(\mu) = \begin{cases} 1 - (1+\delta)\mu_1 & \text{if} \quad \mu_1 \geq 1/2 \\ (1+\delta)\mu_1 - \delta & \text{if} \quad \mu_1 < 1/2. \end{cases}$$

By Remark 1.2.23 and Corollary 2.2.10 we see that for each $\varepsilon \geq 0$,

$$\begin{aligned} \operatorname{Be}(f, X, C_\delta, \varepsilon) = & \bigcup_{\delta/(1+\delta) \leq \mu < 1/2} ((1+\delta)\mu\varepsilon - \delta\varepsilon)\text{-argmin}_X(\mu f_1 + (1-\mu) f_2) \\ & \bigcup \bigcup_{1/2 \leq \mu < 1/(1+\delta)} (\varepsilon - (1+\delta)\mu\varepsilon)\text{-argmin}_X(\mu f_1 + (1-\mu) f_2). \end{aligned}$$

By using calculus $\varepsilon$-subdifferential (see [120, Theorem 2.8.7]) we deduce that

$$\begin{aligned} & x \in ((1+\delta)\mu\varepsilon - \delta\varepsilon)\text{-argmin}_X(\mu f_1 + (1-\mu) f_2) \\ \iff & 0 \in \bigcup_{\substack{\beta_1, \beta_2 \geq 0 \\ \beta_1 + \beta_2 = (1+\delta)\mu\varepsilon - \delta\varepsilon}} \mu\partial_{\beta_1/\mu} f_1(x) + (1-\mu)\partial_{\beta_2/(1-\mu)} f_2(x) \end{aligned}$$

and

$$\begin{aligned} & x \in (\varepsilon - (1+\delta)\mu\varepsilon)\text{-argmin}_X(\mu f_1 + (1-\mu) f_2) \\ \iff & 0 \in \bigcup_{\substack{\beta_1, \beta_2 \geq 0 \\ \beta_1 + \beta_2 = \varepsilon - (1+\delta)\mu\varepsilon}} \mu\partial_{\beta_1/\mu} f_1(x) + (1-\mu)\partial_{\beta_2/(1-\mu)} f_2(x). \end{aligned}$$

Therefore, by Theorem 2.3.3($c$) the following optimality condition is obtained: if $(\varepsilon_n) \subset \mathbb{R}_+$ and $(x_n) \subset X$ satisfy $\varepsilon_n \to 0$, $x_n \to x_0$ and for some $\delta \in (0,1)$ and for each $n$

$$0 \in \bigcup_{\substack{\delta/(1+\delta)\le\mu<1/2 \\ \beta_1,\beta_2\ge 0, \beta_1+\beta_2=(1+\delta)\mu\varepsilon_n-\delta\varepsilon_n}} \mu\partial_{\beta_1/\mu} f_1(x_n) + (1-\mu)\partial_{\beta_2/(1-\mu)} f_2(x_n)$$

$$\bigcup \bigcup_{\substack{1/2\le\mu<1/(1+\delta) \\ \beta_1,\beta_2\ge 0, \beta_1+\beta_2=\varepsilon_n-(1+\delta)\mu\varepsilon_n}} \mu\partial_{\beta_1/\mu} f_1(x_n) + (1-\mu)\partial_{\beta_2/(1-\mu)} f_2(x_n)$$

then $x_0 \in \mathrm{E}(f, X)$.

In the next example we show that the notion of Benson $\varepsilon$-proper solution with respect to $q$ is not a suitable approximate proper efficiency concept, since it corresponds with Benson $(C_q, \varepsilon)$-proper solutions and the sets $C_q$ are not admissible for even simple problems. So in view of Corollary 2.2.16$(a)$ we have that the notions of Benson and Henig $\varepsilon$-proper solution introduced respectively by Rong and El Maghri (see Definitions 1.2.15 and 1.2.17 and statement (2.14)) is not appropriate.

**Example 2.3.6.** Let us consider problem $(\mathcal{P}_S)$ with the following data: $X = Y = \mathbb{R}^2$, $f(y_1, y_2) = (y_1, y_2)$, for all $(y_1, y_2) \in \mathbb{R}^2$, $S = D = \mathbb{R}^2_+$. It is obvious that $\mathrm{E}(f, S) = \{(0, 0)\}$.

Let $C \in \mathcal{F}_Y$ be an arbitrary convex set such that $C + \mathbb{R}^2_+ = C$. By applying Remark 1.2.23 we see that $f - f(y)$ is nearly $(C, \varepsilon)$-subconvexlike on $S$, for all $y \in S$ and for all $\varepsilon \ge 0$, and by Corollary 2.2.10 we see that

$$\mathrm{Be}(f, S, C, \varepsilon) = \bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \varepsilon\tau_C(\mu)\text{-argmin}_S(\mu \circ f). \tag{2.16}$$

Consider $q = (q_1, q_2) \in \mathbb{R}^2$ such that $q \notin -\mathbb{R}^2_+$ (i.e., $q_1 > 0$ or $q_2 > 0$), $C_q = q + \mathbb{R}^2_+$ and suppose that $q_1 > 0$. Let $\mu_n := (1, 1/n)$, $n \in \mathbb{N}$, $n \ge 1$. For all $n \ge n_0$ and $n_0$ large enough it is clear that $\tau_{C_q}(\mu_n) > 0$ and

$$\{(0, y) : y \ge 0\} \subset \bigcap_{\varepsilon>0} \bigcup_{n \ge n_0} \varepsilon\tau_{C_q}(\mu_n)\text{-argmin}_{\mathbb{R}^2_+}(\mu_n \circ f) \subset \bigcap_{\varepsilon>0} \mathrm{Be}(f, S, q, \varepsilon). \tag{2.17}$$

Analogously, if $q_2 > 0$, $\eta_n = (1/n, 1)$, $n \in \mathbb{N}$, $n \ge 1$ and $n_0 \in \mathbb{N}$ large enough such that $\tau_{C_q}(\eta_n) > 0$ for all $n \ge n_0$ it follows that

$$\{(y, 0) : y \ge 0\} \subset \bigcap_{\varepsilon>0} \bigcup_{n \ge n_0} \varepsilon\tau_{C_q}(\eta_n)\text{-argmin}_{\mathbb{R}^2_+}(\eta_n \circ f) \subset \bigcap_{\varepsilon>0} \mathrm{Be}(f, S, q, \varepsilon). \tag{2.18}$$

Then, for each $q \in \mathbb{R}^2\backslash(-\mathbb{R}^2_+)$, from (2.17) and (2.18) we see that the limit superior of the sets $\mathrm{Be}(f, S, C_q, \varepsilon)$ when $\varepsilon$ tends to zero is not included in $\mathrm{E}(f, S)$. In particular, let us observe that

$$\sup_{z\in\limsup_{\varepsilon\to 0}\mathrm{Be}(f,S,q,\varepsilon)} \left(\inf_{y\in\mathrm{E}(f,S)} \|y - z\|\right)$$
$$\geq \sup_{z\in\bigcap_{\varepsilon>0}\mathrm{Be}(f,S,q,\varepsilon)} \left(\inf_{y\in\mathrm{E}(f,S)} \|y - z\|\right) = +\infty$$

and one could obtain as limit of Benson $\varepsilon$-proper solutions with respect to $q$ when $\varepsilon$ tends to zero feasible points whose objective value is so far from the efficient set as one wants.

In order to approximate the efficient set of the problem, one could consider, for example, the following admissible set:

$$C = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq 0,\ y_2 \geq 0,\ y_1 + y_2 \geq 1\}.$$

Indeed, it is obvious that $D^{s+} \cap C^{\tau+} = \operatorname{int}\mathbb{R}^2_+$. For each $\mu = (\mu_1, \mu_2) \in \operatorname{int}\mathbb{R}^2_+$ we have that

$$\tau_C(\mu) = \begin{cases} \mu_2 & \text{if} \quad \mu_1 \geq \mu_2 \\ \mu_1 & \text{if} \quad \mu_1 < \mu_2 \end{cases}$$

and it follows that

$$\varepsilon\tau_C(\mu)\text{-argmin}_S(\mu \circ f) = \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2_+ : (\mu_1/\mu_2)y_1 + y_2 \leq \varepsilon\} & \text{if} \quad \mu_1 \geq \mu_2 \\ \{(y_1, y_2) \in \mathbb{R}^2_+ : y_1 + (\mu_2/\mu_1)y_2 \leq \varepsilon\} & \text{if} \quad \mu_1 < \mu_2. \end{cases}$$

Thus, by (2.16) it is easy to check that

$$\mathrm{Be}(f, S, C, \varepsilon) = \{(y_1, y_2) \in \mathbb{R}^2_+ : y_1 + y_2 \leq \varepsilon\}$$

and then

$$\limsup_{\varepsilon\to 0} \mathrm{Be}(f, S, C, \varepsilon) = \{(0, 0)\} = \mathrm{E}(f, S).$$

# Capítulo 3

# Approximate optimality conditions and saddle points

## 3.1 Introduction

It is well-known that saddle point assertions play an important role in scalar optimization due to their relations with other fundamental tools and theories such as Kuhn-Tucker optimality conditions, duality, minimax theory, etc. (see [58]).

Motivated by these facts, different authors have obtained exact and approximate saddle point results for vector optimization problems with single and set-valued mappings by considering efficient solutions, weakly efficient solutions, Benson proper efficient solutions and approximate efficient solutions in the sense of Kutateladze (see [3, 4, 16, 19, 23, 29, 41, 72, 73, 80, 98, 99, 105, 108, 114, 116]).

These saddle point assertions are usually based on generalized convexity assumptions and they are consequences of linear scalarizations that characterize the exact or approximate solutions of the vector optimization problem through solutions of associated scalar optimization problems. In [3, 4, 19, 29, 33, 48, 72, 73, 93, 94, 97, 98, 104, 108, 110, 116] and the references therein, the reader can find some of these scalarization results.

Approximate saddle point theorems that characterize suboptimal solutions of convex scalar optimization problems with inequality and equality constraints have been obtained in [28, 102].

In [29], these results were stated in convex Pareto multiobjective optimization problems with inequality constraints for approximate weak efficient solutions in the sense of Kutateladze via a scalar Lagrangian function. In [41, 114], the results were generalized for single-valued vector optimization problems with equality and cone constraints for approximate and approximate weak efficient solutions in the Kutateladze sense, and approximate efficient solutions in the Vályi sense (see [48, 49, 70, 114]) via a single-valued vector Lagrangian mapping. In all these papers, the complementary slack condition is not bounded (see [114, Remark 3.1]).

In [16, 105] the authors derive approximate saddle-point assertions for vector-valued location and approximation problems using a vector Lagrangian mapping which is affine in each variable.

In the last years, some of these approximate saddle point results for single valued vector optimization problems have been extended to vector optimization problems with set-valued mappings. For example, in [98] the authors consider a subconvexlike vector optimization problem with set-valued maps in real locally convex Hausdorff topological vector spaces and derive scalarization results and sufficient approximate saddle point assertions for approximate weak solutions in the Kutateladze sense. As in the previous references, the complementary slack condition is not bounded.

This chapter focuses on saddle point theorems for Benson $(C, \varepsilon)$-proper solutions of $(\mathcal{P}_S)$ where $S$ is given as in (1.1), $K$ is a solid, convex cone, $K \neq Z$ and $f : X \to Y$ (i.e., $f$ is not extended). In these results, we consider nearly subconvexlikeness assumptions on the objective and the cone constraint mappings.

In Section 3.2 we state necessary and sufficient scalar Lagrangian optimality conditions for Benson $(C, \varepsilon)$-proper solutions of $(\mathcal{P}_S)$. These optimality conditions will be useful in order to obtain saddle point results related to this type of solutions.

In Section 3.3 we introduce a new set-valued Lagrangian and we define the new concept of $(C, \varepsilon)$-proper saddle point for this Lagrangian. Next, we obtain necessary and sufficient conditions for Benson $(C, \varepsilon)$-proper solutions of $(\mathcal{P}_S)$

through $(C, \varepsilon)$-proper saddle points.

One of the main properties of these approximate proper saddle points is that, for suitable sets $C$, the complementary slack condition is bounded.

Moreover, the obtained results reduce to well-known proper saddle point theorems for exact solutions, and some of them are stated under weaker hypotheses.

The results of this chapter are collected in [44, Sections 3 and 4].

## 3.2 Lagrangian optimality conditions for Benson $(C, \varepsilon)$-proper solutions

Next, we obtain scalar Lagrangian optimality conditions for Benson $(C, \varepsilon)$-proper solutions of problem $(\mathcal{P}_S)$. For this aim, we need the following Lemma.

**Lemma 3.2.1.** Let $x_0 \in S$, $\varepsilon \geq 0$ and $C \in \mathcal{F}_Y$. Suppose that $\operatorname{int} D^+ \neq \emptyset$ and $f - f(x_0)$ is nearly $(C, \varepsilon)$-subconvexlike on $S$. If $x_0 \in \operatorname{Be}(f, S, C, \varepsilon)$, then there exists $\mu \in D^{s+} \cap C^{\tau+}$ such that

$$\operatorname{cl}\operatorname{cone}((f, g)(X) + (C(\varepsilon) \times K) - (f(x_0), 0)) \cap (-H \times -\operatorname{int} K) = \emptyset, \tag{3.1}$$

where $H$ is the open half space defined by $H = \{y \in Y : \langle \mu, y \rangle > 0\}$.

*Proof.* By 2.2.4 there exists $\mu \in D^{s+} \cap C^{\tau+}$ such that

$$\operatorname{cl}\operatorname{cone}(f(S_0) + C(\varepsilon) - f(x_0)) \cap (-H) = \emptyset. \tag{3.2}$$

Then, statement (3.1) is satisfied through this open half space $H$. Indeed, suppose on the contrary that there exist $(z_1, z_2) \in -H \times -\operatorname{int} K$ and nets $(\alpha_i) \subset \mathbb{R}_+$, $(x_i) \subset X$, $(d_i) \subset C(\varepsilon)$ and $(k_i) \subset K$ such that

$$\begin{aligned} \alpha_i(f(x_i) + d_i - f(x_0)) &\to z_1, \\ \alpha_i(g(x_i) + k_i) &\to z_2. \end{aligned} \tag{3.3}$$

As $K \neq Y$, it follows that $0 \notin \operatorname{int} K$ and so we can assume that $\alpha_i \neq 0$. Since $z_2 \in -\operatorname{int} K$ we can suppose that $\alpha_i(g(x_i) + k_i) \in -\operatorname{int} K$. As $\alpha_i \neq 0$, we deduce that

$$g(x_i) \in -k_i - (1/\alpha_i) \operatorname{int} K \subset -K.$$

Hence, $(x_i)$ is a net of feasible points. By (3.3) it follows that $z_1 \in \operatorname{cl}\operatorname{cone}(f(S)+C(\varepsilon)-f(x_0))\cap(-H)$, which contradicts statement (3.2) and the proof is complete.

□

The necessary condition given in Theorem 3.2.2 will be used in Section 3.3 to prove saddle point results for Benson $(C,\varepsilon)$-proper solutions of problem $(\mathcal{P}_S)$.

**Theorem 3.2.2.** Let $x_0 \in S$, $\varepsilon \geq 0$ and $C \in \mathcal{F}_Y$. Suppose that $\operatorname{int} D^+ \neq \emptyset$, $f - f(x_0)$ is nearly $(C,\varepsilon)$-subconvexlike on $S$, $(f - f(x_0), g)$ is nearly $(C \times K, \varepsilon)$-subconvexlike on $X$ and the Slater constraint qualification holds. If $x_0 \in \operatorname{Be}(f, S, C, \varepsilon)$, then there exist $\mu \in D^{s+} \cap C^{\tau+}$ and $\lambda \in K^+$ such that the following two conditions are satisfied:

$(a)$ $x_0 \in \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_X(\mu \circ f + \lambda \circ g)$,

$(b)$ $-\varepsilon\tau_C(\mu) \leq (\lambda \circ g)(x_0) \leq 0$.

Moreover, we can suppose that $\tau_C(\mu) = 1$ if one of the following additional conditions is satisfied:
(A1) $C = B + Q$, where $B \subset D\backslash\{0\}$ is compact and $Q \subset D$,
(A2) $Y$ is normed, $C \subset D\backslash\{0\}$ and $0 \notin \operatorname{cl} C$.

*Proof.* By Lemma 3.2.1 it follows that there exists $\xi \in D^{s+} \cap C^{\tau+}$ such that the open half space $H = \{y \in Y : \langle \xi, y\rangle > 0\}$ satisfies

$$\operatorname{cl}\operatorname{cone}((f,g)(X) + (C(\varepsilon) \times K) - (f(x_0), 0)) \cap (-H \times -\operatorname{int} K) = \emptyset.$$

Since $(f - f(x_0), g)$ is nearly $(C \times K, \varepsilon)$-subconvexlike on $X$, by Eidelheit's separation theorem (see for instance [64, Theorem 3.16 ]), we deduce that there exists a functional $(\mu, \lambda) \in (Y^* \times Z^*)\backslash\{(0,0)\}$ such that

$$\langle \mu, f(x) + d - f(x_0)\rangle + \langle \lambda, g(x) + k\rangle \geq 0, \quad \forall\, x \in X, d \in C(\varepsilon), k \in K, \tag{3.4}$$

$$\langle \mu, h\rangle + \langle \lambda, k\rangle > 0, \quad \forall\, h \in H, k \in \operatorname{int} K. \tag{3.5}$$

By (3.5) it is clear that $\mu \in H^+$ and $\lambda \in K^+$. Suppose that $\mu = 0$. Then, $\lambda \neq 0$ and by (3.4), we obtain that

$$\langle \lambda, g(x)\rangle \geq 0, \quad \forall\, x \in X. \tag{3.6}$$

As Slater constraint qualification holds, there exists $\bar{x} \in X$ such that $g(\bar{x}) \in -\operatorname{int} K$. As $\lambda \neq 0$ we have that $\langle \lambda, g(\bar{x}) \rangle < 0$, which is a contradiction with (3.6). Hence, $\mu \neq 0$. On the other hand, since $\mu \in H^+$ and $D \backslash \{0\} \subset \operatorname{int} H$, then $\langle \mu, d \rangle > 0$, for all $d \in D \backslash \{0\}$, i.e., $\mu \in D^{s+}$. Moreover, as $\langle \lambda, g(x_0) \rangle \leq 0$, considering $x = x_0$ and $k = 0$ in (3.4) we deduce that $\langle \mu, d \rangle \geq 0$, for all $d \in C(\varepsilon)$, and then $\mu \in C^{\tau+}$.

By taking $k = 0$ in (3.4), it is clear that

$$\langle \mu, f(x) \rangle + \inf_{d \in C(\varepsilon)} \{\langle \mu, d \rangle\} - \langle \mu, f(x_0) \rangle + \langle \lambda, g(x) \rangle \geq 0, \quad \forall\, x \in X. \tag{3.7}$$

As $\langle \lambda, g(x_0) \rangle \leq 0$, it follows that

$$(\mu \circ f)(x) + (\lambda \circ g)(x) \geq (\mu \circ f)(x_0) + (\lambda \circ g)(x_0) - \varepsilon \tau_C(\mu), \quad \forall\, x \in X$$

and part $(a)$ is proved.

By taking $x = x_0$ in (3.7), we deduce that $-\varepsilon \tau_C(\mu) \leq (\lambda \circ g)(x_0) \leq 0$ and the proof of part $(b)$ is finished.

Finally, the last statement of the theorem follows by taking $\mu / \tau_C(\mu)$ and $\lambda / \tau_C(\mu)$ instead of $\mu$ and $\lambda$, respectively, and the proof is complete if we check that $\tau_C(\mu) > 0$. Indeed, if (A1) is true, then we have

$$\tau_C(\mu) = \inf\{\langle \mu, d_1 + d_2 \rangle : d_1 \in B, d_2 \in Q\} \geq \inf\{\langle \mu, d_1 \rangle : d_1 \in B\} > 0,$$

since $\mu \in D^{s+}$, $Q \subset D$ and $B \subset D \backslash \{0\}$ is compact.

Suppose that (A2) is true. As $0 \notin \operatorname{cl} C$ there exists $\delta > 0$ such that $\|d\| \geq \delta$ for all $d \in C$. By [64, Lemma 3.21$(d)$] we see that $D^{s+} \subset \operatorname{int} D^+$ and by (2.11) we deduce that

$$\tau_C(\mu) \geq \delta \inf\{\|\xi - \mu\| : \xi \in Y^* \backslash D^+\} > 0,$$

since $\mu \in \operatorname{int} D^+$ and the proof finishes. □

In the next theorem we give a reciprocal result of Theorem 3.2.2.

**Theorem 3.2.3.** Let $x_0 \in S$, $\varepsilon \geq 0$ and $C \in \mathcal{F}_Y$. If there exist $\mu \in D^{s+} \cap C^{\tau+}$ and $\lambda \in K^+$ such that $x_0 \in \varepsilon \tau_C(\mu)$-$\operatorname{argmin}_X(\mu \circ f + \lambda \circ g)$ and $-\varepsilon \tau_C(\mu) \leq (\lambda \circ g)(x_0)$, then $x_0 \in \operatorname{Be}(f, S, C, \bar{\varepsilon})$, where $\bar{\varepsilon} = \varepsilon$ if $(\lambda \circ g)(x_0) = 0$ and $\bar{\varepsilon} = 2\varepsilon$ if $(\lambda \circ g)(x_0) \neq 0$.

*Proof.* By hypothesis, we have that

$$\begin{aligned}(\mu \circ f)(x) &\geq (\mu \circ f)(x) + (\lambda \circ g)(x) \\ &\geq (\mu \circ f)(x_0) + (\lambda \circ g)(x_0) - \varepsilon\tau_C(\mu) \\ &\geq (\mu \circ f)(x_0) - \bar{\varepsilon}\tau_C(\mu), \quad \forall\, x \in S,\end{aligned}$$

since $-\varepsilon\tau_C(\mu) \leq (\lambda \circ g)(x_0)$. Then, $x_0 \in \bar{\varepsilon}\tau_C(\mu)\text{-argmin}_S(\mu \circ f)$ and by Theorem 2.2.8, $x_0 \in \mathrm{Be}(f, S, C, \bar{\varepsilon})$. □

By applying Theorems 3.2.2 and 3.2.3 to $C = D$ and $\varepsilon = 0$ we obtain the following characterization of Benson proper solutions of problem $(\mathcal{P}_S)$ through solutions of an associated scalar Lagrangian optimization problem, which was stated in [99, Corollary 4.1].

**Theorem 3.2.4.** Suppose that $x_0 \in S$, $\operatorname{int} D^+ \neq \emptyset$, $f - f(x_0)$ is nearly $D$-subconvexlike on $S$, $(f - f(x_0), g)$ is nearly $(D \times K)$-subconvexlike on $X$ and the Slater constraint qualification holds. Then $x_0 \in \mathrm{Be}(f, S)$ if and only if there exist $\mu \in D^{s+}$ and $\lambda \in K^+$ such that $x_0 \in \operatorname{argmin}_X(\mu \circ f + \lambda \circ g)$ and $(\lambda \circ g)(x_0) = 0$.

## 3.3 $(C, \varepsilon)$-proper saddle points

In this section, we introduce a new set-valued Lagrangian and we define a new concept of $(C, \varepsilon)$-proper saddle point for this Lagrangian. Then, we obtain necessary and sufficient conditions for Benson $(C, \varepsilon)$-proper solutions of $(\mathcal{P}_S)$ through $(C, \varepsilon)$-proper saddle points.

**Definition 3.3.1.** Consider a nonempty set $B \subset D\backslash\{0\}$. The function $\Phi_B : X \times K^+ \to 2^Y$ defined by

$$\Phi_B(x, \lambda) = f(x) + \langle \lambda, g(x) \rangle B, \quad \forall\, x \in X, \forall\, \lambda \in K^+,$$

is called $B$-Lagrangian associated with problem $(\mathcal{P}_S)$.

**Remark 3.3.2.** Several authors have studied vector Lagrangian mappings $\mathcal{L} : X \times \mathcal{L}_+(Z, Y) \to Y$, where

$$\mathcal{L}_+(Z, Y) = \{T \in \mathcal{L}(Z, Y) : T(K) \subset D\} \tag{3.8}$$

and

$$\mathcal{L}(x, T) = f(x) + T(g(x)), \quad \forall\, x \in X, \forall\, T \in \mathcal{L}_+(Z, Y),$$

that turn problem $(\mathcal{P}_S)$ into an unconstrained vector optimization problem. The functional $T$ is usually defined as follows:

$$T(z) = \langle \lambda, z \rangle q, \quad \forall\, z \in Z,$$

where $\lambda \in K^+$ and $q \in D \backslash \{0\}$. So the following vector Lagrangian mapping $\mathcal{L}_q : X \times K^+ \to Y$ is obtained:

$$\mathcal{L}_q(x, \lambda) = f(x) + \langle \lambda, g(x) \rangle q, \quad \forall\, x \in X, \forall\, \lambda \in K^+. \tag{3.9}$$

Then, the set-valued $B$-Lagrangian of Definition 3.3.1 reduces to $\mathcal{L}_q$ by considering the singleton $B = \{q\}$. On the other hand, by using the set-valued $B$-Lagrangian we can obtain stronger saddle point conditions than the usual ones.

**Definition 3.3.3.** Let $\varepsilon \geq 0$ and $C \in \mathcal{H}_Y$. We say that $(x_0, \lambda_0) \in X \times K^+$ is a $(C, \varepsilon)$-proper saddle point for the set-valued $B$-Lagrangian associated with problem $(\mathcal{P}_S)$ if the following two conditions are satisfied:

$(a)$ $\operatorname{cl}\operatorname{cone}(\Phi_B(X, \lambda_0) + C(\varepsilon) - \Phi_B(x_0, \lambda_0)) \cap (-D) = \{0\}$,

$(b)$ $\Phi_B(x_0, \lambda_0) \subset \operatorname{Max}(\Phi_B(x_0, K^+), C, \varepsilon)$.

**Remark 3.3.4.** $(a)$ Let us observe that condition $(a)$ of Definition 3.3.3 generalizes (2.1) from a vector-valued mapping $f$ to a set-valued mapping $\Phi_B$, so we denote it by $x_0 \in \operatorname{Be}(\Phi_B(\cdot, \lambda_0), X, C, \varepsilon)$, and we say that $\operatorname{Be}(\Phi_B(\cdot, \lambda_0), X, C, \varepsilon)$ is the set of Benson $(C, \varepsilon)$-proper solutions for the set-valued mapping $\Phi_B(\cdot, \lambda_0)$ on $X$.

If $C = D$ or $\varepsilon = 0$ and $C(0) = D$, we use the notation $x_0 \in \operatorname{Be}(\Phi_B(\cdot, \lambda_0), X)$. In both cases we say that $x_0$ is a proper saddle point for the $B$-Lagrangian associated with problem $(\mathcal{P}_S)$.

$(b)$ In the literature, the saddle point concepts associated with vector Lagrangian mappings are usually based on (non necessarily proper) efficient solutions of the related Lagrangian vector optimization problem. However, the saddle point concept of Definition 3.3.3 considers in part $(a)$ a kind of proper minimal point of the set-valued Lagrangian $\Phi_B$. Due to this fact we can obtain necessary optimality conditions for approximate or exact Benson proper solutions of problem $(\mathcal{P}_S)$ via saddle point assertions stronger than the usual ones.

This approach is motivated by a proper saddle point notion introduced by Li [72, Definition 6.2] in the setting of constrained vector optimization with set-valued maps (see also the concept of supper saddle point by Mehra [87, Definition 6.1]). In this sense, let us observe that Definition 3.3.3 reduces to the single valued version of the proper saddle point notion due to Li (see [72, Definition 6.2]) by considering $\varepsilon = 0$ and $C = D$ (see Remark 3.3.8).

The following theorem shows a sufficient condition for the elements of the set $\mathrm{Be}(\Phi_B(\cdot,\lambda_0), X, C, \varepsilon)$ based on suboptimal solutions of associated scalar optimization problems.

**Theorem 3.3.5.** Let $\varepsilon \geq 0$ and $C \in \mathcal{F}_Y$. If there exist $\mu \in D^{s+} \cap C^{\tau+}$ and $\lambda \in K^+$ such that $x_0 \in \varepsilon\tau_C(\mu)\text{-}\mathrm{argmin}_X(\mu \circ f + \lambda \circ g)$, then $x_0 \in \mathrm{Be}(\Phi_{B_\mu}(\cdot,\lambda), X, C, \varepsilon)$.

*Proof.* Suppose on the contrary that $x_0 \notin \mathrm{Be}(\Phi_{B_\mu}(\cdot,\lambda), X, C, \varepsilon)$. Then

$$\mathrm{cl\,cone}(\Phi_{B_\mu}(X,\lambda) + C(\varepsilon) - \Phi_{B_\mu}(x_0,\lambda)) \cap (-D\backslash\{0\}) \neq \emptyset$$

and there exist $w \in -D\backslash\{0\}$ and nets $(\alpha_i) \subset \mathbb{R}_+$, $(y_i) \subset \Phi_{B_\mu}(X,\lambda)$, $(d_i) \subset C(\varepsilon)$ and $(z_i) \subset \Phi_{B_\mu}(x_0,\lambda)$ such that $\alpha_i(y_i + d_i - z_i) \to w$.

For each $i$, there exist $x_i \in X$ and $q_i, p_i \in B_\mu$ with

$$\begin{aligned} y_i &= f(x_i) + \langle \lambda, g(x_i)\rangle q_i, \\ z_i &= f(x_0) + \langle \lambda, g(x_0)\rangle p_i, \end{aligned}$$

and as $x_0 \in \varepsilon\tau_C(\mu)\text{-}\mathrm{argmin}_X(\mu \circ f + \lambda \circ g)$ we deduce that

$$\begin{aligned} &\langle \mu, \alpha_i(f(x_i) + \langle \lambda, g(x_i)\rangle q_i + d_i - f(x_0) - \langle \lambda, g(x_0)\rangle p_i)\rangle \geq \\ &\alpha_i((\mu \circ f + \lambda \circ g)(x_i) + \varepsilon\tau_C(\mu) - (\mu \circ f + \lambda \circ g)(x_0)) \geq 0. \qquad (3.10) \end{aligned}$$

Thus, taking the limit in (3.10) it follows that $\langle \mu, w\rangle \geq 0$. But, on the other hand, $w \in -D \setminus \{0\}$ and, since $\mu \in D^{s+}$, we deduce that $\langle \mu, w\rangle < 0$, obtaining a contradiction. □

As a corollary, we obtain a necessary condition for Benson $(C,\varepsilon)$-proper solutions of problem $(\mathcal{P}_S)$ via Benson $(C,\varepsilon)$-proper solutions for unconstrained $B$-Lagrangians associated with $(\mathcal{P}_S)$.

**Corollary 3.3.6.** Consider $x_0 \in S$, $\varepsilon \geq 0$ and $C \in \mathcal{F}_Y$. Suppose that $\operatorname{int} D^+ \neq \emptyset$, $f - f(x_0)$ is nearly $(C,\varepsilon)$-subconvexlike on $S$, $(f - f(x_0), g)$ is nearly $(C \times K, \varepsilon)$-subconvexlike on $X$ and the Slater constraint qualification holds. If $x_0 \in \mathrm{Be}(f,S,C,\varepsilon)$, then there exist $\mu \in D^{s+} \cap C^{\tau+}$ and $\lambda \in K^+$ such that $x_0 \in \mathrm{Be}(\Phi_{B_\mu}(\cdot,\lambda),X,C,\varepsilon)$.

*Proof.* By Theorem 3.2.2 there exist $\mu \in D^{s+} \cap C^{\tau+}$ and $\lambda \in K^+$ such that $x_0 \in \varepsilon\tau_C(\mu)\text{-argmin}_X(\mu \circ f + \lambda \circ g)$. By Theorem 3.3.5 we deduce that $x_0 \in \mathrm{Be}(\Phi_{B_\mu}(\cdot,\lambda),X,C,\varepsilon)$, and the proof is complete. □

In the following result, we obtain a characterization of condition $(b)$ in Definition 3.3.3 in terms of the feasibility of the point $x_0$ and an approximate complementary slack condition.

**Lemma 3.3.7.** Suppose that $K$ is closed. Consider $\varepsilon \geq 0$, $x_0 \in X$, $\lambda_0 \in K^+$ and $B \subset D\backslash\{0\}$, $C \subset Y$ such that $\operatorname{cone} B \cap \operatorname{cone} C \neq \{0\}$ and $C + \operatorname{cone} B = C$. Then $\Phi_B(x_0,\lambda_0) \subset \mathrm{Max}(\Phi_B(x_0,K^+),C,\varepsilon)$ if and only if $g(x_0) \in -K$ and $\langle \lambda_0, g(x_0)\rangle B \subset -(D \cap C_0(\varepsilon)^c)$.

*Proof.* It is clear that $\Phi_B(x_0,\lambda_0) \subset \mathrm{Max}(\Phi_B(x_0,K^+),C,\varepsilon)$ if and only if

$$\langle \lambda, g(x_0)\rangle b - \langle \lambda_0, g(x_0)\rangle b_0 \notin C_0(\varepsilon), \quad \forall\, \lambda \in K^+, \forall\, b, b_0 \in B. \tag{3.11}$$

In particular, by taking $b = b_0$ it follows that

$$\langle \lambda - \lambda_0, g(x_0)\rangle b_0 \notin C_0(\varepsilon), \quad \forall\, \lambda \in K^+, \forall\, b_0 \in B \tag{3.12}$$

and by considering $\lambda = 0$ we deduce that

$$\langle \lambda_0, g(x_0)\rangle B \cap (-C_0(\varepsilon)) = \emptyset. \tag{3.13}$$

Let us suppose that $g(x_0) \notin -K$. By applying a standard separation argument (see for instance [64, Theorem 3.18]) we deduce that there exists $\bar{\lambda} \in K^+$ such that $\langle\bar{\lambda}, g(x_0)\rangle > 0$. As $\operatorname{cone} B \cap \operatorname{cone} C \neq \{0\}$ there exists $\bar{b} \in B$ and $\alpha > 0$ such that $\alpha\bar{b} \in C_0(\varepsilon)$. Consider the functional $\lambda' := \lambda_0 + (\alpha/\langle\bar{\lambda}, g(x_0)\rangle)\bar{\lambda}$. It is clear that $\lambda' \in K^+$ and

$$\langle\lambda' - \lambda_0, g(x_0)\rangle\bar{b} = \alpha\bar{b} \in C_0(\varepsilon),$$

contrary to (3.12). Then $g(x_0) \in -K$ and so

$$\langle\lambda_0, g(x_0)\rangle B \subset -D, \tag{3.14}$$

since $\lambda_0 \in K^+$ and $B \subset D$. By (3.13) and (3.14) we have that

$$\langle\lambda_0, g(x_0)\rangle B \subset -(D \cap C_0(\varepsilon)^c). \tag{3.15}$$

Reciprocally, suppose that $g(x_0) \in -K$ and (3.15) is true. Then

$$\langle\lambda, g(x_0)\rangle b \in -\operatorname{cone} B, \quad \forall\, \lambda \in K^+, \forall\, b \in B \tag{3.16}$$

and

$$-\langle\lambda_0, g(x_0)\rangle b_0 \notin C_0(\varepsilon), \quad \forall\, b_0 \in B. \tag{3.17}$$

If there exist $\lambda \in K^+$, $b$, $b_0 \in B$ such that

$$\langle\lambda, g(x_0)\rangle b - \langle\lambda_0, g(x_0)\rangle b_0 \in C_0(\varepsilon)$$

then by (3.16) we see that

$$\begin{aligned} -\langle\lambda_0, g(x_0)\rangle b_0 &= \langle\lambda, g(x_0)\rangle b - \langle\lambda_0, g(x_0)\rangle b_0 - \langle\lambda, g(x_0)\rangle b \\ &\in C_0(\varepsilon) + \operatorname{cone} B = C_0(\varepsilon) \end{aligned}$$

contrary to (3.17). Therefore statement (3.11) is true and we have $\Phi_B(x_0, \lambda_0) \subset \operatorname{Max}(\Phi_B(x_0, K^+), C, \varepsilon)$. □

**Remark 3.3.8.** $(a)$ Let us observe that the closedness of the cone $K$ is not required to prove the sufficient condition of Lemma 3.3.7.

$(b)$ When $\varepsilon = 0$ and $D \subset \operatorname{cone} C$, statement $\langle\lambda_0, g(x_0)\rangle B \subset -(D \cap C_0(\varepsilon)^c)$ of Lemma 3.3.7 reduces to the well-known complementary slack condition. Indeed,

$$D \cap C_0(0)^c = D \cap (D\backslash\{0\})^c = \{0\}$$

and so

$$\langle \lambda_0, g(x_0)\rangle B \subset -(D \cap C_0(0)^c) \iff \langle \lambda_0, g(x_0)\rangle = 0,$$

since $0 \notin B$.

($c$) By Lemma 3.3.7 and [72, Proposition 6.1] we deduce that the concept of $(C,\varepsilon)$-proper saddle point for the $B$-Lagrangian associated with problem $(\mathcal{P}_S)$ reduces to the single valued version of the proper saddle point notion due to Li (see [72, Definition 6.2]) by considering $\varepsilon = 0$ and $C = D$.

($d$) Let us observe that Lemma 3.3.7 essentially reduces to [114, Proposition 3.1] by considering a singleton $B = \{q\}$, $e \in D\backslash\{0\}$ and $C = e + D$. In this case, the complementary slack condition for $\varepsilon > 0$ ensures that

$$\langle \lambda_0, g(x_0)\rangle q \in -D \cap (-\varepsilon e - D)^c$$

and the set $-D \cap (-\varepsilon e - D)^c$ is not bounded (see [114, Remark 3.1]). In general, all approximate saddle point concepts in the literature associated with vector optimization problems give unbounded complementary slack conditions (see, for instance [29, 41, 98]).

However, the set $D \cap C_0(\varepsilon)^c$ can be bounded if we consider a suitable set $C$. Thus, the notion of $B$-Lagrangian and the $(C,\varepsilon)$-proper saddle point concept introduced in Definition 3.3.3 overcome this drawback. For example, if $Y$ is normed and the norm $\| \|$ is $D$-monotone on $D$ (i.e., $0 \leq_D d_1 \leq_D d_2 \Rightarrow \|d_1\| \leq \|d_2\|$) then $B = \mathcal{B} \cap (D\backslash\{0\})$, where $\mathcal{B} \subset Y$ denotes the unit open ball, and $C = \mathcal{B}^c \cap D$ satisfy $\operatorname{cone} B = \operatorname{cone} C = D$, $C + \operatorname{cone} B = C$ and $D \cap C_0(\varepsilon)^c = \varepsilon\mathcal{B} \cap D$ is bounded.

($e$) With respect to the assumptions of Lemma 3.3.7, let us observe that condition $C + \operatorname{cone} B = C$ does not imply that $\operatorname{cone} B \cap \operatorname{cone} C \neq \{0\}$. Indeed, consider for example $Y = \mathbb{R}^2$, $D = \mathbb{R}^2_+$, $C = (1,1) + \mathbb{R}^2_+$ and $B = \{(1,0)\}$. It is clear that $\operatorname{cone} B = \{(\alpha, 0) : \alpha \geq 0\}$ and $\operatorname{cone} C = \operatorname{int} \mathbb{R}^2_+ \cup \{0\}$. Thus, $C + \operatorname{cone} B = C$ but $\operatorname{cone} B \cap \operatorname{cone} C = \{0\}$.

However, if $\operatorname{cone} C$ is closed then

$$\begin{aligned} C + \operatorname{cone} B = C &\Rightarrow \operatorname{cone} B \subset \operatorname{cone} C \qquad (3.18)\\ &\Rightarrow \operatorname{cone} B \cap \operatorname{cone} C = \operatorname{cone} B \neq \{0\}. \end{aligned}$$

Let us check (3.18). Fix a point $q \in C$ and consider $\alpha > 0$ and an arbitrary element $b \in B$. Then

$$\begin{aligned}\alpha b &= \lim_{n\to\infty}((1/n)q + \alpha b) = \lim_{n\to\infty}(1/n)(q + (n\alpha)b)\\ &\in \operatorname{cl}\operatorname{cone}(C + \operatorname{cone} B) = \operatorname{cone} C\end{aligned}$$

and we have that $\operatorname{cone} B \subset \operatorname{cone} C$.

**Lemma 3.3.9.** Let $\mu \in D^{s+}$, $\alpha_0 \geq 0$, $\varepsilon \geq 0$ and consider $C_\mu = B_\mu + (D\backslash\{0\})$. Then

$$\alpha_0 B_\mu \subset (C_\mu)_0(\varepsilon)^c \iff \alpha_0 \leq \varepsilon. \tag{3.19}$$

*Proof.* It is easy to check that $(C_\mu)_0(0) = D\backslash\{0\}$. Then

$$\alpha_0 B_\mu \subset (C_\mu)_0(0)^c \iff \alpha_0 B_\mu \subset D \cap (D\backslash\{0\})^c = \{0\} \iff \alpha_0 = 0$$

and relation (3.19) is true if $\varepsilon = 0$.

Consider that $\varepsilon \neq 0$. It is clear that $(C_\mu)_0(\varepsilon) = C_\mu(\varepsilon)$. Moreover, without loss of generality we can suppose that $\varepsilon = 1$ since $C_\mu(\varepsilon)^c = \varepsilon C_\mu^c$. Assume that $\alpha_0 B_\mu \subset C_\mu^c$. If $\alpha_0 > 1$ and $b \in B_\mu$ is arbitrary we have

$$\alpha_0 b = b + (\alpha_0 - 1)b \in B_\mu + (D\backslash\{0\}) = C_\mu,$$

that is contrary to $\alpha_0 B_\mu \subset C_\mu^c$. Thus, $\alpha \leq 1$ and the necessary condition is true. Reciprocally, consider $\alpha_0 \leq 1$ and suppose that there exists $b \in B_\mu$ such that $\alpha_0 b \in C_\mu$. Then $\langle \mu, \alpha_0 b\rangle = \alpha_0\langle \mu, b\rangle = \alpha_0 > 1$, since $\langle \mu, d\rangle > 1$ for all $d \in C_\mu$, and a contradiction is obtained. Thus, the sufficient condition holds and the proof is complete. □

Next we obtain a necessary condition for Benson $(C, \varepsilon)$-proper solutions of $(\mathcal{P}_S)$ in terms of $(C_\mu, \varepsilon)$-proper saddle points of $B_\mu$-Lagrangians, $\mu \in D^{s+} \cap C^{\tau+}$.

**Theorem 3.3.10.** Consider $x_0 \in S$, $\varepsilon \geq 0$ and $C \in \mathcal{F}_Y$. Suppose that $\operatorname{int} D^+ \neq \emptyset$, $f - f(x_0)$ is nearly $(C, \varepsilon)$-subconvexlike on $S$, $(f - f(x_0), g)$ is nearly $(C \times K, \varepsilon)$-subconvexlike on $X$ and the Slater constraint qualification holds. If $x_0 \in \operatorname{Be}(f, S, C, \varepsilon)$ then there exist $\mu \in D^{s+} \cap C^{\tau+}$ and $\lambda_0 \in K^+$ such that $(x_0, \lambda_0)$ is a $(C_\mu, \varepsilon)$-proper saddle point for the $B_\mu$-Lagrangian associated with problem $(\mathcal{P}_S)$.

*Proof.* By Theorem 3.2.2 we deduce that there exist $\mu \in D^{s+} \cap C^{\tau+}$ and $\lambda_0 \in K^+$ such that

$$\begin{aligned} x_0 \in \varepsilon\tau_C(\mu)\text{-argmin}_X(\mu \circ f + \lambda_0 \circ g), \\ -\varepsilon\tau_C(\mu) \leq (\lambda_0 \circ g)(x_0). \end{aligned}$$

If $\tau_C(\mu) > 0$, then replacing $\mu$ by $\mu/\tau_C(\mu) \in D^{s+} \cap C^{\tau+}$ and $\lambda_0$ by $\lambda_0/\tau_C(\mu) \in K^+$, it follows that

$$x_0 \in \varepsilon\text{-argmin}_X(\mu \circ f + \lambda_0 \circ g), \tag{3.20}$$

$$-\varepsilon \leq (\lambda_0 \circ g)(x_0). \tag{3.21}$$

Thus, by (3.20) and Theorem 3.3.5 we have that

$$x_0 \in \text{Be}(\Phi_{B_\mu}(\cdot, \lambda_0), X, C_\mu, \varepsilon), \tag{3.22}$$

since

$$\begin{aligned} \tau_{C_\mu}(\mu) &= \inf\{\langle \mu, q + d\rangle : q \in B_\mu, d \in D\backslash\{0\}\} \\ &= 1 + \inf\{\langle \mu, d\rangle : d \in D\backslash\{0\}\} = 1. \end{aligned}$$

On the other hand, it is obvious that $\langle \lambda_0, g(x_0)\rangle B_\mu \subset -D$, since $x_0$ is feasible, $\lambda_0 \in K^+$ and $B_\mu \subset D$. By (3.19) and (3.21) we see that $\langle \lambda_0, g(x_0)\rangle B_\mu \subset -(D \cap (C_\mu)_0(\varepsilon)^c)$ and by applying the sufficient condition of Lemma 3.3.7 (see part $(a)$ of Remark 3.3.8) we have that

$$\Phi_{B_\mu}(x_0, \lambda_0) \subset \text{Max}(\Phi_{B_\mu}(x_0, K^+), C_\mu, \varepsilon) \tag{3.23}$$

and the result follows by (3.22) and (3.23).

If $\tau_C(\mu) = 0$, then $x_0 \in \text{argmin}_X(\mu \circ f + \lambda \circ g)$ and $\langle \lambda_0, g(x_0)\rangle = 0$ which in particular implies (3.20) and (3.21) and, therefore, (3.22) holds. On the other hand, it is clear that

$$\langle \lambda, g(x_0)\rangle B_\mu = \{0\} \subset -(D \cap (C_\mu)_0(\varepsilon)^c),$$

and by Lemma 3.3.7 and Remark 3.3.8$(a)$ we see that condition (3.23) also holds, concluding the proof. $\square$

In the following result we give an exact version of Theorem 3.3.10.

**Theorem 3.3.11.** Consider $x_0 \in S$, $q \in D\backslash\{0\}$ and suppose that $\operatorname{int} D^+ \neq \emptyset$, $K$ is closed, $f - f(x_0)$ is nearly $D$-subconvexlike on $S$, $(f - f(x_0), g)$ is nearly $(D \times K)$-subconvexlike on $X$ and the Slater constraint qualification holds. If $x_0 \in \mathrm{Be}(f, S)$ then there exist $\mu \in D^{s+}$ and $\lambda_0 \in K^+$ such that $(x_0, \lambda_0)$ is a proper saddle point for the $B_\mu$ and $\{q\}$-Lagrangians associated with problem $(\mathcal{P}_S)$.

*Proof.* For $C = C_q$ and $\varepsilon = 0$ the hypotheses of Theorem 3.3.10 are satisfied. Indeed, from (2.2) it follows that $f - f(x_0)$ is nearly $(C_q, 0)$-subconvexlike on $S$, since $f-f(x_0)$ is nearly $D$-subconvexlike on $S$ and also $\mathrm{Be}(f, S) = \mathrm{Be}(f, S, C_q, 0)$. Analogously, $(f - f(x_0), g)$ is nearly $(C_q \times K, 0)$-subconvexlike on $X$, since $(f - f(x_0), g)$ is nearly $(D \times K)$-subconvexlike on $X$ and $K$ is closed.

Then, by Theorem 3.3.10 we know that there exists $\mu \in D^{s+}$ and $\lambda_0 \in K^+$ such that $(x_0, \lambda_0)$ is a $(C_\mu, 0)$-proper saddle point for the $B_\mu$-Lagrangian associated with problem $(\mathcal{P}_S)$. As $C_\mu(0) = D$ we deduce that $(x_0, \lambda_0)$ is a proper saddle point for the $B_\mu$-Lagrangian associated with problem $(\mathcal{P}_S)$.

On the other hand, since $q \in D\backslash\{0\}$ and $\mu \in D^{s+}$, following the proof of Theorem 3.3.10 we can suppose that $\tau_{C_q}(\mu) = \langle \mu, q\rangle = 1$, which implies that $q \in B_\mu$, and we see that

$$\begin{aligned}&\operatorname{cl}\operatorname{cone}(\Phi_q(X, \lambda_0) + D - \Phi_q(x_0, \lambda_0)) \cap (-D)\\ &\subset \operatorname{cl}\operatorname{cone}(\Phi_{B_\mu}(X, \lambda_0) + C_\mu(0) - \Phi_{B_\mu}(x_0, \lambda_0)) \cap (-D) = \{0\}\end{aligned}$$

and

$$\begin{aligned}\Phi_q(x_0, \lambda_0) \in \Phi_{B_\mu}(x_0, \lambda_0) &\subset \mathrm{Max}(\Phi_{B_\mu}(x_0, K^+), C_\mu, 0)\\ &= \mathrm{Max}(\Phi_{B_\mu}(x_0, K^+)).\end{aligned}$$

Since $\Phi_q(x_0, \lambda_0) \in \Phi_q(x_0, K^+) \subset \Phi_{B_\mu}(x_0, K^+)$ it follows that

$$\Phi_q(x_0, \lambda_0) \in \mathrm{Max}(\Phi_q(x_0, K^+)).$$

Thus, $(x_0, \lambda_0)$ is a proper saddle point for the $\{q\}$-Lagrangian associated with problem $(\mathcal{P}_S)$ and the proof is complete. □

**Remark 3.3.12.** The (exact) saddle point result of Theorem 3.3.11 is stronger than other similar saddle point results in the literature based on vector valued Lagrangian functions $\mathcal{L} : X \times \Gamma \to Y$ and Benson proper efficient solutions of problem $(\mathcal{P}_S)$, since it considers Benson proper efficient solutions of Lagrangian mappings instead of efficient solutions (see Remarks 3.3.2 and 3.3.4$(b)$, and compare Theorem 3.3.11 with [3, Corollary 4.2], [73, Theorem 4.4], [80, Theorem 3.2] and [108, Theorem 4.1]).

In the following result we obtain a sufficient condition for Benson $(C, \bar{\varepsilon})$-proper solutions of problem $(\mathcal{P}_S)$ based on Benson $(C, \varepsilon)$-proper solutions of unconstrained $B$-Lagrangian mappings.

**Theorem 3.3.13.** Consider $\varepsilon \geq 0$, $\lambda \in K^+$, $B \subset D\backslash\{0\}$ such that $C = B + P$, where $P \in \mathcal{H}_Y$ satisfies $P_0(0) \subset P$ and $P + D = P$. If $x_0 \in S$, $-\varepsilon \leq \langle \lambda, g(x_0) \rangle$ and $x_0 \in \mathrm{Be}(\Phi_B(\cdot, \lambda), X, C, \varepsilon)$ then $x_0 \in \mathrm{Be}(f, S, C, \varepsilon - \langle \lambda, g(x_0) \rangle)$.

*Proof.* Suppose that $\bar{\varepsilon} := \varepsilon - \langle \lambda, g(x_0) \rangle > 0$. As $-\varepsilon \leq \langle \lambda, g(x_0) \rangle \leq 0$, if $\varepsilon = 0$ then $\langle \lambda, g(x_0) \rangle = 0$ and so $\bar{\varepsilon} = 0$, which is a contradiction. Thus $\varepsilon > 0$.

Since $C(\bar{\varepsilon}) \subset \varepsilon B - \langle \lambda, g(x_0) \rangle B + P$ and $0 \in \langle \lambda, g(x) \rangle B + D$ for all $x \in S$ we have that

$$\begin{aligned}
& f(x) - f(x_0) + C(\bar{\varepsilon}) \\
& \subset f(x) - f(x_0) + \varepsilon B - \langle \lambda, g(x_0) \rangle B + P \\
& \subset f(x) + \langle \lambda, g(x) \rangle B + D - f(x_0) + \varepsilon B - \langle \lambda, g(x_0) \rangle B + P \\
& = \Phi_B(x, \lambda) - \Phi_B(x_0, \lambda) + \varepsilon B + P + D \\
& = \Phi_B(x, \lambda) - \Phi_B(x_0, \lambda) + C(\varepsilon), \ \forall x \in S,
\end{aligned}$$

since $P + D = P$, $\varepsilon > 0$ and $\varepsilon B + P = \varepsilon(B + P) = \varepsilon C$. Therefore,

$$f(S) - f(x_0) + C(\bar{\varepsilon}) \subset \Phi_B(X, \lambda) - \Phi_B(x_0, \lambda) + C(\varepsilon). \tag{3.24}$$

If $\bar{\varepsilon} = 0$ then $\varepsilon = \langle \lambda, g(x_0) \rangle = 0$ and it follows that

$$\begin{aligned} & f(x) - f(x_0) + C_0(0) \\ & \subset f(x) + \langle \lambda, g(x) \rangle B + D - f(x_0) - \langle \lambda, g(x_0) \rangle B + C_0(0) \\ & = \Phi_B(x, \lambda) - \Phi_B(x_0, \lambda) + C_0(0) + D \\ & = \Phi_B(x, \lambda) - \Phi_B(x_0, \lambda) + C_0(0), \ \forall x \in S. \end{aligned}$$

Then,

$$f(S) - f(x_0) + C_0(0) \subset \Phi_B(X, \lambda) - \Phi_B(x_0, \lambda) + C_0(0). \tag{3.25}$$

Since $x_0 \in \mathrm{Be}(\Phi_B(\cdot, \lambda), X, C, \varepsilon)$, taking into account (3.24), (3.25) and (2.2) we deduce that

$$\begin{aligned} & \mathrm{cl\,cone}(f(S) - f(x_0) + C(\bar{\varepsilon})) \cap (-D) \\ & \subset \mathrm{cl\,cone}(\Phi_B(X, \lambda) - \Phi_B(x_0, \lambda) + C(\varepsilon)) \cap (-D) = \{0\}, \end{aligned}$$

i.e., $x_0 \in \mathrm{Be}(f, S, C, \varepsilon - \langle \lambda, g(x_0) \rangle)$, which finishes the proof. □

**Remark 3.3.14.** With respect to the assumptions of Theorem 3.3.13, let us observe that if $P \subset D$ and $0 \in P$, then condition $P + D = P$ imply $P = D$. However $P = D \backslash \{0\}$ or $P = \mathrm{int}\, D$ (when $D$ is solid) satisfy all these conditions and $P \neq D$ in both cases.

In the next corollary we give a sufficient condition for Benson $(C, \bar{\varepsilon})$-proper solutions of problem $(\mathcal{P}_S)$ through $(C_\mu, \varepsilon)$-proper saddle points of the $B_\mu$-Lagrangian.

**Corollary 3.3.15.** Consider $\varepsilon \geq 0$, $\mu \in D^{s+}$ and suppose that $K$ is closed.

$(a)$ If $(x_0, \lambda_0) \in X \times K^+$ is a $(C_\mu, \varepsilon)$-proper saddle point for the $B_\mu$-Lagrangian associated with problem $(\mathcal{P}_S)$, then $x_0 \in \mathrm{Be}(f, S, C_\mu, \varepsilon - \langle \lambda_0, g(x_0) \rangle)$.

$(b)$ If $(x_0, \lambda_0) \in X \times K^+$ is a proper saddle point for the $B_\mu$-Lagrangian associated with problem $(\mathcal{P}_S)$, then $x_0 \in \mathrm{Be}(f, S)$.

*Proof.* Let $(x_0, \lambda_0) \in X \times K^+$ be a $(C_\mu, \varepsilon)$-proper saddle point for the $B_\mu$-Lagrangian associated with problem $(\mathcal{P}_S)$. By Lemma 3.3.7 we deduce that $x_0 \in S$ and $\langle \lambda_0, g(x_0) \rangle B_\mu \subset -(D \cap (C_\mu)_0(\varepsilon)^c)$. By (3.19) we see that

$$-\varepsilon \leq \langle \lambda_0, g(x_0) \rangle \leq 0. \tag{3.26}$$

Then part $(a)$ follows by applying Theorem 3.3.13 to $\lambda = \lambda_0$, $B = B_\mu$ and $P = D\backslash\{0\}$.

Suppose that $(x_0, \lambda_0) \in X \times K^+$ is a proper saddle point for the $B_\mu$-Lagrangian associated with problem $(\mathcal{P}_S)$. It is clear that $(x_0, \lambda_0) \in X \times K^+$ is a $(C_\mu, 0)$-saddle point for the $B_\mu$-Lagrangian associated with problem $(\mathcal{P}_S)$, since $C_\mu(0) = D$ (see Remark 3.3.4$(a)$). Then, by part $(a)$ and (3.26) we deduce that $x_0 \in \mathrm{Be}(f, S, C_\mu, 0) = \mathrm{Be}(f, S)$. $\square$

Next we characterize the set of Benson proper efficient solutions of problem $(\mathcal{P}_S)$ through saddle points for $B_\mu$-Lagrangians. The result is a direct consequence of Theorem 3.3.11 and Corollary 3.3.15.

**Corollary 3.3.16.** Let $x_0 \in X$ and suppose that $\operatorname{int} D^+ \neq \emptyset$, $K$ is closed, $f - f(x_0)$ is nearly $D$-subconvexlike on $S$, $(f - f(x_0), g)$ is nearly $(D \times K)$-subconvexlike on $X$ and the Slater constraint qualification holds. Then $x_0 \in \mathrm{Be}(f, S)$ if and only if there exist $\mu \in D^{s+}$ and $\lambda_0 \in K^+$ such that $(x_0, \lambda_0)$ is a proper saddle point for the $B_\mu$-Lagrangian associated with problem $(\mathcal{P}_S)$.

**Remark 3.3.17.** In [72, Theorems 6.1 and 6.2], Li characterized the set of Benson proper efficient solutions of a subconvexlike vector optimization problem with set-valued mappings via proper saddle points by assuming that the ordering cone is solid. Then Corollary 3.3.16 improves the vector valued version of these theorems, since its assumptions are weaker.

# Capítulo 4

# Approximate duality

## 4.1 Introduction

In vector optimization, like in scalar optimization, duality theories have been developed with the aim of providing alternative formulations to solve a so-called primal optimization problem. Roughly speaking, they entail defining, for a given primal minimization problem, a related maximization problem based on a dual objective mapping that involves the objective and the constraint mappings of the primal problem, in such a way that the optimal values of this dual problem coincide with the optimal values of the primal problem, under certain assumptions.

In multiobjective optimization, some of the first works about duality were given by Tanino and Sawaragi [108] and Luc [80], and for vector optimization problems by Corley [23] and Jahn [63, 64].

In the literature, most of the works about this topic provide duality results for proper efficient solutions of the primal problem. Papers [12, 14, 15, 63, 72, 80, 108] follow this line (see also [13, 64]).

Jahn [63,64] defined a dual problem through a scalar Lagrangian and obtained weak and strong duality results that relate the maximal points of the dual objective set with proper minimal elements of the primal objective set under stability and cone-convexity assumptions.

In Pareto multiobjective optimization, Boţ and Wanka [14] introduced several dual Pareto multiobjective problems and gave duality results for proper efficient solutions in the sense of Geoffrion. These dual problems are associated with scalar conjugate dual problems corresponding to the usual linear scalarization of the primal Pareto multiobjective problem. The relationships between these dual problems were obtained in [15] (see also [13]).

In vector optimization, Boţ and Grad [12] introduced several vector dual problems via a very general approach based on the conjugate of an scalarization scheme on a generic vector perturbation mapping, and they obtained duality results for proper efficient solutions of the primal problem.

Also, in the setting of vector optimization problems with set-valued mappings, Li [72] defined a dual mapping by means of proper efficient solutions in the sense of Benson of a set-valued Lagrangian, and derived duality results for Benson proper efficient solutions of the primal problem under cone-subconvexlikeness hypotheses.

As it is known, the importance of studying approximate solutions of vector optimization problems lies in the difficulty of obtaining exact solutions by means of the usual numerical techniques employed for solving a vector optimization problem. This fact makes relevant the analysis of scalarization processes from an approximate point of view and also the developing of approximate duality theories that facilitates the calculus of approximate solutions of the original (primal) vector optimization problem. Some works in this line are [31, 40, 50, 51, 65, 96, 98, 99, 114].

One of the first papers on approximate duality in vector optimization was due to Vályi [114], who obtained Lagrangian duality results for a kind of approximate solutions introduced by himself through linear scalarization. Later, Rong and Wu [98] applied the ideas and methods of Vályi in the setting of vector optimization problems with cone-subconvexlike set-valued mappings and they derived weak and strong duality results for approximate weak efficient solutions in the sense of Kutateladze (see [70]) of the primal problem. In the same framework, Jia and Li [65] established an approximate conjugate duality theory by means of approximate weak efficient solutions in the sense of Kutateladze. Recently, Sach, Tuan and Minh [100] obtained approximate duality results for vector quasi-equilibrium problems with set-valued mappings by considering Benson $(C_q, \varepsilon)$-proper solutions, with $q \in D\backslash\{0\}$.

In this chapter we consider problem $(\mathcal{P}_S)$, with the feasible set given by a cone constraint as in (1.1) and we introduce two new approximate dual problems, deriving weak and strong duality results for Benson $(C, \varepsilon)$-proper solutions of the primal problem $(\mathcal{P}_S)$.

In Section 4.2, we introduce an approximate dual problem denoted $(\mathcal{D}_{C,\varepsilon})$ by means of a scalar Lagrangian and an approximation set $C$, which reduces to the well-known dual problem defined by Jahn [63,64] for the exact case. By assuming stability and generalized convexity assumptions we obtain several relationships between Benson $(C, \varepsilon)$-proper solutions of problem $(\mathcal{P}_S)$ and aproximate solutions of the dual problem $(\mathcal{D}_{C,\varepsilon})$. In particular, we provide weak and strong duality results.

In Section 4.3, we introduce another approximate dual problem denoted $(\mathcal{D}^{\mathrm{Be}}_{C,\varepsilon})$, whose objective mapping is defined via Benson $(C, \varepsilon)$-proper solutions of the set-valued Lagrangian given in Definition 3.3.1. As in the previous case, we obtain relationships between Benson $(C, \varepsilon)$-proper solutions of problem $(\mathcal{P}_S)$ and approximate solutions of the dual problem $(\mathcal{D}^{\mathrm{Be}}_{C,\varepsilon})$, in particular weak and strong duality results.

The results of this chapter are included in [45].

## 4.2 Lagrange dual problem

In this section, we consider problem $(\mathcal{P}_S)$ with $f : X \to Y$ and the feasible set given as in (1.1). Next, we introduce an approximate Lagrange dual problem of $(\mathcal{P}_S)$ and by it we state weak and strong duality results for Benson $(C, \varepsilon)$-proper solutions of problem $(\mathcal{P}_S)$.

Let $\varepsilon \geq 0$, $C \in \mathcal{F}_Y$ and the following approximate Lagrange dual problem associated with $(\mathcal{P}_S)$:

$$\text{Maximize } \ \Xi_{C,\varepsilon}, \qquad (\mathcal{D}_{C,\varepsilon})$$

where the $(C,\varepsilon)$-dual set $\Xi_{C,\varepsilon} \subset Y$ is defined as

$$\Xi_{C,\varepsilon} := \bigcup_{\lambda\in K^+} \bigcup_{\mu\in D^{s+}\cap C^{\tau+}} \{y \in Y : \langle \mu, y\rangle \leq (\mu\circ f)(x)+(\lambda\circ g)(x)+\varepsilon\tau_C(\mu),\ \forall x \in X\}.$$

In other words, problem $(\mathcal{D}_{C,\varepsilon})$ asks about maximal points of the set $\Xi_{C,\varepsilon}$.

**Remark 4.2.1.** $(a)$ If $Y = \mathbb{R}$ and $D = \mathbb{R}_+$, then $\mathcal{F}_Y = 2^{[0,+\infty)}\backslash\{\emptyset\}$ and $(\mathcal{D}_{C,\varepsilon})$ reduces to the problem

$$\text{Maximize } \ h(\lambda) := \inf_{x\in X}\{f(x) + (\lambda \circ g)(x) + \varepsilon \inf C\} \text{ subject to } \lambda \in K^+,$$

which has the same solutions as the classical Lagrange dual problem. However, their optimum values differ in the value $\varepsilon \inf C$.

$(b)$ If $Y$ is normed, then the $(C,\varepsilon)$-dual set can be reformulated as follows:

$$\Xi_{C,\varepsilon} = \bigcup_{\lambda\in K^+} \bigcup_{\substack{\mu\in D^{s+}\cap C^{\tau+}\\ \|\mu\|=1}} \{y \in Y : \langle \mu, y\rangle \leq (\mu\circ f)(x)+(\lambda\circ g)(x)+\varepsilon\tau_C(\mu),\ \forall x \in X\}.$$

$(c)$ When $C = D$ or $\varepsilon = 0$ and $C \subset D$, problem $(\mathcal{D}_{C,\varepsilon})$ reduces to the following exact Lagrange dual problem, which was defined by Jahn in [63] (see also [64, Section 8.2]):

$$\text{Maximize } \ \Xi := \bigcup_{\lambda\in K^+} \bigcup_{\mu\in D^{s+}} \{y \in Y : \langle \mu, y\rangle \leq (\mu\circ f)(x)+(\lambda\circ g)(x),\ \forall x \in X\}. \tag{$\mathcal{D}$}$$

Let us observe that, in general, $\Xi_{C,0} \neq \Xi$. Moreover, problem $(\mathcal{D}_{C,\varepsilon})$ is feasible if and only if problem $(\mathcal{D})$ is feasible, i.e.,

$$\Xi_{C,0} \neq \emptyset \iff \Xi_{C,\varepsilon} \neq \emptyset, \quad \forall\, \varepsilon \geq 0.$$

In the following proposition we establish several basic properties of the $(C,\varepsilon)$-dual set $\Xi_{C,\varepsilon}$. The proof is trivial, so we omit it.

**Proposition 4.2.2.** One has

$(a)$ If $C \subset D$, then $\Xi \subset \Xi_{C,\varepsilon}$.

$(b)$ $\Xi_{C',\delta} \subset \Xi_{C,\varepsilon}$, for all $\delta \geq 0$ and $C' \in \mathcal{F}_Y$ such that $C(\varepsilon) \subset \mathrm{cl}(C'(\delta))$.

$(c)$ If $C$ is coradiant, $\Xi_{C,\delta} \subset \Xi_{C,\varepsilon}$, for all $0 \leq \delta \leq \varepsilon$.

$(d)$ $\Xi_{C+C',\varepsilon} = \Xi_{\mathrm{cl}\, C+C',\varepsilon}$, for all $C' \subset D$.

$(e)$ $\Xi_{C,\varepsilon} \subset \Xi_{C+C',\varepsilon}$, for all $C' \subset D$. If additionally $0 \in \mathrm{cl}\, C'$, then $\Xi_{C,\varepsilon} = \Xi_{C+C',\varepsilon}$.

$(f)$ $\Xi_{C,\varepsilon} = \Xi_{\mathrm{cl}\, C,\varepsilon} = \Xi_{\mathrm{co}\, C,\varepsilon} = \Xi_{\mathrm{shw}\, C,\varepsilon}$.

**Remark 4.2.3.** The previous properties of the set $\Xi_{C,\varepsilon}$ match with other similar ones of the set $\mathrm{Be}(f, S, C, \varepsilon)$, which are satisfied under certain convexity assumptions (see Propositions 2.2.3 and 2.2.14). In particular, observe that under these convexity assumptions, the set of Benson $(C, \varepsilon)$-proper solutions of problem $(\mathcal{P}_S)$ and the $(C, \varepsilon)$-dual set of problem $(\mathcal{D}_{C,\varepsilon})$ do not change, for example, by considering $C' = \mathrm{cl}(\mathrm{shw}\,\mathrm{co}\, C + D)$ instead of $C$.

In [63, 64], Jahn provided weak, strong and converse duality results for $(\mathcal{D})$ through solutions of the scalar optimization problems $(\mathcal{P}_\mu)$, $\mu \in D^{s+}$. From Theorems 2.2.6 and 2.2.8 we know that by assuming generalized convexity assumptions, the set of Benson $(C, \varepsilon)$-proper solutions of $(\mathcal{P}_S)$ is characterized through approximate solutions of the scalar problems $(\mathcal{P}_\mu)$ for $\mu \in D^{s+}$. Thus, problem $(\mathcal{D}_{C,\varepsilon})$ will let us derive approximate duality results which involve Benson $(C, \varepsilon)$-proper solutions of problem $(\mathcal{P}_S)$. In the next theorem we present a weak duality result.

**Theorem 4.2.4.** It follows that

$$(\Xi_{C,\varepsilon} - f(S)) \cap (\mathrm{cl}(C(\varepsilon)) + D\backslash\{0\}) = \emptyset. \tag{4.1}$$

*Proof.* Reasoning by contradiction, suppose that there exist $y_0 \in \Xi_{C,\varepsilon}$ and $x_0 \in S$ such that

$$y_0 - f(x_0) \in \mathrm{cl}(C(\varepsilon)) + D\backslash\{0\}. \tag{4.2}$$

From the definition of $\Xi_{C,\varepsilon}$, there exist $\mu \in D^{s+} \cap C^{\tau+}$ and $\lambda \in K^+$ verifying

$$\langle \mu, y_0 \rangle \leq (\mu \circ f)(x) + (\lambda \circ g)(x) + \varepsilon \tau_C(\mu),\ \forall x \in X.$$

In particular, for $x = x_0$ and taking into account that $(\lambda \circ g)(x_0) \leq 0$ we obtain from the statement above the following inequality:

$$\langle \mu, y_0 - f(x_0) \rangle \leq \varepsilon \tau_C(\mu). \tag{4.3}$$

On the other hand, since (4.2) holds, there exist $c \in \mathrm{cl}(C(\varepsilon))$ and $d \in D \backslash \{0\}$ such that $y_0 - f(x_0) = c + d$, so

$$\langle \mu, y_0 - f(x_0) \rangle = \langle \mu, c \rangle + \langle \mu, d \rangle \geq \varepsilon \tau_C(\mu) + \langle \mu, d \rangle > \varepsilon \tau_C(\mu),$$

obtaining a contradiction with (4.3). □

**Remark 4.2.5.** $(a)$ It is easy to check that $0 \notin \mathrm{cl}\, C + D \backslash \{0\}$, for all $C \in \mathcal{F}_Y$. Then $(\mathrm{cl}\, C + D \backslash \{0\})_0(0) \subset \mathrm{cl}(C(0)) + D \backslash \{0\}$ and $(\mathrm{cl}\, C + D \backslash \{0\})_0(\varepsilon) = \mathrm{cl}(C(\varepsilon)) + D \backslash \{0\}$, for all $\varepsilon > 0$. Therefore, statement (4.1) implies that

$$(\Xi_{C,\varepsilon} - f(S)) \cap (\mathrm{cl}\, C + D \backslash \{0\})_0(\varepsilon) = \emptyset.$$

$(b)$ Theorem 4.2.4 reduces to [64, Lemma 8.5$(b)(i)$] and [13, Theorem 4.3.5$(a)$] when $C = D$ or $\varepsilon = 0$ and $\mathrm{cone}\, C = D$.

We denote in the following

$$\begin{aligned} T^1_{C,\varepsilon} &:= \mathrm{Min}(\mathrm{cl}(C(\varepsilon))) = \mathrm{cl}(C(\varepsilon)) \cap (\mathrm{cl}(C(\varepsilon)) + D \backslash \{0\})^c, \\ T^2_{C,\varepsilon} &:= D \cap (\mathrm{cl}(C(\varepsilon)) + D \backslash \{0\})^c, \\ T^3_{C,\varepsilon} &:= \mathrm{cl}(C(0)) \cap (\mathrm{cl}(C(\varepsilon)) + D \backslash \{0\})^c. \end{aligned}$$

If $C \subset D$, then $T^1_{C,\varepsilon} \subset T^3_{C,\varepsilon} \subset T^2_{C,\varepsilon}$, and if $C = D$, then $T^i_{C,\varepsilon} = \{0\}$, for all $i$. Moreover, if $C + D = C$, then $D \subset \mathrm{cl}(C(0))$ and so $T^2_{C,\varepsilon} \subset T^3_{C,\varepsilon}$.

Next theorem gives a sufficient condition for Benson $(C, \varepsilon)$-proper solutions of problem $(\mathcal{P}_S)$ and approximate maximal points of $(\mathcal{D}_{C,\varepsilon})$. It is an extension of [64, Lemma 8.5$(b)(ii)$] to the approximate case.

**Theorem 4.2.6.** Let $x_0 \in S$ and $y_0 \in \Xi_{C,\varepsilon}$ such that $y_0 - f(x_0) \in T^2_{C,\varepsilon}$. Then $x_0 \in \mathrm{Be}(f, S, C, \varepsilon)$ and $y_0 \in \mathrm{Max}(\Xi_{C,\varepsilon}, \mathrm{cl}\, C + D \backslash \{0\}, \varepsilon)$.

*Proof.* Since $y_0 - f(x_0) \in T^2_{C,\varepsilon}$, there exists $d_0 \in T^2_{C,\varepsilon}$ such that $y_0 = f(x_0) + d_0 \in \Xi_{C,\varepsilon}$ and then there exist $\mu \in D^{s+} \cap C^{\tau+}$ and $\lambda \in K^+$ such that for all $x \in X$,

$$(\mu \circ f)(x_0) \leq (\mu \circ f)(x_0) + \langle \mu, d_0 \rangle \leq (\mu \circ f)(x) + (\lambda \circ g)(x) + \varepsilon\tau_C(\mu).$$

In particular, since $(\lambda \circ g)(x) \leq 0$ for all $x \in S$, from the inequality above it follows that

$$(\mu \circ f)(x_0) \leq (\mu \circ f)(x) + \varepsilon\tau_C(\mu), \ \forall x \in S,$$

and by applying Theorem 2.2.8 we have that $x_0 \in \mathrm{Be}(f, S, C, \varepsilon)$.

On the other hand, by Theorem 4.2.4 we know that $(\Xi_{C,\varepsilon} - f(x_0)) \cap (\mathrm{cl}(C(\varepsilon)) + D\backslash\{0\}) = \emptyset$. Thus,

$$(\Xi_{C,\varepsilon} - f(x_0) - d_0) \cap (-d_0 + \mathrm{cl}(C(\varepsilon)) + D\backslash\{0\}) = \emptyset.$$

Since $D$ is pointed, it is easy to see that $\mathrm{cl}(C(\varepsilon)) + D\backslash\{0\} \subset -d_0 + \mathrm{cl}(C(\varepsilon)) + D\backslash\{0\}$. Then,

$$(\Xi_{C,\varepsilon} - y_0) \cap (\mathrm{cl}(C(\varepsilon)) + D\backslash\{0\}) = \emptyset$$

and by Remark 4.2.5$(a)$ we conclude that

$$(\Xi_{C,\varepsilon} - y_0) \cap (\mathrm{cl}\, C + D\backslash\{0\})_0(\varepsilon) = \emptyset,$$

i.e., $y_0 \in \mathrm{Max}(\Xi_{C,\varepsilon}, \mathrm{cl}\, C + D\backslash\{0\}, \varepsilon)$. □

Next, we give two strong duality results, which provide a characterization of the set of Benson $(C, \varepsilon)$-proper solutions of problem $(\mathcal{P}_S)$ through approximate maximal points of the $(C, \varepsilon)$-dual set.

**Theorem 4.2.7.** Suppose that $\mathrm{int}\, D^+ \neq \emptyset$, $x_0 \in S$, $f - f(x_0)$ is nearly $(C, \varepsilon)$-subconvexlike on $S$, $(f - f(x_0), g)$ is nearly $(C \times K, \varepsilon)$-subconvexlike on $X$ and the Slater constraint qualification holds. Then, $x_0 \in \mathrm{Be}(f, S, C, \varepsilon)$ if and only if $f(x_0) \in \mathrm{Max}(\Xi_{C,\varepsilon}, \mathrm{cl}\, C + D\backslash\{0\}, \varepsilon)$.

*Proof.* Suppose that $x_0 \in \mathrm{Be}(f, S, C, \varepsilon)$. Following the proof of Theorem 3.2.2 (see statement (3.7)) we deduce that there exist $\mu \in D^{s+} \cap C^{\tau+}$ and $\lambda \in K^+$ such that

$$(\mu \circ f)(x_0) \leq (\mu \circ f)(x) + (\lambda \circ g)(x) + \varepsilon\tau_C(\mu), \ \forall x \in X,$$

so $f(x_0) \in \Xi_{C,\varepsilon}$ and by applying Theorem 4.2.6 we obtain that $f(x_0) \in \mathrm{Max}(\Xi_{C,\varepsilon}, \mathrm{cl}\, C + D\backslash\{0\}, \varepsilon)$. Reciprocally, since $f(x_0) \in \mathrm{Max}(\Xi_{C,\varepsilon}, \mathrm{cl}\, C + D\backslash\{0\}, \varepsilon)$ we have in particular that $f(x_0) \in \Xi_{C,\varepsilon}$ and by applying again Theorem 4.2.6 we deduce that $x_0 \in \mathrm{Be}(f, S, C, \varepsilon)$. □

In the sequel we denote

$$\mathbb{B} := \{\mu \in D^{s+} : \inf_{x\in S}\{(\mu \circ f)(x)\} > -\infty\}.$$

**Theorem 4.2.8.** Suppose that $\mathrm{int}\, D^+ \neq \emptyset$, $S$ is convex, $f$ is $D$-convex on $S$ and $C$ is convex and verifies that $C + D = C$.

(*a*) If $\tau_C(\mu) > 0$ and problem $(\mathcal{P}_\mu)$ is normal, for all $\mu \in D^{s+} \cap C^{\tau+} \cap \mathbb{B}$, then $x_0 \in \mathrm{Be}(f, S, C, \varepsilon)$ implies $f(x_0) \in \mathrm{Max}(\Xi_{C,\varepsilon'}, \mathrm{cl}\, C + D\backslash\{0\}, \varepsilon')$, for all $\varepsilon' > \varepsilon$.

(*b*) If problem $(\mathcal{P}_\mu)$ is stable, for all $\mu \in D^{s+} \cap C^{\tau+} \cap \mathbb{B}$, then $x_0 \in \mathrm{Be}(f, S, C, \varepsilon)$ if and only if $f(x_0) \in \mathrm{Max}(\Xi_{C,\varepsilon}, \mathrm{cl}\, C + D\backslash\{0\}, \varepsilon)$.

*Proof.* By Remark 1.2.23 it follows that $f - f(x_0)$ is nearly $(C, \varepsilon)$-subconvexlike on $S$. Then, if $x_0 \in \mathrm{Be}(f, S, C, \varepsilon)$, by applying Theorem 2.2.6 we deduce that there exists $\mu \in D^{s+} \cap C^{\tau+} \cap \mathbb{B}$ such that

$$(\mu \circ f)(x_0) \leq \inf_{x\in S}\{(\mu \circ f)(x)\} + \varepsilon\tau_C(\mu). \tag{4.4}$$

(*a*) As $(\mathcal{P}_\mu)$ is normal it follows that for each $\varepsilon' > \varepsilon$ there exists $\lambda' \in K^+$ such that

$$\begin{aligned}\inf_{x\in S}\{(\mu \circ f)(x)\} &= \sup_{\lambda\in K^+} \inf_{x\in X}\{(\mu \circ f + \lambda \circ g)(x)\} \\ &\leq \inf_{x\in X}\{(\mu \circ f)(x) + (\lambda' \circ g)(x)\} + (\varepsilon' - \varepsilon)\tau_C(\mu)\end{aligned}$$

and by (4.4) we obtain

$$(\mu \circ f)(x_0) \leq (\mu \circ f)(x) + (\lambda' \circ g)(x) + \varepsilon' \tau_C(\mu), \quad \forall\, x \in X,$$

i.e., $f(x_0) \in \Xi_{C,\varepsilon'}$. Then, by applying Theorem 4.2.6 we deduce that $f(x_0) \in \operatorname{Max}(\Xi_{C,\varepsilon'}, \operatorname{cl} C + D\backslash\{0\}, \varepsilon')$, and the proof of part $(a)$ finishes.

$(b)$ Since $(\mathcal{P}_\mu)$ is stable, there exists $\lambda \in K^+$ verifying

$$\inf_{x\in S}\{(\mu \circ f)(x)\} = \inf_{x\in X}\{(\mu \circ f)(x) + (\lambda \circ g)(x)\}$$

and by (4.4) we see that

$$(\mu \circ f)(x_0) \leq (\mu \circ f)(x) + (\lambda \circ g)(x) + \varepsilon \tau_C(\mu), \quad \forall\, x \in X,$$

i.e., $f(x_0) \in \Xi_{C,\varepsilon}$. Then, by applying Theorem 4.2.6 we deduce that $f(x_0) \in \operatorname{Max}(\Xi_{C,\varepsilon}, \operatorname{cl} C + D\backslash\{0\}, \varepsilon)$. The reciprocal statement follows in the same way as in Theorem 4.2.7 and the proof is complete. □

**Remark 4.2.9.** Suppose that $S$ is convex, $f$ is $D$-convex on $S$, $g$ is $K$-convex on $X$ and $\mu \in D^+$. If $\operatorname{argmin}_S(\mu \circ f) \neq \emptyset$, then each regularity condition $(RC_i^{C_L})$, $i \in \{1, 2, 3, 4\}$, of [13, pg. 80–82] implies that problem $(\mathcal{P}_\mu)$ is stable (see [13, Theorem 3.3.16]).

Theorems 4.2.7 and 4.2.8$(b)$ reduce to the following corollaries by considering $\varepsilon = 1$ and $C = D$ (i.e., in the exact case).

**Corollary 4.2.10.** Suppose that $\operatorname{int} D^+ \neq \emptyset$, $x_0 \in S$, $f - f(x_0)$ is nearly $D$-subconvexlike on $S$, $(f - f(x_0), g)$ is nearly $D \times K$-subconvexlike on $X$ and the Slater constraint qualification holds. Then, $x_0 \in \operatorname{Be}(f, S)$ if and only if $f(x_0) \in \operatorname{Max}(\Xi)$.

**Corollary 4.2.11.** Suppose that $\operatorname{int} D^+ \neq \emptyset$, $S$ is convex and $f$ is $D$-convex on $S$.

$(a)$ If problem $(\mathcal{P}_\mu)$ is stable for each $\mu \in D^{s+} \cap \mathbb{B}$, then $x_0 \in \operatorname{Be}(f, S)$ if and only if $f(x_0) \in \operatorname{Max}(\Xi)$.

$(b)$ If $\mu \in D^{s+} \cap \mathbb{B}$, $(\mathcal{P}_\mu)$ is stable and $x_0 \in \operatorname{argmin}_S(\mu \circ f)$, then $f(x_0) \in \operatorname{Max}(\Xi)$.

*Proof.* Part $(a)$ follows easily by considering $\varepsilon = 1$ and $C = D$ in Theorem 4.2.8$(b)$. Part $(b)$ is a direct consequence of the proof of Theorem 4.2.8$(b)$ by taking $\varepsilon = 1$ and $C = D$. □

**Remark 4.2.12.** Part $(b)$ of Corollary 4.2.11 was stated by Jahn in [64, Theorem 8.7$(b)$].

On the other hand, suppose that $\operatorname{int} D^+ \neq \emptyset$, $S$ is convex, $f$ is $D$-convex on $S$ and $g$ is $K$-convex on $X$. By Remark 4.2.9 we see that the Slater constraint qualification (denoted by $(RC_1^{C_L})$ in [13]) implies that problem $(\mathcal{P}_\mu)$, $\mu \in D^{s+}$, is stable whenever $\operatorname{argmin}_S(\mu \circ f) \neq \emptyset$. Then Corollary 4.2.11$(b)$ reduces to [13, Theorem 4.3.5$(b)$] by assuming the Slater constraint qualification instead of the stability of $(\mathcal{P}_\mu)$, and Corollary 4.2.11$(a)$ generalizes [13, Theorem 4.3.5$(b)$], since it states a characterization of the maximal points of $\Xi$ (instead of a sufficient condition) in terms of Benson (exact) proper solutions of problem $(\mathcal{P}_S)$.

The following lemma, which is required to state converse duality results, extends [63, Lemma 2.4] to the approximate case. We omit the proof, since it is similar to the proof of [63, Lemma 2.4] (see also [64, Lemma 8.8$(b)$] and [13, Theorem 4.3.3]).

**Lemma 4.2.13.** Suppose that $C$ is convex, coradiant and such that $C + D = C$, $\Xi_{C,\varepsilon} \neq \emptyset$, $S$ is convex, $f$ is $D$-convex on $S$ and the scalar optimization problem $(\mathcal{P}_\mu)$ is normal for each $\mu \in D^{s+} \cap C^{\tau+} \cap \mathbb{B}$. Then, $Y \setminus \operatorname{cl}(f(S) + C(\varepsilon)) \subset \Xi_{C,\varepsilon}$.

In the next theorem, we characterize the set of maximal points of $\Xi_{C,\varepsilon}$.

**Theorem 4.2.14.** It follows that

$(a)$ $(f(S) + T^1_{C,\varepsilon}) \cap \Xi_{C,\varepsilon} \subset \Xi_{C,\varepsilon} \cap \operatorname{bd} \Xi_{C,\varepsilon}$.

$(b)$ Suppose that $f(S) + \operatorname{cl} C(\varepsilon)$ is closed and the assumptions of Lemma 4.2.13 are fulfilled. Then,

$$\operatorname{Max}(\Xi_{C,\varepsilon}) = \Xi_{C,\varepsilon} \cap \operatorname{bd} \Xi_{C,\varepsilon} = (f(S) + T^1_{C,\varepsilon}) \cap \Xi_{C,\varepsilon}.$$

*Proof.* $(a)$ Let $y_0 \in (f(S) + T^1_{C,\varepsilon}) \cap \Xi_{C,\varepsilon}$. Then, there exist $x_0 \in S$ and $d_0 \in T^1_{C,\varepsilon}$ such that $y_0 = f(x_0) + d_0 \in \Xi_{C,\varepsilon}$. If $y_0 \notin \operatorname{bd} \Xi_{C,\varepsilon}$, it follows that $y_0 \in \operatorname{int} \Xi_{C,\varepsilon}$ and in particular there exists $\bar{d} \in D\backslash\{0\}$ such that $y_0 + \bar{d} \in \Xi_{C,\varepsilon}$. Hence, there exist $\mu \in D^{s+} \cap C^{\tau+}$ and $\lambda \in K^+$ such that for all $x \in X$,

$$(\mu \circ f)(x_0) + \langle \mu, d_0 + \bar{d} \rangle = \langle \mu, y_0 + \bar{d} \rangle \leq (\mu \circ f)(x) + (\lambda \circ g)(x) + \varepsilon \tau_C(\mu).$$

Considering $x = x_0$ in the statement above and taking into account that $(\lambda \circ g)(x_0) \leq 0$ we have that

$$\langle \mu, d_0 + \bar{d} \rangle \leq \varepsilon \tau_C(\mu). \tag{4.5}$$

But, on the other hand, since $d_0 \in \mathrm{cl}(C(\varepsilon))$ and $\bar{d} \in D\backslash\{0\}$ it follows that $\langle \mu, d_0 + \bar{d} \rangle > \varepsilon \tau_C(\mu)$, which contradicts (4.5). Hence, $y_0 \in \mathrm{bd}\,\Xi_{C,\varepsilon}$.

$(b)$ First, we prove the second equality. Inclusion "$\supset$" follows by $(a)$. For the reciprocal inclusion, consider $y_0 \in \Xi_{C,\varepsilon} \cap \mathrm{bd}\,\Xi_{C,\varepsilon}$. By Lemma 4.2.13,

$$Y\backslash(f(S) + \mathrm{cl}\, C(\varepsilon)) = Y\backslash\, \mathrm{cl}(f(S) + C(\varepsilon)) \subset \mathrm{int}\,\Xi_{C,\varepsilon},$$

since $Y\backslash\, \mathrm{cl}(f(S) + C(\varepsilon))$ is open. Thus, $y_0 \in f(S) + \mathrm{cl}(C(\varepsilon))$. Moreover, from Theorem 4.2.4 we see that $y_0 \notin f(S) + \mathrm{cl}(C(\varepsilon)) + D\backslash\{0\}$. Thus, we have that $y_0 \in f(S) + T^1_{C,\varepsilon}$ and $\Xi_{C,\varepsilon} \cap \mathrm{bd}\,\Xi_{C,\varepsilon} \subset (f(S) + T^1_{C,\varepsilon}) \cap \Xi_{C,\varepsilon}$.

For the first equality, inclusion $\mathrm{Max}(\Xi_{C,\varepsilon}) \subset \Xi_{C,\varepsilon} \cap \mathrm{bd}\,\Xi_{C,\varepsilon}$ is clear. Consider $y_0 \in \Xi_{C,\varepsilon} \cap \mathrm{bd}\,\Xi_{C,\varepsilon} = (f(S) + T^1_{C,\varepsilon}) \cap \Xi_{C,\varepsilon}$. Then there exist $x_0 \in S$ and $d_0 \in T^1_{C,\varepsilon}$ such that $y_0 = f(x_0) + d_0 \in \Xi_{C,\varepsilon}$. If $y_0 \notin \mathrm{Max}(\Xi_{C,\varepsilon})$, there exists $\bar{y} \in \Xi_{C,\varepsilon}$ such that $\bar{y} - f(x_0) - d_0 \in D\backslash\{0\}$, so $\bar{y} - f(x_0) \in d_0 + D\backslash\{0\} \subset \mathrm{cl}(C(\varepsilon)) + D\backslash\{0\}$, which contradicts Theorem 4.2.4. □

**Remark 4.2.15.** Observe that under the corresponding assumptions, Theorem 4.2.14 says that for each $y_0 \in \Xi_{C,\varepsilon}$,

$$y_0 \in \mathrm{Max}(\Xi_{C,\varepsilon}) \Longleftrightarrow y_0 \in f(S) + T^1_{C,\varepsilon}.$$

In particular, for $C = D$ this equivalence implies that $\mathrm{Max}(\Xi) \subset f(S)$ and by Theorem 4.2.6 we see that

$$S \cap f^{-1}(\mathrm{Max}(\Xi)) \subset \mathrm{Be}(f, S).$$

This converse strong duality result was stated by Jahn [64, Theorem 8.9$(b)$] (see also [13, Theorem 4.3.5$(c)$] and Remark 4.2.9.

In the following theorem we present a converse strong duality result that relates the maximal points of $\Xi_{C,\varepsilon}$ with Benson proper efficient solutions and Benson $(C,\varepsilon)$-proper solutions of the primal problem.

**Theorem 4.2.16.** Suppose that $f(S) + \operatorname{cl} C(\varepsilon)$ is closed and the hypotheses of Lemma 4.2.13 are fulfilled. If $y_0 \in \operatorname{Max}(\Xi_{C,\varepsilon})$, then there exists $x_0 \in \operatorname{Be}(f,S)$ such that $y_0 - f(x_0) \in T^1_{C,\varepsilon}$. Additionally, if $f(S)+T^1_{C,\varepsilon} \subset f(S)$, then there exists $x_0 \in \operatorname{Be}(f,S,C,\varepsilon)$ such that $y_0 = f(x_0)$.

*Proof.* Since $y_0 \in \operatorname{Max}(\Xi_{C,\varepsilon})$, by Theorem 4.2.14$(b)$ there exist $x_0 \in S$ and $d_0 \in T^1_{C,\varepsilon}$ such that $y_0 = f(x_0)+d_0 \in \Xi_{C,\varepsilon}$. Then, it follows that $x_0 \in \operatorname{Be}(f,S)$. Indeed, from the definition of $\Xi_{C,\varepsilon}$ there exist $\mu \in D^{s+} \cap C^{\tau+}$ and $\lambda \in K^+$ such that, for all $x \in X$,

$$(\mu \circ f)(x_0) + \langle \mu, d_0\rangle = \langle \mu, y_0\rangle \leq (\mu \circ f)(x) + (\lambda \circ g)(x) + \varepsilon\tau_C(\mu). \tag{4.6}$$

Since $d_0 \in \operatorname{cl}(C(\varepsilon))$, it follows that $\varepsilon\tau_C(\mu) \leq \langle \mu, d_0\rangle$, and statement (4.6) implies that

$$(\mu \circ f)(x_0) \leq (\mu \circ f)(x),\ \forall x \in S,$$

so applying Theorem 2.2.8 we deduce that $x_0 \in \operatorname{Be}(f,S)$.

Finally, if $f(S) + T^1_{C,\varepsilon} \subset f(S)$, then there exists $x_0' \in S$ such that $y_0 = f(x_0')$ and by (4.6) and Theorem 2.2.8 we conclude that $x_0' \in \operatorname{Be}(f,S,C,\varepsilon)$. □

In the next theorem we extend the converse strong duality result stated above to the approximate case. The following lemma is necessary.

**Lemma 4.2.17.** For each $\delta \geq 0$, it fulfills that

$$\operatorname{Max}(\Xi_{C,\varepsilon+\delta}, \operatorname{cl} C + D\backslash\{0\}, \varepsilon) \subset Y \backslash \operatorname{int} \Xi_{C,\delta}.$$

*Proof.* Let $y_0 \in \operatorname{Max}(\Xi_{C,\varepsilon+\delta}, \operatorname{cl} C + D\backslash\{0\}, \varepsilon)$ and suppose that $y_0 \in \operatorname{int} \Xi_{C,\delta}$. Then there exist $\mu \in D^{s+} \cap C^{\tau+}$ and $\lambda \in K^+$ such that

$$\langle \mu, y_0\rangle < \inf_{x\in X}\{(\mu \circ f + \lambda \circ g)(x)\} + \delta\tau_C(\mu).$$

Let

$$\alpha := \inf_{x\in X}\{(\mu\circ f+\lambda\circ g)(x)\}+\delta\tau_C(\mu)-\langle\mu,y_0\rangle.$$

As $\alpha>0$, there exists $d\in(\operatorname{cl}C+D\backslash\{0\})(\varepsilon)$ such that

$$\langle\mu,d\rangle<\alpha+\varepsilon\tau_{\operatorname{cl}C+D\backslash\{0\}}(\mu)=\alpha+\varepsilon\tau_C(\mu)$$

and so

$$\begin{aligned}\langle\mu,y_0+d\rangle&<\langle\mu,y_0\rangle+\alpha+\varepsilon\tau_C(\mu)\\&=\inf_{x\in X}\{(\mu\circ f+\lambda\circ g)(x)\}+(\varepsilon+\delta)\tau_C(\mu).\end{aligned}$$

Then $y_0+d\in\Xi_{C,\varepsilon+\delta}$ and we obtain that $y_0\notin\operatorname{Max}(\Xi_{C,\varepsilon+\delta},\operatorname{cl}C+D\backslash\{0\},\varepsilon)$, which is a contradiction. Therefore, $y_0\notin\operatorname{int}\Xi_{C,\delta}$. □

**Theorem 4.2.18.** Let $\delta\geq0$. Suppose that $C$ is convex, coradiant and $C+D=C$, $\Xi_{C,\delta}\neq\emptyset$, $S$ is convex, $f$ is $D$-convex on $S$, $f(S)+\operatorname{cl}C(\delta)$ is closed, and problem $(\mathcal{P}_\mu)$ is normal for each $\mu\in D^{s+}\cap C^{\tau+}\cap\mathbb{B}$. If $y_0\in\operatorname{Max}(\Xi_{C,\varepsilon+\delta},\operatorname{cl}C+D\backslash\{0\},\varepsilon)$, then there exists $x_0\in\operatorname{Be}(f,S,C,\varepsilon)$ such that $y_0-f(x_0)\in\operatorname{cl}(C(\delta))\cap(\operatorname{cl}(C(\varepsilon+\delta))+D\backslash\{0\})^c$.

*Proof.* By Lemma 4.2.17 we see that $y_0\notin\operatorname{int}\Xi_{C,\delta}$. On the other hand, by Lemma 4.2.13 we deduce that

$$Y\backslash(f(S)+\operatorname{cl}C(\delta))=Y\backslash\operatorname{cl}(f(S)+C(\delta))\subset\operatorname{int}\Xi_{C,\delta},$$

since $f(S)+\operatorname{cl}C(\delta)$ is closed, and then by Theorem 4.2.4 we obtain that $y_0\in f(S)+\operatorname{cl}(C(\delta))\cap(\operatorname{cl}(C(\varepsilon+\delta))+D\backslash\{0\})^c$. In particular, there exist $x_0\in S$ and $d_0\in\operatorname{cl}C(\delta)$ such that $y_0=f(x_0)+d_0$. As $y_0\in\Xi_{C,\varepsilon+\delta}$, there exist $\mu\in D^{s+}\cap C^{\tau+}\cap\mathbb{B}$ and $\lambda\in K^+$ verifying, for all $x\in X$,

$$(\mu\circ f)(x_0)+\langle\mu,d_0\rangle=\langle\mu,y_0\rangle\leq(\mu\circ f)(x)+(\lambda\circ g)(x)+(\varepsilon+\delta)\tau_C(\mu).\quad(4.7)$$

Since $d_0\in\operatorname{cl}(C(\delta))$, $\delta\tau_C(\mu)\leq\langle\mu,d_0\rangle$ and statement (4.7) implies that

$$(\mu\circ f)(x_0)\leq(\mu\circ f)(x)+\varepsilon\tau_C(\mu),\ \forall x\in S,$$

so by applying Theorem 2.2.8 we deduce that $x_0\in\operatorname{Be}(f,S,C,\varepsilon)$, as we want to prove. □

The following result follows by applying Theorem 4.2.18 to $\delta = 0$.

**Corollary 4.2.19.** Suppose that $C$ is convex, coradiant and such that $C+D = C$, $\Xi_{C,0} \neq \emptyset$, $S$ is convex, $f$ is $D$-convex on $S$, $f(S) + \operatorname{cl}\operatorname{cone} C$ is closed, and problem $(\mathcal{P}_\mu)$ is normal for each $\mu \in D^{s+} \cap C^{\tau+} \cap \mathbb{B}$. If $y_0 \in \operatorname{Max}(\Xi_{C,\varepsilon}, \operatorname{cl} C + D\backslash\{0\}, \varepsilon)$, then there exists $x_0 \in \operatorname{Be}(f, S, C, \varepsilon)$ such that $y_0 - f(x_0) \in T^3_{C,\varepsilon}$.

The next two corollaries are direct consequences of the previous corollary and Theorem 4.2.6.

**Corollary 4.2.20.** Suppose that $C \subset D$, $C$ is convex, $C + D = C$, $\Xi_{C,\varepsilon} \neq \emptyset$, $S$ is convex, $f$ is $D$-convex on $S$, $f(S) + D$ is closed, and problem $(\mathcal{P}_\mu)$ is normal for each $\mu \in D^{s+} \cap \mathbb{B}$. If $y_0 \in \operatorname{Max}(\Xi_{C,\varepsilon}, \operatorname{cl} C + D\backslash\{0\}, \varepsilon)$, then there exists $x_0 \in \operatorname{Be}(f, S, C, \varepsilon)$ such that $y_0 - f(x_0) \in T^2_{C,\varepsilon}$. If additionally $f(S) + T^2_{C,\varepsilon} \subset f(S)$, then there exists $x_0 \in \operatorname{Be}(f, S, C, \varepsilon)$ such that $y_0 = f(x_0)$.

**Corollary 4.2.21.** Suppose that $C$ is convex, $C + D = C$, $\Xi_{C,0} \neq \emptyset$, $S$ is convex, $f$ is $D$-convex on $S$ and problem $(\mathcal{P}_\mu)$ is normal for each $\mu \in D^{s+} \cap C^{\tau+} \cap \mathbb{B}$.

$(a)$ If $C$ is coradiant and $f(S) + \operatorname{cl}\operatorname{cone} C$ is closed, then

$$\begin{aligned}(f(S) + T^2_{C,\varepsilon}) \cap \Xi_{C,\varepsilon} &= (f(\operatorname{Be}(f, S, C, \varepsilon)) + T^2_{C,\varepsilon}) \cap \Xi_{C,\varepsilon} \\ &\subset \operatorname{Max}(\Xi_{C,\varepsilon}, \operatorname{cl} C + D\backslash\{0\}, \varepsilon) \\ &\subset (f(\operatorname{Be}(f, S, C, \varepsilon)) + T^3_{C,\varepsilon}) \cap \Xi_{C,\varepsilon} \\ &\subset (f(S) + T^3_{C,\varepsilon}) \cap \Xi_{C,\varepsilon}.\end{aligned}$$

$(b)$ If $C \subset D$ and $f(S) + D$ is closed, then

$$\begin{aligned}(f(S) + T^2_{C,\varepsilon}) \cap \Xi_{C,\varepsilon} &= (f(\operatorname{Be}(f, S, C, \varepsilon)) + T^2_{C,\varepsilon}) \cap \Xi_{C,\varepsilon} \\ &= \operatorname{Max}(\Xi_{C,\varepsilon}, \operatorname{cl} C + D\backslash\{0\}, \varepsilon).\end{aligned}$$

**Remark 4.2.22.** Let us observe that for suitable sets $C \in \mathcal{F}_Y$, the sets $T^i_{C,\varepsilon}$, $i = 1, 2, 3$, can be bounded. Indeed, if $(Y, \|\ \|)$ is normed and the norm $\|\ \|$ is $D$-monotone on $D$ (i.e., $0 \leq_D d_1 \leq_D d_2 \Longrightarrow \|d_1\| \leq \|d_2\|$), denoting $\mathcal{B} \subset Y$ the unit open ball, the set $C = \mathcal{B}^c \cap D$ satisfies that $C = C + D$,

$$\operatorname{cl}(C(\varepsilon)) + D\backslash\{0\} = ((\varepsilon\mathcal{B})^c \cap D) + D\backslash\{0\} = (\operatorname{cl}(\varepsilon\mathcal{B}))^c \cap D$$

and

$$T^1_{C,\varepsilon} \subset T^2_{C,\varepsilon} = T^3_{C,\varepsilon} = \mathrm{cl}(\varepsilon\mathcal{B}) \cap D,$$

so the sets $T^i_{C,\varepsilon}$ are bounded. As a consequence, the gap $\|y_0 - f(x_0)\|$ resulting from Theorems 4.2.6 and 4.2.14, and Corollaries 4.2.19-4.2.21 satisfies that $\|y_0 - f(x_0)\| \leq \varepsilon$, so it is also bounded. Moreover, if $\varepsilon$ tends to zero this gap becomes null.

Next, we give an illustrative example with the purpose of clarifying the stated results and show the importance of choosing a convenient set $C$ for obtaining an appropriate set of Benson $(C, \varepsilon)$-proper solutions of problem $(\mathcal{P}_S)$.

**Example 4.2.23.** Consider $X = Y = \mathbb{R}^2$, $D = \mathbb{R}^2_+$, $Z = \mathbb{R}^3$ and $K = \mathbb{R}^3_+$. Let $\varepsilon \geq 0$, $q = (1, 1)$, $C_q = q + \mathbb{R}^2_+$, $C = \mathcal{B}^c \cap \mathbb{R}^2_+$, where $\mathcal{B}$ is the unit open ball with respect to $\|\cdot\|_1$, $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the identity mapping and $g : \mathbb{R}^2 \to \mathbb{R}^3$ be the mapping defined as

$$g(x_1, x_2) = (-x_1 - 1, -x_2 - 1, -x_1 - x_2 - 1).$$

It follows that $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq -1, x_2 \geq -1, x_1 + x_2 \geq -1\}$, and it is easy to see that $\mathrm{E}(f, S) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = -1, -1 \leq x_1 \leq 0\}$. We are going to obtain the sets $\mathrm{Be}(f, S, C_q, \varepsilon)$ and $\mathrm{Be}(f, S, C, \varepsilon)$ via strong duality from the approximate Lagrange dual problems $(\mathcal{D}_{C_q,\varepsilon})$ and $(\mathcal{D}_{C,\varepsilon})$, respectively.

By Theorem 4.2.4 we deduce that

$$\begin{aligned}\Xi_{C_q,\varepsilon} &\subset \mathbb{R}^2\backslash(f(S) + (C_q(\varepsilon)\backslash\{\varepsilon q\})) \\ &= \mathbb{R}^2\backslash\{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq -1 + \varepsilon, y_2 \geq -1 + \varepsilon, y_1 + y_2 > -1 + 2\varepsilon\}. \quad (4.8)\end{aligned}$$

On the other hand, we have that

$$E := \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 = -1 + 2\varepsilon, -1 + \varepsilon \leq y_1 \leq \varepsilon\} \subset \Xi_{C_q,\varepsilon}. \quad (4.9)$$

Indeed, let $y = (y_1, y_2) \in E$ and consider $\mu = (1, 1) \in \operatorname{int} \mathbb{R}^2_+ \cap C_q^{\tau+}$ and $\lambda = (0, 0, 1) \in \mathbb{R}^3_+$. Then, $\langle \mu, y \rangle = y_1 + y_2 = -1 + 2\varepsilon$, and for all $(x_1, x_2) \in \mathbb{R}^2$ it follows that

$$(\mu \circ f)(x_1, x_2) + (\lambda \circ g)(x_1, x_2) + \varepsilon\tau_{C_q}(\mu) = -1 + 2\varepsilon.$$

Thus, $\langle \mu, y\rangle \leq (\mu \circ f)(x_1, x_2) + (\lambda \circ g)(x_1, x_2) + \varepsilon \tau_{C_q}(\mu)$, for all $(x_1, x_2) \in \mathbb{R}^2$, concluding that $y \in \Xi_{C_q,\varepsilon}$. Also, it is clear that the hypotheses of Lemma 4.2.13 are satisfied (observe that Slater constraint qualification holds, and then by Remark 4.2.9 we deduce that problem $(\mathcal{P}_\mu)$ is stable for all $\mu \in \mathbb{R}^2_+$). Thus, applying this lemma we obtain that

$$\begin{aligned} &\mathbb{R}^2 \backslash \operatorname{cl}(f(S) + C_q(\varepsilon)) \\ &= \mathbb{R}^2 \backslash \{(y_1, y_2) : y_1 \geq -1 + \varepsilon, y_2 \geq -1 + \varepsilon, y_1 + y_2 \geq -1 + 2\varepsilon\} \subset \Xi_{C_q,\varepsilon}. \end{aligned} \tag{4.10}$$

Hence, by (4.8)-(4.10) we conclude that

$$\begin{aligned} \Xi_{C_q,\varepsilon} &= \mathbb{R}^2 \backslash \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq -1 + \varepsilon, y_2 \geq -1 + \varepsilon, y_1 + y_2 > -1 + 2\varepsilon\} \\ &= \{(y_1, y_2) \in \mathbb{R}^2 : y_1 < -1 + \varepsilon\} \cup \{(y_1, y_2) \in \mathbb{R}^2 : y_2 < -1 + \varepsilon\} \\ &\cup \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 \leq -1 + 2\varepsilon\}. \end{aligned}$$

In this way, by Corollary 4.2.21($b$) and Theorem 4.2.7 we have that

$$\operatorname{Be}(f, S, C_q, \varepsilon) = \operatorname{Max}(\Xi_{C_q,\varepsilon}, \operatorname{cl} C_q + D \backslash \{0\}, \varepsilon) = \bigcup_{0 \leq \delta < \varepsilon} (\operatorname{bd} S + \delta q) \cup E.$$

Observe that this set is unbounded and so it contains approximate proper solutions as far as one wants from the efficient set and for all $\varepsilon > 0$. That shows that the set of Benson $(C_q, \varepsilon)$-proper solutions (see also Definition 1.2.15) is not suitable for approximating the efficient set.

Now, if we consider the set $C$ instead of $C_q$, following an analogous procedure as above, we obtain that

$$\begin{aligned} \Xi_{C,\varepsilon} &= \mathbb{R}^2 \backslash \{(y_1, y_2) \in \mathbb{R}^2 : y_1 \geq -1, y_2 \geq -1, y_1 + y_2 > -1 + \varepsilon\} \\ &= \{(y_1, y_2) \in \mathbb{R}^2 : y_1 < -1\} \cup \{(y_1, y_2) \in \mathbb{R}^2 : y_2 < -1\} \\ &\cup \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 \leq -1 + \varepsilon\} \end{aligned}$$

and by Corollary 4.2.21($b$) and Theorem 4.2.7 we deduce that

$$\begin{aligned} \operatorname{Be}(f, S, C, \varepsilon) &= \operatorname{Max}(\Xi_{C,\varepsilon}, \operatorname{cl} C + D \backslash \{0\}, \varepsilon) \\ &= \{(y_1, y_2) \in \mathbb{R}^2 : y_1, y_2 \geq -1, -1 \leq y_1 + y_2 \leq -1 + \varepsilon\}. \end{aligned}$$

Therefore we obtain in this case a bounded set of approximate Benson proper solutions of $(\mathcal{P}_S)$, where each of these solutions is close to the efficient set, and it tends to the efficient set when $\varepsilon$ tends to zero.

## 4.3 Set-valued Lagrange approach

In this section, we define a new type of approximate Lagrange dual problem by means of Benson $(C,\varepsilon)$-proper solutions of the set-valued Lagrangian $\Phi_{B_\mu}$ (see Definition 3.3.1). We give weak and strong duality results and we establish relationships between this dual problem and the dual problem studied in Section 4.2.

Throughout we consider again a set $C \in \mathcal{F}_Y$, $\varepsilon \geq 0$ and the following set-valued $(C,\varepsilon)$-dual mapping $\Theta_{C,\varepsilon} : (D^{s+} \cap C^{\tau+}) \times K^+ \to 2^Y$:

$$\begin{aligned}\Theta_{C,\varepsilon}(\mu,\lambda) &:= \{y \in \Phi_{B_\mu}(x,\lambda) : x \in \mathrm{Be}(\Phi_{B_\mu}(\cdot,\lambda), X, C, \varepsilon)\}\\ &= \bigcup_{x \in \mathrm{Be}(\Phi_{B_\mu}(\cdot,\lambda),X,C,\varepsilon)} \Phi_{B_\mu}(x,\lambda), \quad \forall\, \mu \in D^{s+} \cap C^{\tau+}, \forall\, \lambda \in K^+.\end{aligned}$$

If $C = D$ or $\varepsilon = 0$ and $\operatorname{cone} C = D$, then we denote the $(C,\varepsilon)$-dual mapping by $\Theta$.

**Remark 4.3.1.** $(a)$ In the scalar case (i.e., $Y = \mathbb{R}$ and $D = \mathbb{R}_+$) it is not hard to check that $C \subset \mathbb{R}_+$ and $D^{s+} \cap C^{\tau+} = (0,+\infty)$, for all $C \in \mathcal{F}_Y$, and for all $\mu > 0$ and $\lambda \in K^+$ it follows that

$$\Theta_{C,\varepsilon}(\mu,\lambda) = \{f(x) + \langle \lambda, g(x)\rangle/\mu : x \in \varepsilon\tau_C(1)\text{-argmin}_X\{f(z) + \langle \lambda, g(z)\rangle/\mu\}\}.$$

$(b)$ In [44, Section 4] it is showed that the sets $\mathrm{Be}(\Phi_{B_\mu}(\cdot,\lambda), X, C, \varepsilon)$ and $\mathrm{Be}(f, S, C, \varepsilon)$ are closely related to each other. As a consequence we will obtain through the $(C,\varepsilon)$-dual mapping $\Theta_{C,\varepsilon}$ weak and strong duality results associated with Benson $(C,\varepsilon)$-proper solutions of the primal problem $(\mathcal{P}_S)$.

$(c)$ The $(C,\varepsilon)$-dual mapping $\Theta_{C,\varepsilon}$ is motivated by the (exact) dual mapping defined by Li in [72, Definition 7.1] in the setting of vector optimization problems with set-valued mappings. To be precise, Li considered (in the single-valued case) the following dual mapping $\bar{\Theta} : \mathcal{L}_+(Z,Y) \to 2^Y$, (for the definition of $\mathcal{L}_+(Z,Y)$ see (3.8)):

$$\bar{\Theta}(T) := \{f(x) + T(g(x)) : x \in \mathrm{Be}(f + T \circ g, X)\}, \quad \forall T \in \mathcal{L}_+(Z,Y).$$

Then, for each $q \in D\backslash\{0\}$ and $\lambda \in K^+$, the linear mapping $T_{q,\lambda} \in \mathcal{L}_+(Z,Y)$ given by $T_{q,\lambda}(z) = \langle \lambda, z \rangle q$, for all $z \in Z$, satisfies that

$$\bar{\Theta}(T_{q,\lambda}) = \{\Phi_{\{q\}}(x,\lambda) : x \in \mathrm{Be}(\Phi_{\{q\}}(\cdot,\lambda), X)\}.$$

Therefore, $\Theta_{C,\varepsilon}$ generalizes $\bar{\Theta}$ to a set-valued Lagrangian and also to approximate solutions.

The $(C,\varepsilon)$-dual problem associated with $\Theta_{C,\varepsilon}$ is defined as follows:

$$\text{Maximize } \ \Xi^{\mathrm{Be}}_{C,\varepsilon} := \bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \bigcup_{\lambda \in K^+} \Theta_{C,\varepsilon}(\mu,\lambda). \tag{$\mathcal{D}^{\mathrm{Be}}_{C,\varepsilon}$}$$

If $C = D$ or $\varepsilon = 0$ and $\mathrm{cone}\, C = D$ we denote the dual set $\Xi^{\mathrm{Be}}_{C,\varepsilon}$ by $\Xi^{\mathrm{Be}}$ and problem $(\mathcal{D}^{\mathrm{Be}}_{C,\varepsilon})$ by $(\mathcal{D}^{\mathrm{Be}})$. When problem $(\mathcal{P}_S)$ is scalar, the $(C,\varepsilon)$-dual problem $(\mathcal{D}^{\mathrm{Be}}_{C,\varepsilon})$ is

$$\text{Maximize } \bigcup_{\lambda \in K^+} \{f(x) + \langle \lambda, g(x) \rangle : x \in \varepsilon\tau_C(1)\text{-}\mathrm{argmin}_X\{f(z) + \langle \lambda, g(z) \rangle\}\} \tag{4.11}$$

The following lemma is necessary in order to prove some results of this section.

**Lemma 4.3.2.** Let $\emptyset \neq H \subset Y$ be a compact set and $\emptyset \neq F \subset Y$. Suppose that $\mathrm{cl\,cone}\, F \cap (-H) = \emptyset$. Then

$$\mathrm{cl\,cone}(F + \mathrm{cone}\, H) = \mathrm{cl\,cone}\, F + \mathrm{cone}\, H.$$

*Proof.* It is easy to check that

$$\mathrm{cl\,cone}\, F + \mathrm{cone}\, H \subset \mathrm{cl\,cone}(F + \mathrm{cone}\, H).$$

Reciprocally, let $y \in \mathrm{cl\,cone}(F + \mathrm{cone}\, H)$. Then, there exist nets $(\alpha_i) \subset \mathbb{R}_+$, $(y_i) \subset F$ and $(v_i) \subset \mathrm{cone}\, H$ such that

$$\alpha_i(y_i + v_i) \to y. \tag{4.12}$$

It is clear that $(\alpha_i v_i) \subset \operatorname{cone} H$. Thus, there exist $(\beta_i) \subset \mathbb{R}_+$ and $(h_i) \subset H$ such that $\alpha_i v_i = \beta_i h_i$, for all $i$. Moreover, as $H$ is compact, we can suppose that $(h_i)$ converges to a point $h \in H$.

Let us check that $(\beta_i)$ is bounded. Indeed, suppose on the contrary that $(\beta_i)$ is not bounded. In this case we can assume without losing generality that $\beta_i \to +\infty$. By statement (4.12) it follows that

$$0 = \lim_i (\alpha_i/\beta_i)(y_i + v_i) = \lim_i (\alpha_i/\beta_i) y_i + \lim_i h_i = \lim_i (\alpha_i/\beta_i) y_i + h$$

and then $0 \in \operatorname{cl}\operatorname{cone} F + H$, which is a contradiction, since $\operatorname{cl}\operatorname{cone} F \cap (-H) = \emptyset$.

As $(\beta_i)$ is bounded, we can assume without losing generality that $(\beta_i)$ is convergent, so let $\beta := \lim_i \beta_i$. Hence, by statement (4.12) we have that

$$y = \lim_i (\alpha_i(y_i + v_i)) = \lim_i \alpha_i y_i + \lim_i \beta_i h_i = \lim_i \alpha_i y_i + \beta h$$

and then $y \in \operatorname{cl}\operatorname{cone} F + \operatorname{cone} H$, which finishes the proof. □

In the following proposition we establish several basic properties of the set $\Xi^{\mathrm{Be}}_{C,\varepsilon}$.

**Proposition 4.3.3.** One has

($a$) If $C \subset D$ then $\Xi^{\mathrm{Be}} \subset \Xi^{\mathrm{Be}}_{C,\varepsilon}$.

($b$) Let $\delta \geq 0$ and $C' \in \mathcal{F}_Y$. If $C(\varepsilon) \subset \operatorname{cl}(C'(\delta))$, then $\Xi^{\mathrm{Be}}_{C',\delta} \subset \Xi^{\mathrm{Be}}_{C,\varepsilon}$.

($c$) If $C$ is coradiant, $\Xi^{\mathrm{Be}}_{C,\delta} \subset \Xi^{\mathrm{Be}}_{C,\varepsilon}$, for all $0 \leq \delta \leq \varepsilon$. If additionally $C$ is convex, then $\Xi^{\mathrm{Be}}_{C,\varepsilon} = \Xi^{\mathrm{Be}}_{C+C(0),\varepsilon}$.

($d$) $\Xi^{\mathrm{Be}}_{C+C',\varepsilon} = \Xi^{\mathrm{Be}}_{\operatorname{cl} C+C',\varepsilon}$, for all $C' \subset D$.

($e$) Let $H \subset D \backslash \{0\}$ be a nonempty compact set. Then $\Xi^{\mathrm{Be}}_{C,\varepsilon} = \Xi^{\mathrm{Be}}_{C+\operatorname{cone} H,\varepsilon}$.

*Proof.* We only prove part ($e$), since parts ($a$)-($d$) are direct consequences of the definitions.

It is clear that $C + \operatorname{cone} H \in \mathcal{F}_Y$, since $C \in \mathcal{F}_Y$ and $D^{s+} \cap C^{\tau+} \subset D^{s+} \cap (C + \operatorname{cone} H)^{\tau+}$. Thus, as $C(\varepsilon) \subset (C + \operatorname{cone} H)(\varepsilon)$, by part ($b$) we deduce that $\Xi^{\mathrm{Be}}_{C+\operatorname{cone} H,\varepsilon} \subset \Xi^{\mathrm{Be}}_{C,\varepsilon}$.

Reciprocally, let $y \in \Xi^{\mathrm{Be}}_{C,\varepsilon}$. Then there exists $\mu \in D^{s+} \cap C^{\tau+}$, $\lambda \in K^+$ and $x \in \mathrm{Be}(\Phi_{B_\mu}(\cdot,\lambda), X, C, \varepsilon)$ such that $y \in \Phi_{B_\mu}(x,\lambda)$. If $x \in \mathrm{Be}(\Phi_{B_\mu}(\cdot,\lambda), X, C + \operatorname{cone} H, \varepsilon)$, then the result follows. Suppose on the contrary that the point $x \notin \mathrm{Be}(\Phi_{B_\mu}(\cdot,\lambda), X, C + \operatorname{cone} H, \varepsilon)$, i.e.,

$$F_1 := \operatorname{cl}\operatorname{cone}(\Phi_{B_\mu}(X,\lambda) + (C + \operatorname{cone} H)(\varepsilon) - \Phi_{B_\mu}(x,\lambda)) \cap (-D) \neq \{0\}$$

and define $F_2 := \Phi_{B_\mu}(X,\lambda) + C(\varepsilon) - \Phi_{B_\mu}(x,\lambda)$. It is easy to check that

$$F_1 = \operatorname{cl}\operatorname{cone}(F_2 + \operatorname{cone} H) \cap (-D).$$

On the other hand, as $x \in \mathrm{Be}(\Phi_{B_\mu}(\cdot,\lambda), X, C, \varepsilon)$ we have that

$$\operatorname{cl}\operatorname{cone} F_2 \cap (-H) \subset \operatorname{cl}\operatorname{cone} F_2 \cap (-D\backslash\{0\}) = \emptyset \tag{4.13}$$

and by Lemma 4.3.2 we deduce that

$$\begin{aligned}\{0\} &\neq \operatorname{cl}\operatorname{cone}(F_2 + \operatorname{cone} H) \cap (-D)\\ &= (\operatorname{cl}\operatorname{cone} F_2 + \operatorname{cone} H) \cap (-D)\end{aligned}$$

and since $D$ is pointed and $\operatorname{cone} H \subset D$ we obtain that

$$\emptyset \neq \operatorname{cl}\operatorname{cone} F_2 \cap (-\operatorname{cone} H - D\backslash\{0\}) \subset \operatorname{cl}\operatorname{cone} F_2 \cap (-D\backslash\{0\}),$$

which is contrary to (4.13) and the proof is complete. □

Next, we give two weak duality results.

**Theorem 4.3.4.** Let $\mu \in D^{s+} \cap C^{\tau+}$ and $\lambda \in K^+$. If $x_0 \in \mathrm{Be}(\Phi_{B_\mu}(\cdot,\lambda), X, C, \varepsilon)$, then

$$\operatorname{cl}\operatorname{cone}(\Phi_{B_\mu}(x_0,\lambda) - C(\varepsilon) - f(x)) \cap D = \{0\}, \quad \forall\, x \in S. \tag{4.14}$$

*Proof.* As $x_0 \in \mathrm{Be}(\Phi_{B_\mu}(\cdot,\lambda), X, C, \varepsilon)$, it follows that

$$\operatorname{cl}\operatorname{cone}(\Phi_{B_\mu}(X,\lambda) + C(\varepsilon) - \Phi_{B_\mu}(x_0,\lambda)) \cap (-D) = \{0\}. \tag{4.15}$$

Suppose on the contrary that there exists $x \in S$ such that

$$F_1 := \operatorname{cl}\operatorname{cone}(f(x) + C(\varepsilon) - \Phi_{B_\mu}(x_0,\lambda)) \cap (-D) \neq \{0\}$$

and define $F_2 := f(x) + \lambda(g(x))b + C(\varepsilon) - \Phi_{B_\mu}(x_0, \lambda)$, where $b$ is an arbitrary point of $B_\mu$. By (4.15) it follows that

$$\operatorname{cl}\operatorname{cone} F_2 \cap (-\{b\}) \subset \operatorname{cl}\operatorname{cone} F_2 \cap (-D\backslash\{0\}) = \emptyset. \tag{4.16}$$

On the other hand, as $x \in S$, we have that $-\lambda(g(x))b \in \operatorname{cone}\{b\}$. Then, by applying Lemma 4.3.2 with $H = \{b\}$ we obtain

$$\begin{aligned}\{0\} \neq F_1 &= \operatorname{cl}\operatorname{cone}(F_2 - \lambda(g(x))b) \cap (-D)\\ &\subset \operatorname{cl}\operatorname{cone}(F_2 + \operatorname{cone}\{b\}) \cap (-D)\\ &= (\operatorname{cl}\operatorname{cone} F_2 + \operatorname{cone}\{b\}) \cap (-D)\end{aligned}$$

and as $D$ is pointed we deduce that

$$\begin{aligned}\emptyset &\neq \operatorname{cl}\operatorname{cone} F_2 \cap (-\operatorname{cone}\{b\} - D\backslash\{0\})\\ &\subset \operatorname{cl}\operatorname{cone} F_2 \cap (-D\backslash\{0\}),\end{aligned}$$

which is contrary to (4.16) and the proof finishes. □

**Corollary 4.3.5.** It follows that $(\Xi^{\mathrm{Be}}_{C,\varepsilon} - f(S)) \cap (\operatorname{cl}(C(\varepsilon)) + D\backslash\{0\}) = \emptyset$.

*Proof.* Consider $y \in \Xi^{\mathrm{Be}}_{C,\varepsilon}$ and $x \in S$. There exist $\mu \in D^{s+} \cap C^{\tau+}$, $\lambda \in K^+$ and $x_0 \in \mathrm{Be}(\Phi_{B_\mu}(\cdot, \lambda), X, C, \varepsilon)$ such that $y \in \Phi_{B_\mu}(x_0, \lambda)$. By (4.14) we see that

$$(y - \operatorname{cl}(C(\varepsilon)) - f(x)) \cap (D\backslash\{0\}) = \emptyset,$$

i.e., $y - f(x) \notin \operatorname{cl}(C(\varepsilon)) + D\backslash\{0\}$ and the proof is complete. □

The following theorem gives a sufficient condition for Benson $(C, \varepsilon)$-proper solutions of problem $(\mathcal{P}_S)$ and approximate maximal points of $(\mathcal{D}^{\mathrm{Be}}_{C,\varepsilon})$.

**Theorem 4.3.6.** Let $x \in S$ and $y \in \Xi^{\mathrm{Be}}_{C,\varepsilon}$ such that $y - f(x) \in T^2_{C,\varepsilon}$. Then $x \in \mathrm{Be}(f, S, C, \varepsilon)$ and $y \in \mathrm{Max}(\Xi^{\mathrm{Be}}_{C,\varepsilon}, \operatorname{cl} C + D\backslash\{0\}, \varepsilon)$.

*Proof.* Since $y \in \Xi^{\mathrm{Be}}_{C,\varepsilon}$, there exist $\mu \in D^{s+} \cap C^{\tau+}$, $\lambda \in K^+$ and $x_0 \in \mathrm{Be}(\Phi_{B_\mu}(\cdot, \lambda), X, C, \varepsilon)$ such that $y \in \Phi_{B_\mu}(x_0, \lambda)$. Then,

$$\operatorname{cl}\operatorname{cone}(\Phi_{B_\mu}(X, \lambda) + C(\varepsilon) - y) \cap (-D) = \{0\}. \tag{4.17}$$

As $y - f(x) \in T^2_{C,\varepsilon}$ there exists $d \in D$ such that $y = f(x) + d$. Let us prove that

$$\operatorname{cl}\operatorname{cone}(\Phi_{B_\mu}(X,\lambda) + C(\varepsilon) - f(x)) \cap (-D) = \{0\}. \tag{4.18}$$

Reasoning by contradiction suppose that $d \neq 0$ and

$$\operatorname{cl}\operatorname{cone}(\Phi_{B_\mu}(X,\lambda) + C(\varepsilon) - f(x)) \cap (-D) \neq \{0\}.$$

Let us define $F_1 := \Phi_{B_\mu}(X,\lambda) + C(\varepsilon) - y$. By Lemma 4.3.2 we deduce that

$$\begin{aligned}\{0\} &\neq \operatorname{cl}\operatorname{cone}(F_1 + d) \cap (-D)\\ &\subset \operatorname{cl}\operatorname{cone}(F_1 + \operatorname{cone}\{d\}) \cap (-D)\\ &= (\operatorname{cl}\operatorname{cone} F_1 + \operatorname{cone}\{d\}) \cap (-D).\end{aligned}$$

Therefore, $\operatorname{cl}\operatorname{cone} F_1 \cap (-D\backslash\{0\}) \neq \emptyset$, which is contrary to (4.17), and so (4.18) is true.

Suppose by contradiction that $x \notin \operatorname{Be}(f,S,C,\varepsilon)$ and consider an arbitrary $b \in B_\mu$. Then

$$\begin{aligned}\{0\} &\neq \operatorname{cl}\operatorname{cone}(f(S) + C(\varepsilon) - f(x)) \cap (-D)\\ &\subset \operatorname{cl}\operatorname{cone}(\Phi_{B_\mu}(X,\lambda) + \operatorname{cone}\{b\} + C(\varepsilon) - f(x)) \cap (-D).\end{aligned} \tag{4.19}$$

Let us define $F := \Phi_{B_\mu}(X,\lambda) + C(\varepsilon) - f(x)$. By (4.18) we have that

$$\operatorname{cl}\operatorname{cone} F \cap (-\{b\}) \subset \operatorname{cl}\operatorname{cone} F \cap (-D\backslash\{0\}) = \emptyset \tag{4.20}$$

and so by applying Lemma 4.3.2 to (4.19) it follows that

$$\begin{aligned}\{0\} &\neq \operatorname{cl}\operatorname{cone}(F + \operatorname{cone}\{b\}) \cap (-D)\\ &= (\operatorname{cl}\operatorname{cone} F + \operatorname{cone}\{b\}) \cap (-D).\end{aligned}$$

Then, $\operatorname{cl}\operatorname{cone} F \cap (-D\backslash\{0\}) \neq \emptyset$, that is contrary to (4.20). Thus, $x \in \operatorname{Be}(f,S,C,\varepsilon)$.

Finally, by Corollary 4.3.5 and Remark 4.2.5$(a)$ we obtain that

$$\begin{aligned}\Xi^{\mathrm{Be}}_{C,\varepsilon} - y &= \Xi^{\mathrm{Be}}_{C,\varepsilon} - f(x) - d\\ &\subset (\operatorname{cl}(C(\varepsilon)) + D\backslash\{0\} - d)^c\\ &\subset (\operatorname{cl}(C(\varepsilon)) + D\backslash\{0\})^c\\ &\subset ((\operatorname{cl} C + D\backslash\{0\})_0(\varepsilon))^c,\end{aligned}$$

which implies that $y \in \operatorname{Max}(\Xi^{\mathrm{Be}}_{C,\varepsilon}, \operatorname{cl} C + D\backslash\{0\}, \varepsilon)$ and the proof finishes. $\square$

Next we provide strong duality results for Benson $(C,\varepsilon)$-proper solutions of problem $(\mathcal{P}_S)$ through (perhaps not feasible) approximate maximal points of the dual problem $(\mathcal{D}^{\mathrm{Be}}_{C,\varepsilon})$.

**Theorem 4.3.7.** Consider that $\operatorname{int} D^+ \neq \emptyset$, $x_0 \in S$, $f - f(x_0)$ is nearly $(C,\varepsilon)$-subconvexlike on $S$, $(f - f(x_0), g)$ is nearly $(C \times K, \varepsilon)$-subconvexlike on $X$ and the Slater constraint qualification holds. If $x_0 \in \mathrm{Be}(f, S, C, \varepsilon)$, then there exist $\mu \in D^{s+} \cap C^{\tau+}$ and $\lambda \in K^+$ such that

$(a)$ $-\varepsilon\tau_C(\mu) \leq (\lambda \circ g)(x_0) \leq 0$.

$(b)$ $\Phi_{B_\mu}(x_0, \lambda) \subset \Xi^{\mathrm{Be}}_{C,\varepsilon}$.

$(c)$ $(\Xi^{\mathrm{Be}}_{C,\varepsilon} - f(x_0)) \cap (\operatorname{cl} C + D\backslash\{0\})_0(\varepsilon) = \emptyset$.

If additionally $Y$ is normed, then $\mu$ can be chosen to be unitary.

*Proof.* By Theorem 3.2.2, there exist $\mu \in D^{s+} \cap C^{\tau+}$ and $\lambda \in K^+$ such that $x_0 \in \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_X(\mu \circ f + \lambda \circ g)$ and $-\varepsilon\tau_C(\mu) \leq (\lambda \circ g)(x_0) \leq 0$. Thus part $(a)$ is proved and it is clear that $\mu$ can be chosen such that $\|\mu\| = 1$ whenever $Y$ is normed (by considering $\mu/\|\mu\|$ and $\lambda/\|\mu\|$ instead of $\mu$ and $\lambda$ if it is necessary). Moreover, by Theorem 3.3.5 we obtain that $x_0 \in \mathrm{Be}(\Phi_{B_\mu}(\cdot, \lambda), X, C, \varepsilon)$ and part $(b)$ follows. Finally, by Corollary 4.3.5 and Remark 4.2.5$(a)$ we know that

$$(\Xi^{\mathrm{Be}}_{C,\varepsilon} - f(x_0)) \cap (\operatorname{cl} C + D\backslash\{0\})_0(\varepsilon) = \emptyset$$

and so part $(c)$ is proved and the proof finishes. □

**Theorem 4.3.8.** Consider that $\operatorname{int} D^+ \neq \emptyset$. Suppose that $C$ is convex, $C = C + D$, $S$ is convex, $f$ is $D$-convex on $S$ and problem $(\mathcal{P}_\mu)$ is stable for each $\mu \in D^{s+} \cap C^{\tau+}$. If $x_0 \in \mathrm{Be}(f, S, C, \varepsilon)$, then there exist $\mu \in D^{s+} \cap C^{\tau+}$ and $\lambda \in K^+$ such that conditions $(a)$-$(c)$ of Theorem 4.3.7 hold.

*Proof.* As $C$ is convex, $C = C + D$, $S$ is convex and $f$ is $D$-convex on $S$, by Remark 1.2.23 it follows that $f - f(x_0)$ is nearly $(C,\varepsilon)$-subconvexlike on $S$. Then, by Theorem 2.2.6 there exists $\mu \in D^{s+} \cap C^{\tau+}$ such that

$$(\mu \circ f)(x_0) \leq \inf_{x \in S}\{(\mu \circ f)(x)\} + \varepsilon\tau_C(\mu).$$

As $(\mathcal{P}_\mu)$ is stable, there exists $\lambda \in K^+$ verifying

$$(\mu \circ f)(x_0) \leq (\mu \circ f)(x) + (\lambda \circ g)(x) + \varepsilon\tau_C(\mu), \ \forall x \in X. \tag{4.21}$$

Since $x_0 \in S$, we have that $(\lambda \circ g)(x_0) \leq 0$ and by statement (4.21) we deduce that

$$(\mu \circ f)(x_0) + (\lambda \circ g)(x_0) \leq (\mu \circ f)(x) + (\lambda \circ g)(x) + \varepsilon\tau_C(\mu), \ \forall x \in X.$$

Applying Theorem 3.3.5 we obtain condition $(b)$. Moreover, by considering $x = x_0$ in (4.21) it follows that $-\varepsilon\tau_C(\mu) \leq (\lambda \circ g)(x_0) \leq 0$ and condition $(a)$ is satisfied. Finally, part $(c)$ is proved by applying the same reasoning as in Theorem 4.3.7. □

**Remark 4.3.9.** Theorems 4.3.7 and 4.3.8 reduce to genuine exact strong duality results by taking $C = D$ and $\varepsilon = 1$. Indeed, by part $(a)$ we see that $\Phi_{B_\mu}(x_0, \lambda) = \{f(x_0)\}$ and so $f(x_0)$ is a feasible dual point. Then, part $(c)$ implies that $f(x_0) \in \mathrm{Max}(\Xi^{\mathrm{Be}})$.

In the approximate case, the point $f(x_0)$ could not be dual feasible, i.e., it could happen that $f(x_0) \notin \Xi^{\mathrm{Be}}_{C,\varepsilon}$, as the next simple example shows.

**Example 4.3.10.** Consider problem $(\mathcal{P}_S)$ with the following data: $X = Y = Z = \mathbb{R}$, $D = \mathbb{R}_+$, $K = -\mathbb{R}_+$, $f(x) = g(x) = x$ for all $x \in \mathbb{R}$, $C = [1, +\infty)$ and $\varepsilon > 0$.

It is obvious that $\mathrm{Be}(f, S, C, \varepsilon) = [0, \varepsilon]$ and by (4.11) we see that $\Xi^{\mathrm{Be}}_{C,\varepsilon} = \{0\}$. Thus $f(x_0) \notin \Xi^{\mathrm{Be}}_{C,\varepsilon}$, for all $x_0 \in \mathrm{Be}(f, S, C, \varepsilon)\backslash\{0\}$.

However, $f(x_0)$ is near to the dual feasible set since

$$\Phi_{B_\mu}(x_0, \lambda) - f(x_0) = \bigcup_{b \in B_\mu} (-\langle\lambda, g(x_0)\rangle)b$$

and $-\langle\lambda, g(x_0)\rangle \in [0, \varepsilon\tau_C(\mu)]$. For instance, let $(\mathcal{P}_S)$ be a Pareto multiobjective problem and consider the $\ell_1$ norm in $\mathbb{R}^p$ and $C = \{(y_1, y_2, \ldots, y_p) \in \mathbb{R}^p_+ : \sum_{i=1}^p y_i \geq 1\}$. For each $\mu = (\mu_1, \mu_2, \ldots, \mu_p) \in \mathbb{R}^p_+$ with $\|\mu\| = 1$ there exists $\mu_{i_0} \geq 1/p$ and so $b_{i_0} := (1/\mu_{i_0})e_{i_0} \in B_\mu$, where $e_{i_0}$ is the $i_0$-canonical vector and

$$\|\langle\lambda, g(x_0)\rangle b_{i_0}\| \leq p\,\varepsilon\tau_C(\mu) \leq p\,\varepsilon \min_{1 \leq i \leq p}\{\mu_i/p\} \leq \varepsilon/p. \tag{4.22}$$

Moreover, let us observe that essentially (4.22) holds because the Lagrangian $\Phi_{B_\mu}$ is set-valued.

The following theorem will let us characterize the approximate dual set $\Xi^{\mathrm{Be}}_{C,\varepsilon}$ in terms of approximate solutions of a scalar Lagrangian by assuming generalized convexity assumptions. For this aim, we use the vector Lagrangian $\mathcal{L}_q$, $q \in D\backslash\{0\}$, given in (3.9).

**Theorem 4.3.11.** Suppose that $\operatorname{int} D^+ \neq \emptyset$ and $\mathcal{L}_b(\cdot,\lambda) - \mathcal{L}_b(x,\lambda)$ is nearly $(C,\varepsilon)$-subconvexlike on $X$, for all $b \in D\backslash\{0\}$, $x \in X$ and $\lambda \in K^+$. Then,

$$\begin{aligned}
&\Xi^{\mathrm{Be}}_{C,\varepsilon}\\
&= \bigcup_{\mu\in D^{s+}\cap C^{\tau+}} \bigcup_{\lambda\in K^+} \{\mathcal{L}_b(x_0,\lambda) : b \in B_\mu, x_0 \in \varepsilon\tau_C(\mu)\text{-}\mathrm{argmin}_X(\mu\circ f + \lambda\circ g)\}\\
&= \bigcup_{\mu\in D^{s+}\cap C^{\tau+}} \bigcup_{\lambda\in K^+} \{y \in \Phi_{B_\mu}(x_0,\lambda) : x_0 \in \varepsilon\tau_C(\mu)\text{-}\mathrm{argmin}_X(\mu\circ f + \lambda\circ g)\}.
\end{aligned}$$

*Proof.* Consider $\mu \in D^{s+}\cap C^{\tau+}$, $\lambda \in K^+$ and $x_0 \in \varepsilon\tau_C(\mu)\text{-}\mathrm{argmin}_X(\mu\circ f + \lambda\circ g)$. By applying Theorem 3.3.5 we have that $x_0 \in \mathrm{Be}(\Phi_{B_\mu}(\cdot,\lambda), X, C, \varepsilon)$, so $\Phi_{B_\mu}(x_0,\lambda) \subset \Theta_{C,\varepsilon}(\mu,\lambda)$ and inclusion "$\supset$" is proved.

Reciprocally, let $y_0 \in \Xi^{\mathrm{Be}}_{C,\varepsilon}$. Then, there exist $\mu \in D^{s+}\cap C^{\tau+}$, $\lambda \in K^+$, $b \in B_\mu$ and $x_0 \in \mathrm{Be}(\Phi_{B_\mu}(\cdot,\lambda), X, C, \varepsilon)$ such that $y_0 = \mathcal{L}_b(x_0,\lambda)$. We have that

$$\operatorname{cl}\operatorname{cone}(\Phi_{B_\mu}(X,\lambda) + C(\varepsilon) - \Phi_{B_\mu}(x_0,\lambda)) \cap (-D) = \{0\}$$

and in particular

$$\operatorname{cl}\operatorname{cone}(\mathcal{L}_b(X,\lambda) + C(\varepsilon) - \mathcal{L}_b(x_0,\lambda)) \cap (-D) = \{0\}. \tag{4.23}$$

Since $\mathcal{L}_b(\cdot,\lambda) - \mathcal{L}_b(x_0,\lambda)$ is nearly $(C,\varepsilon)$-subconvexlike on $X$, the first set of the intersection in statement (4.23) is a closed and convex cone. By applying [64, Theorem 3.22], there exists a functional $\bar{\mu} \in D^{s+}$ such that

$$\langle \bar{\mu}, f(x) + \lambda(g(x))b + d - f(x_0) - \lambda(g(x_0))b\rangle \geq 0, \quad \forall\, x \in X, d \in C(\varepsilon). \tag{4.24}$$

In particular, by considering $x = x_0$ we deduce that $\bar{\mu} \in C^{\tau+}$. Moreover, since $b \in D\backslash\{0\}$ and $\bar{\mu} \in D^{s+}$, we have that $\langle\bar{\mu}, b\rangle > 0$, so we can consider the functional $\hat{\mu} := (1/\langle\bar{\mu}, b\rangle)\bar{\mu} \in D^{s+}\cap C^{\tau+}$. Thus, $b \in B_{\hat{\mu}}$ and from (4.24) we deduce for all $x \in X$ and $d \in C(\varepsilon)$,

$$(\hat{\mu}\circ f)(x) + (\lambda\circ g)(x) + \langle\hat{\mu}, d\rangle - (\hat{\mu}\circ f)(x_0) - (\lambda\circ g)(x_0) \geq 0,$$

which is equivalent to

$$(\hat{\mu} \circ f)(x) + (\lambda \circ g)(x) + \varepsilon\tau_C(\hat{\mu}) - (\hat{\mu} \circ f)(x_0) - (\lambda \circ g)(x_0) \geq 0,\ \forall x \in X,$$

that is, $x_0 \in \varepsilon\tau_C(\hat{\mu})\text{-argmin}_X(\hat{\mu} \circ f + \lambda \circ g)$ and $y_0 \in \Phi_{B_{\hat{\mu}}}(x_0, \lambda)$. The proof is complete. □

**Remark 4.3.12.** It is easy to see that if $C$ is convex with $C + D = C$, $f$ is $D$-convex on $X$ and $g$ is $K$-convex on $X$, then $\mathcal{L}_b(\cdot, \lambda) - \mathcal{L}_b(x, \lambda)$ is nearly $(C, \varepsilon)$-subconvexlike on $X$, for all $b \in D\backslash\{0\}$, $x \in X$ and $\lambda \in K^+$.

The following corollary is a direct consequence of Theorem 4.3.11 and the proof of Proposition 2.2.14. Let us underline that part $(c)$ relates the problems $(\mathcal{D}_{C,\varepsilon})$ and $(\mathcal{D}^{\text{Be}}_{C,\varepsilon})$.

**Corollary 4.3.13.** Suppose that $\operatorname{int} D^+ \neq \emptyset$ and $\mathcal{L}_b(\cdot, \lambda) - \mathcal{L}_b(x, \lambda)$ is nearly $(C, \varepsilon)$-subconvexlike on $X$, for all $b \in D\backslash\{0\}$, $x \in X$ and $\lambda \in K^+$. Then,

$(a)$ $\Xi^{\text{Be}}_{C,\varepsilon} = \Xi^{\text{Be}}_{\operatorname{co} C,\varepsilon} = \Xi^{\text{Be}}_{\operatorname{shw} C,\varepsilon}$.

$(b)$ $\Xi^{\text{Be}}_{C,\varepsilon} \subset \Xi^{\text{Be}}_{C+C',\varepsilon}$, for all $C' \subset D$. If additionally $0 \in \operatorname{cl} C'$, then $\Xi^{\text{Be}}_{C,\varepsilon} = \Xi^{\text{Be}}_{C+C',\varepsilon}$.

$(c)$ $\Xi^{\text{Be}}_{C,\varepsilon} \subset \Xi_{C,\varepsilon}$.

We finish this section by relating the set of maximal points of $\Xi_{C,\varepsilon}$ with a small perturbation of the set $f(S) \cap \Xi^{\text{Be}}$.

**Theorem 4.3.14.** Suppose that $\operatorname{int} D^+ \neq \emptyset$, $C$ is convex, coradiant and $C + D = C$, $\Xi_{C,\varepsilon} \neq \emptyset$, $f$ is $D$-convex on $X$, $g$ is $K$-convex on $X$, $f(S) + \operatorname{cl} C(\varepsilon)$ is closed and the scalar optimization problem $(\mathcal{P}_\mu)$ is normal for each $\mu \in D^{s+} \cap C^{\tau+} \cap \mathbb{B}$. Then,

$$(f(S) + T^1_{C,\varepsilon}) \cap \Xi^{\text{Be}}_{C,\varepsilon} \subset \operatorname{Max}(\Xi_{C,\varepsilon}) \subset (f(S) + T^1_{C,\varepsilon}) \cap (\Xi^{\text{Be}} + T^1_{C,\varepsilon}).$$

*Proof.* By Corollary 4.3.13$(c)$ and Theorem 4.2.14$(b)$ it follows that

$$(f(S) + T^1_{C,\varepsilon}) \cap \Xi^{\text{Be}}_{C,\varepsilon} \subset (f(S) + T^1_{C,\varepsilon}) \cap \Xi_{C,\varepsilon} = \operatorname{Max}(\Xi_{C,\varepsilon}). \tag{4.25}$$

On the other hand, let $y_0 \in \mathrm{Max}(\Xi_{C,\varepsilon})$. By statement (4.25), there exist $x_0 \in S$ and $d_0 \in T^1_{C,\varepsilon}$ such that $y_0 = f(x_0) + d_0 \in \Xi_{C,\varepsilon}$. Moreover, from the definition of $\Xi_{C,\varepsilon}$ there exist $\mu \in D^{s+} \cap C^{\tau+}$ and $\lambda \in K^+$ such that

$$(\mu \circ f)(x_0) + \langle \mu, d_0 \rangle \leq (\mu \circ f)(x) + (\lambda \circ g)(x) + \varepsilon\tau_C(\mu), \ \forall x \in X. \tag{4.26}$$

Since $d_0 \in \mathrm{cl}(C(\varepsilon))$, it follows that $\varepsilon\tau_C(\mu) \leq \langle \mu, d_0 \rangle$ and statement (4.26) implies that

$$(\mu \circ f)(x_0) \leq (\mu \circ f)(x) + (\lambda \circ g)(x), \ \forall x \in X. \tag{4.27}$$

In particular, for $x = x_0$ in (4.27) we deduce that $(\lambda \circ g)(x_0) \geq 0$. Since $x_0 \in S$, $(\lambda \circ g)(x_0) \leq 0$, so $(\lambda \circ g)(x_0) = 0$ and from statement (4.27) we have that

$$(\mu \circ f)(x_0) + (\lambda \circ g)(x_0) \leq (\mu \circ f)(x) + (\lambda \circ g)(x), \ \forall x \in X.$$

Taking into account Theorem 4.3.11 it follows that $f(x_0) \in \Xi^{\mathrm{Be}}$ and so $y_0 = f(x_0) + d_0 \in \Xi^{\mathrm{Be}} + T^1_{C,\varepsilon}$, concluding the proof. □

**Remark 4.3.15.** We know that the set $T^1_{C,\varepsilon}$ is bounded for suitable sets $C$ (see Remark 4.2.22). Thus, roughly speaking, Theorem 4.3.14 shows that the set of maximal points of $\Xi_{C,\varepsilon}$ is essentially a small perturbation of the set $f(S) \cap \Xi^{\mathrm{Be}}$. In particular, if $C$ is convex, $C \subset D$ and $C + D = C$, then the set of maximal points of $\Xi_{C,0} = \Xi$ reduces exactly to $f(S) \cap \Xi^{\mathrm{Be}}$, since $T^1_{C,0} = \{0\}$.

**Corollary 4.3.16.** Suppose that $\mathrm{int}\, D^+ \neq \emptyset$, $\Xi \neq \emptyset$, $f$ is $D$-convex on $X$, $g$ is $K$-convex on $X$, $f(S) + D$ is closed and the scalar optimization problem $(\mathcal{P}_\mu)$ is normal for each $\mu \in D^{s+} \cap \mathbb{B}$. Then,

$$\mathrm{Max}(\Xi) = f(S) \cap \Xi^{\mathrm{Be}} \subset \mathrm{Max}(\Xi^{\mathrm{Be}}).$$

*Proof.* The equality follows by applying Theorem 4.3.14 to $C = D$ (see Remark 4.3.15). In particular, $\mathrm{Max}(\Xi) \subset \Xi^{\mathrm{Be}}$ and as $\Xi^{\mathrm{Be}} \subset \Xi$ (see Corollary 4.3.13($c$)) we deduce that $\mathrm{Max}(\Xi) \subset \mathrm{Max}(\Xi^{\mathrm{Be}})$, which finishes the proof. □

# Capítulo 5

# Proper $\varepsilon$-subdifferentials of vector mappings

## 5.1 Introduction

The $\varepsilon$-subdifferential of an extended real-valued convex mapping is one of the most important concepts in Convex Analysis and Optimization. Since it was introduced by Brøndsted and Rockafellar (see Definition 1.2.24), it has become an essential tool in Convex Programming, since it lets characterize the suboptimal solutions of convex optimization problems, admits a rich calculus (see [56]), satisfies suitable continuity properties (see [55,90]) and the existence of $\varepsilon$-subgradients is guaranteed by weak conditions.

Moreover, by using $\varepsilon$-subdifferentials one can provide calculus rules on subdifferentials without any qualification condition (see [60,61]), and it also allows to characterize the solutions of DC optimization problems (see [57]). In [26,59,120], the reader can find a complete description of this important concept from a theoretical and practical (minimization algorithms) point of view.

The first $\varepsilon$-subdifferential concept for extended vector mappings and some calculus rules were introduced by Kutateladze in [70] (see also [68,69,71]). This notion is based on approximate strong solutions of vector optimization problems and so these $\varepsilon$-subgradients are called "strong" $\varepsilon$-subgradients.

Five years later, Loridan [78] defined an $\varepsilon$-subdifferential for vector mappings whose image space is finite dimensional via the Pareto order and by considering $(C_q, \varepsilon)$-efficient solutions. By using the same approach as Loridan but considering other approximate solution concepts, several authors have defined new $\varepsilon$-subdifferentials for extended single-valued vector mappings and set-valued mappings. For example, Taa [104] introduced a "weak" $\varepsilon$-subdifferential for set-valued mappings through Pareto weak $\varepsilon$-optimal solutions (see [70]) of vector optimization problems with set-valued objectives, and El Maghri [83] and Tuan [110] introduced two "proper" $\varepsilon$-subdifferentials for extended single-valued vector mappings and set-valued mappings respectively, by considering new concepts of Henig and Benson $\varepsilon$-proper efficient solutions of single-valued and set-valued vector optimization problems, respectively (see Definitions 1.2.25 and 1.2.26).

The proper $\varepsilon$-subdifferentials introduced by El Maghri and Tuan are not suitable to deal with minimizing sequences. To be precise, it could happen that the limit point of a convergent minimizing sequence is as far as one wants from the efficient set.

The reason for this abnormal limit behaviour is that the proper $\varepsilon$-subdifferentials given by El Maghri and Tuan are based on the proper $\varepsilon$-efficiency concepts stated in Definitions 1.2.17 and 1.2.15, respectively, from which it is not possible to approximate the proper efficient set as the outer limit of proper $\varepsilon$-efficient sets when the error tends to zero (see Example 2.3.6). Essentially, this outer approximation does not work since the used proper $\varepsilon$-efficiency notions quantify the error by considering a unique vector, which arises sets of proper $\varepsilon$-efficient solutions too big.

The main aim of this chapter is to introduce and to study a proper $\varepsilon$-subdifferential for extended single-valued vector mappings that overcomes this drawback. This subdifferential is defined in terms of Benson $(C, \varepsilon)$-proper solutions of unconstrained vector optimization problems and it is called Benson $(C, \varepsilon)$-proper subdifferential. As it was shown in Section 2.3, the set Benson $(C, \varepsilon)$-proper solutions of $(\mathcal{P}_S)$ has a good limit behaviour when $\varepsilon$ tends to zero and because of this, the Benson $(C, \varepsilon)$-proper subdifferential is suitable to deal with minimizing sequences.

Moreover, we obtain Moreau-Rockafellar type theorems, i.e., some exact rules to calculate the Benson $(C, \varepsilon)$-proper subdifferential of the sum of two mappings. For this aim, we introduce and study a strong $\varepsilon$-subdifferential related to a new concept of approximate strong solution of vector optimization problems, and a regularity condition that extends the well-known regularity condition due to Raffin [95] (see also [83, 86, 112, 122]) to the Benson $(C, \varepsilon)$-proper subdifferential.

As it happened with the Benson $(C, \varepsilon)$-proper solutions, the main results about Benson $(C, \varepsilon)$-proper subdifferential are obtained by scalarization, under nearly $(C, \varepsilon)$-subconvexlikeness hypotheses, via well-known properties of the Brøndsted-Rockafellar $\varepsilon$-subdifferential of an extended real-valued convex mapping.

Several results of this chapter extend other similar in [83, 86] referred to exact and proper $\varepsilon$-efficient solutions of vector optimization problems and to proper subdifferentials and $\varepsilon$-subdifferentials of extended vector mappings, since they are based on a more general concept of proper $\varepsilon$-efficiency and also because they have been obtained under weaker convexity assumptions.

The chapter is structured as follows. In Section 5.2, we define the notion of Benson $(C, \varepsilon)$-proper subdifferential through Benson $(C, \varepsilon)$-proper solutions of vector optimization problems. We provide its basic properties and we characterize it under nearly $(C, \varepsilon)$-subconvexlikeness assumptions via $\varepsilon$-subgradients of extended real-valued convex mappings. In particular, we show that the limit of a minimizing sequence, in the sense of this subdifferential, is an efficient point, whenever the set $C$ satisfies suitable properties. Moreover, in the last part, we obtain some conditions in order to check if a mapping is Benson $(C, \varepsilon)$-proper subdifferentiable (existence theorems).

In Section 5.3, we define and study a new approximate strong solution concept of vector optimization problems, which generalizes the approximate strong solution concept due to Kutateladze (see [70]). These solutions are characterized through linear scalarizations without assuming any convexity assumption.

In Section 5.4, we introduce and study a strong $\varepsilon$-subdifferential for extended vector-valued mappings in connection with the approximate strong solution notion defined in Section 5.3. In particular, we characterize it through $\varepsilon$-subgradients of extended real-valued mappings without assuming any convexity assumption, and we obtain a simple formula for it when the objective space of the mapping is finite-dimensional and the ordering cone is the nonnegative orthant.

This strong $\varepsilon$-subdifferential, called $(C, \varepsilon)$-strong subdifferential, motivates a regularity condition that is defined and studied in Section 5.5. It is more general than the regularity condition introduced by El Maghri [86]. However, we show some conditions under which both coincide.

Finally, in Section 5.6, we prove two Moreau-Rockafellar type theorems for Benson $(C, \varepsilon)$-proper subdifferentials. The first one is based on the regularity condition due to El Maghri [86] and involves strong $\varepsilon$-subgradients associated with approximate strong solutions in the sense of Kutateladze. The second one considers the regularity condition defined in Section 5.5 and involves $(C, \varepsilon)$-strong subgradients. Both sum rules are exact and from them we state the gap between the Benson $(C, \varepsilon)$-proper subdifferential and the $(C, \varepsilon)$-strong subdifferential.

This chapter is based on [42, Section 4] and [43].

## 5.2 Benson $(C, \varepsilon)$-proper subdifferential

In this section we introduce the concept of Benson $(C, \varepsilon)$-proper subdifferential for vector mappings, we study its properties and we characterize it through $\varepsilon$-subgradients of associated scalar mappings under convexity assumptions. Moreover, we establish several results in order to know when a vector mapping is Benson $(C, \varepsilon)$-proper subdifferentiable.

**Definition 5.2.1.** Let $x_0 \in \operatorname{dom} f$, $\varepsilon \geq 0$ and $C \in \mathcal{H}_Y$. We define the Benson $(C, \varepsilon)$-proper subdifferential of $f$ at $x_0$ as follows:

$$\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) := \{T \in \mathcal{L}(X, Y) : x_0 \in \mathrm{Be}(f - T, X, C, \varepsilon)\}.$$

The elements of $\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0)$ are called Benson $(C, \varepsilon)$-proper subgradients of $f$ at $x_0$.

**Remark 5.2.2.** $(a)$ For each $C \in \mathcal{H}_Y$ such that $\operatorname{cl}\operatorname{cone} C = D$, from Remark 2.2.2$(b)$ we have that

$$\partial^{\mathrm{Be}}_{C,0} f(x_0) := \{T \in \mathcal{L}(X,Y) : x_0 \in \mathrm{Be}(f - T, X)\},$$

which is based on exact Benson proper solutions. In the following, we denote this subdifferential by $\partial^{\mathrm{Be}} f(x_0)$.

$(b)$ The Benson $(C,\varepsilon)$-proper subdifferential of $f$ at $x_0$ reduces to the $\varepsilon q$-Benson proper subdifferential introduced by Tuan (see Definition 1.2.25) by considering $C = C_q := q + D$ or $C = C_q^0 := q + D\backslash\{0\}$, $q \in D$, i.e.,

$$\partial^{B}_{\varepsilon q} f(x_0) = \partial^{\mathrm{Be}}_{C_q,\varepsilon} f(x_0) = \partial^{\mathrm{Be}}_{C_q^0,\varepsilon} f(x_0).$$

The next proposition shows some basic properties of the Benson $(C,\varepsilon)$-proper subdifferential. For each $C \in \mathcal{F}_Y$ and $\mu \in D^{s+} \cap C^{\tau+}$ we denote

$$\mathrm{C}^{\tau}_{\mu} = \{y \in Y : \langle \mu, y\rangle \geq \tau_C(\mu)\}.$$

**Proposition 5.2.3.** Let $x_0 \in \operatorname{dom} f$, $\varepsilon \geq 0$ and $C \in \mathcal{H}_Y$. We have that

$(a)$ $\partial^{\mathrm{Be}}_{C',\delta} f(x_0) \subset \partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) = \partial^{\mathrm{Be}}_{\operatorname{cl} C,\varepsilon} f(x_0)$, for all $C' \in \mathcal{H}_Y$ and $\delta \geq 0$ such that $C(\varepsilon) \subset \operatorname{cl}(C'(\delta))$.

$(b)$ $\partial^{\mathrm{Be}}_{C+C',\varepsilon} f(x_0) = \partial^{\mathrm{Be}}_{\operatorname{cl} C+C',\varepsilon} f(x_0)$, for all $C' \subset Y$ such that $C + C' \in \mathcal{H}_Y$.

$(c)$ If $C \subset D$ then $\partial^{\mathrm{Be}} f(x_0) \subset \partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0)$.

$(d)$ If $C$ is coradiant, then $\partial^{\mathrm{Be}}_{C,\delta} f(x_0) \subset \partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0)$, for all $0 \leq \delta \leq \varepsilon$.

If additionally $C$ is convex, then $\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) = \partial^{\mathrm{Be}}_{C+C(0),\varepsilon} f(x_0)$.

$(e)$ If $D$ has a compact base, then $\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) = \partial^{\mathrm{Be}}_{C+D,\varepsilon} f(x_0)$.

$(f)$ If $C \in \mathcal{F}_Y$, then $\partial^{\mathrm{Be}}_{C^{\tau}_{\mu},\varepsilon} f(x_0) = \bigcap_{\substack{C' \in \mathcal{F}_Y \\ \tau_{C'}(\mu) = \tau_C(\mu)}} \partial^{\mathrm{Be}}_{C',\varepsilon} f(x_0) \subset \partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0)$, for all $\mu \in D^{s+} \cap C^{\tau+}$.

*Proof.* We prove part $(f)$, since parts $(a)$-$(e)$ are a direct consequence of Proposition 2.2.3.

The inclusion $\subset$ is obvious. Moreover, it is easy to check that $C^\tau_\mu \in \mathcal{F}_Y$ and $\tau_{C^\tau_\mu}(\mu) = \tau_C(\mu)$. Hence, we obtain that

$$\partial^{\mathrm{Be}}_{C^\tau_\mu,\varepsilon} f(x_0) \supset \bigcap_{\substack{C' \in \mathcal{F}_Y \\ \tau_{C'}(\mu)=\tau_C(\mu)}} \partial^{\mathrm{Be}}_{C',\varepsilon} f(x_0).$$

On the other hand, if $C' \in \mathcal{F}_Y$, $\tau_{C'}(\mu) = \tau_C(\mu)$, then it is obvious that $C' \subset C^\tau_\mu$, and by part $(a)$ we see that $\partial^{\mathrm{Be}}_{C^\tau_\mu,\varepsilon} f(x_0) \subset \partial^{\mathrm{Be}}_{C',\varepsilon} f(x_0)$, which finishes the proof. □

Next we characterize the Benson $(C,\varepsilon)$-proper subgradients under nearly $(C,\varepsilon)$-subconvexlikeness assumptions. From now on, let us denote $f_T := f - T$, where $T \in \mathcal{L}(X,Y)$.

**Theorem 5.2.4.** Suppose that $\operatorname{int} D^+ \neq \emptyset$. Let $x_0 \in \operatorname{dom} f$, $\varepsilon \geq 0$, and $C \in \mathcal{F}_Y$. If $T \in \partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0)$ and $f_T - f_T(x_0)$ is nearly $(C,\varepsilon)$-subconvexlike on $X$, then there exists $\mu \in D^{s+} \cap C^{\tau+}$ such that

$$\mu \circ T \in \partial_{\varepsilon\tau_C(\mu)}(\mu \circ f)(x_0).$$

*Proof.* As $T \in \partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0)$, we know from Definition 5.2.1 that

$$x_0 \in \mathrm{Be}(f_T, X, C, \varepsilon).$$

Since $f_T - f_T(x_0)$ is nearly $(C,\varepsilon)$-subconvexlike on $X$, from Theorem 2.2.6, there exists $\mu \in D^{s+} \cap C^{\tau+}$ such that

$$x_0 \in \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_X(\mu \circ (f - T))$$

and the result follows by applying (1.7). □

**Theorem 5.2.5.** Let $x_0 \in \operatorname{dom} f$, $\varepsilon \geq 0$ and $C \in \mathcal{F}_Y$. It follows that

$$\bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \{T \in \mathcal{L}(X,Y) : \mu \circ T \in \partial_{\varepsilon\tau_C(\mu)}(\mu \circ f)(x_0)\} \subset \partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0).$$

*Proof.* Let $\mu \in D^{s+} \cap C^{\tau+}$ and $T \in \mathcal{L}(X,Y)$ such that $\mu \circ T \in \partial_{\varepsilon\tau_C(\mu)}(\mu \circ f)(x_0)$. By (1.7) it follows that $x_0 \in \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_X(\mu \circ (f - T))$ and by Theorem 2.2.8 we see that $x_0 \in \mathrm{Be}(f - T, X, C, \varepsilon)$. Hence, by Definition 5.2.1, we conclude that $T \in \partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0)$. □

**Corollary 5.2.6.** Let $x_0 \in \operatorname{dom} f$, $\varepsilon \geq 0$ and $C \in \mathcal{F}_Y$. Suppose that $\operatorname{int} D^+ \neq \emptyset$ and $f_T - f_T(x_0)$ is nearly $(C,\varepsilon)$-subconvexlike on $X$, for all $T \in \mathcal{L}(X,Y)$. Then,

$$\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) = \bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \{T \in \mathcal{L}(X,Y) : \mu \circ T \in \partial_{\varepsilon \tau_C(\mu)}(\mu \circ f)(x_0)\}. \tag{5.1}$$

**Remark 5.2.7.** $(a)$ By applying Remark 5.2.2$(a)$ and Corollary 5.2.6 to $C = D$, we deduce that

$$\partial^{\mathrm{Be}} f(x_0) = \partial^{\mathrm{Be}}_{D,0} f(x_0) = \bigcup_{\mu \in D^{s+}} \{T \in \mathcal{L}(X,Y) : \mu \circ T \in \partial(\mu \circ f)(x_0)\}.$$

Hence, we obtain the same exact subdifferential given by El Maghri and Laghdir (see [86]) and so, by Remark 1.2.23 we see that Corollary 5.2.6 generalizes [86, Theorem 3.2, part $\sigma = p$].

$(b)$ By applying Corollary 5.2.6 to the set $C_q$, where $q \in Y \backslash (-D \backslash \{0\})$ is arbitrary, and $\varepsilon = 1$ we see that

$$\partial^{\mathrm{Be}}_{C_q,1} f(x_0) = \bigcup_{\substack{\mu \in D^{s+} \\ \langle \mu, q \rangle \geq 0}} \{T \in \mathcal{L}(X,Y) : \mu \circ T \in \partial_{\langle \mu, q \rangle}(\mu \circ f)(x_0)\}$$

coincides with the $q$-proper subdifferential $\partial^p_q f(x_0)$ introduced by El Maghri (see Definition 1.2.26) and Corollary 5.2.6 generalizes [83, Theorem 3.2, part $\sigma = p$]. Let us observe that $C_q \in \mathcal{F}_Y$ (see Example 2.2.5). Moreover, Corollary 5.2.6 can be applied to $C_q$ when $f$ is $D$-convex on $X$, since by Remark 1.2.23, this assumption implies that $f_T - f_T(x_0)$ is nearly $(C_q, 1)$-subconvexlike on $X$, for all $T \in \mathcal{L}(X,Y)$.

$(c)$ Let $Y = \mathbb{R}$, $D = \mathbb{R}_+$, $x_0 \in \operatorname{dom} f$, $\varepsilon \geq 0$ and $C \in \mathcal{F}_Y$, $C \subset \mathbb{R}_+$. With the same hypotheses as in Corollary 5.2.6, using (5.1) it is easy to check that

$$\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) = \partial_{\varepsilon \inf C} f(x_0),$$

and so the Benson $(C,\varepsilon)$-proper subdifferential reduces to the $\varepsilon$-subdifferential of a scalar convex mapping when $\inf C = 1$.

The following proposition is a direct consequence of Proposition 2.2.14.

**Proposition 5.2.8.** Let $x_0 \in \operatorname{dom} f$, $\varepsilon \geq 0$ and $C \in \mathcal{F}_Y$. Suppose that $\operatorname{int} D^+ \neq \emptyset$ and $f_T - f_T(x_0)$ is nearly $(C,\varepsilon)$-subconvexlike on $X$, for all $T \in \mathcal{L}(X,Y)$. Then,

$(a)$ $\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) \subset \bigcap_{\delta \geq \varepsilon} \partial^{\mathrm{Be}}_{C,\delta} f(x_0)$.

$(b)$ $\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) = \partial^{\mathrm{Be}}_{\mathrm{co}\, C,\varepsilon} f(x_0) = \partial^{\mathrm{Be}}_{\mathrm{shw}\, C,\varepsilon} f(x_0)$.

$(c)$ Let $C' \subset D$. Then $\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) \subset \partial^{\mathrm{Be}}_{C+C',\varepsilon} f(x_0)$. If additionally $0 \in \mathrm{cl}\, C'$ then $\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) = \partial^{\mathrm{Be}}_{C+C',\varepsilon} f(x_0)$.

Under certain assumptions on the set $C$, the Benson $(C, \varepsilon)$-proper subdifferential can be used to deal with minimizing sequences, as it is showed in the following result.

**Definition 5.2.9.** Let $C \in \mathcal{H}_Y$. We say that $(x_n) \subset \mathrm{dom}\, f$ is a $C$-minimizing sequence of problem $(\mathcal{P})$ if there exists $(\varepsilon_n) \subset \mathbb{R}_+ \backslash \{0\}$, $\varepsilon_n \to 0$, such that $0 \in \partial^{\mathrm{Be}}_{C,\varepsilon_n} f(x_n)$, for all $n$.

Observe that the minimizing sequence concept defined above is based on the usual $\varepsilon$-stationary point condition. Moreover, it reduces to the minimizing sequence notion that motivates the Tykhonov well-posedness property of a scalar optimization problem (see [27]). Indeed, if $Y = \mathbb{R}$, $D = \mathbb{R}_+$, $f$ is convex and $C = [1, +\infty)$, by Remark 5.2.7$(c)$ we see that $(x_n) \subset \mathrm{dom}\, f$ is a $C$-minimizing sequence of problem $(\mathcal{P})$ if there exists $(\varepsilon_n) \in \mathbb{R}_+ \backslash \{0\}$ such that $\varepsilon_n \to 0$ and $x_n$ is an $\varepsilon_n$-suboptimal solution of problem $(\mathcal{P})$ (i.e., $x_n \in \varepsilon_n\text{-}\mathrm{argmin}_X f$).

**Proposition 5.2.10.** Consider $C \in \mathcal{H}_Y$ and let $(x_n) \subset \mathrm{dom}\, f$ be a $C$-minimizing sequence of problem $(\mathcal{P})$ such that $x_n \to x_0$. If $f : \mathrm{dom}\, f \subset X \to Y$ is continuous, $\mathrm{dom}\, f$ is closed, $C + D\backslash\{0\}$ is solid and coradiant and $D \subset \mathrm{cone}(\mathrm{int}(C + D\backslash\{0\}))$ then $x_0 \in \mathrm{E}(f, X)$.

*Proof.* As $(x_n)$ is a $C$-minimizing sequence of problem $(\mathcal{P})$, there exists $(\varepsilon_n) \subset \mathbb{R}_+ \backslash \{0\}$, $\varepsilon_n \to 0$, such that $0 \in \partial^{\mathrm{Be}}_{C,\varepsilon_n} f(x_n)$, for all $n$. By Definition 5.2.1 it is clear that

$$0 \in \partial^{\mathrm{Be}}_{C,\varepsilon_n} f(x_n) \iff x_n \in \mathrm{Be}(f, X, C, \varepsilon_n), \quad \forall\, n,$$

and since $x_n \to x_0$ we see that $x_0 \in \limsup_{\varepsilon \to 0} \mathrm{Be}(f, S, C, \varepsilon)$. Then the result follows by Theorem 2.3.3$(c)$. □

**Remark 5.2.11.** In general, the proper $\varepsilon$-subdifferential introduced by El Maghri (see Definiton 1.2.26) is not suitable to approximate the efficient set of a convex vector optimization problem via limits of minimizing sequences, since it is essentially based on the Benson $(C_q, \varepsilon)$-proper solution concept (see Remark 5.2.7$(b)$), and we know that the sets $C_q$ are not admissible for even simple convex problems (see Example 2.3.6).

For instance, consider in Example 2.3.6 a vector $q = (q_1, q_2) \in \mathbb{R}^2 \backslash (-\mathbb{R}^2_+)$. If $q_1 > 0$ (resp. $q_2 > 0$) then it is easy to check that $0 \in \partial^{\mathrm{Be}}_{C_q,\varepsilon} f(0, y)$ (resp. $0 \in \partial^{\mathrm{Be}}_{C_q,\varepsilon} f(y, 0)$), for all $y \in \mathbb{R}_+$, for all $\varepsilon > 0$. Thus, for each $y \in \mathbb{R}_+$, $(0, y)$ (resp. $(y, 0)$) is the limit of a $C_q$-minimizing sequence and we see that the limit of $C_q$-minimizing sequences can be as far from the efficient set as one wants.

We finish this section providing some conditions for the existence of Benson $(C, \varepsilon)$-proper subgradients. For this aim, we need the following result (see [120, Theorem 2.4.4]).

**Lemma 5.2.12.** Let $h : X \to \overline{\mathbb{R}}$ be a proper convex mapping and $x_0 \in \operatorname{dom} h$. Then, $\partial_\varepsilon h(x_0) \neq \emptyset$ for all $\varepsilon > 0$ if and only if $h$ is lower semicontinuous at $x_0$.

**Theorem 5.2.13.** Let $x_0 \in \operatorname{dom} f$ and suppose that $f$ is $D$-convex on $X$. If there exists $\mu \in D^{s+}$ such that $\mu \circ f$ is lower semicontinuous at $x_0$, then

$$\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) \neq \emptyset, \quad \forall\, \varepsilon > 0, \quad \forall\, C \in \mathcal{F}_Y \text{ such that } \tau_C(\mu) > 0.$$

*Proof.* Let $\mu \in D^{s+}$ be such that $\mu \circ f$ is lower semicontinuous at $x_0$, $\varepsilon > 0$ and let $C \in \mathcal{F}_Y$, $\tau_C(\mu) > 0$. Since $f$ is $D$-convex on $X$ and $\mu \in D^{s+}$ it follows that $\mu \circ f$ is a convex mapping, with $\operatorname{dom} \mu \circ f = \operatorname{dom} f$. By applying Lemma 5.2.12, we obtain that $\partial_{\varepsilon\tau_C(\mu)}(\mu \circ f)(x_0) \neq \emptyset$. Consider the mapping $T : X \to Y$, defined as

$$T(x) = \langle x^*, x \rangle \bar{d}, \quad \forall\, x \in X,$$

where $x^* \in \partial_{\varepsilon\tau_C(\mu)}(\mu \circ f)(x_0)$ is arbitrary and $\bar{d}$ is an arbitrary point of $D$ such that $\langle \mu, \bar{d} \rangle = 1$. Clearly, $T \in \mathcal{L}(X, Y)$ and $\mu \circ T \in \partial_{\varepsilon\tau_C(\mu)}(\mu \circ f)(x_0)$. Hence from Theorem 5.2.5, $T \in \partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0)$, concluding the proof. □

**Remark 5.2.14.** In view of Remark 5.2.7($b$), the result above reduces to [83, Proposition 3.1] when $C = q + D$, $q \notin -D\backslash\{0\}$ and improves it, since in [83, Proposition 3.1] it is also assumed that $D$ has a compact base and $\mu \circ f$ is lower semicontinuous at $x_0$, for all $\mu \in D^+$.

Next theorem provides a sufficient condition for the lower semicontinuity of the mapping $\mu \circ f$ for $\mu \in D^{s+}$ at a point of its domain, through the existence of exact Benson proper subgradients of $f$ at this point.

**Theorem 5.2.15.** Let $x_0 \in \operatorname{dom} f$. Suppose that $\operatorname{int} D^+ \neq \emptyset$ and $f$ is $D$-convex on $X$. If $\partial^{\mathrm{Be}} f(x_0) \neq \emptyset$, then there exists $\mu \in D^{s+}$ such that $\mu \circ f$ is lower semicontinuous at $x_0$.

*Proof.* Suppose on the contrary that $\mu \circ f$ is not lower semicontinuous at $x_0$ for all $\mu \in D^{s+}$. As $f$ is $D$-convex on $X$, for each $\mu \in D^{s+}$ we have that $\mu \circ f$ is a convex mapping, with $\operatorname{dom} \mu \circ f = \operatorname{dom} f$. Hence, applying Lemma 5.2.12, for each $\mu \in D^{s+}$, there exists $\varepsilon_\mu > 0$ such that $\partial_{\varepsilon_\mu}(\mu \circ f)(x_0) = \emptyset$. In particular, since $\partial(\mu \circ f)(x_0) \subset \partial_{\varepsilon_\mu}(\mu \circ f)(x_0)$ for each $\mu \in D^{s+}$, it follows that

$$\partial(\mu \circ f)(x_0) = \emptyset, \ \forall \mu \in D^{s+}. \tag{5.2}$$

By Remark 1.2.23 with $C = D$, we have that $f_T - f_T(x_0)$ is nearly $(D, 0)$-subconvexlike on $X$, for all $T \in \mathcal{L}(X, Y)$. Hence, taking into account (5.2) and Remark 5.2.7($a$) it follows that $\partial^{\mathrm{Be}} f(x_0) = \emptyset$, which is a contradiction. $\square$

**Theorem 5.2.16.** Consider $x_0 \in \operatorname{dom} f$ and $\varepsilon > 0$. Let $C \in \mathcal{F}_Y$ be a convex and compact set such that $0 \notin C$ and $D \subset \operatorname{cone} C$, and suppose that $f$ is $D$-convex on $X$. If

$$((x_0, f(x_0)) - (\{0\} \times C(\varepsilon))) \cap \operatorname{cl} \operatorname{epi} f = \emptyset,$$

then $\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) \neq \emptyset$.

*Proof.* Since $f$ is $D$-convex on $X$, it is known that $\operatorname{epi} f$ is a convex set. Moreover, as $C$ is convex and compact, $(x_0, f(x_0)) - (\{0\} \times C(\varepsilon))$ is also a convex and compact set. Then, by [64, Theorem 3.20], there exists $(x^*, y^*) \in X^* \times Y^* \backslash \{(0,0)\}$ such that

$$\langle x^*, x_0 \rangle + \langle y^*, f(x_0) - q \rangle < \langle x^*, x \rangle + \langle y^*, y \rangle, \quad \forall\, (x, y) \in \operatorname{epi} f, \forall\, q \in C(\varepsilon).$$

In particular, we have that

$$\langle x^*, x_0\rangle + \langle y^*, f(x_0) - q\rangle < \langle x^*, x\rangle + \langle y^*, f(x)\rangle, \quad \forall\, x \in X, \forall\, q \in C(\varepsilon). \quad (5.3)$$

For $x = x_0$ in (5.3), it follows that

$$\langle y^*, q\rangle > 0, \quad \forall\, q \in C(\varepsilon).$$

Since $D \subset \operatorname{cone} C$, we have that $\langle y^*, d\rangle > 0$, for all $d \in D\backslash\{0\}$. Hence $y^* \in D^{s+} \cap C^{\tau+}$. Let $T \in \mathcal{L}(X,Y)$ be the mapping

$$T(x) = -\langle x^*, x\rangle \bar{d}, \quad \forall\, x \in X,$$

where $\bar{d} \in D$ is an arbitrary point such that $\langle y^*, \bar{d}\rangle = 1$. Clearly, $y^* \circ T = -x^*$. Therefore, from (5.3) we have that

$$\langle -y^* \circ T, x_0\rangle + \langle y^*, f(x_0) - q\rangle < \langle -y^* \circ T, x\rangle + \langle y^*, f(x)\rangle, \quad \forall\, x \in X, \forall\, q \in C(\varepsilon),$$

which is equivalent to the following inequality

$$(y^* \circ f)(x) > (y^* \circ f)(x_0) - \langle y^*, q\rangle + (y^* \circ T)(x - x_0), \quad \forall\, x \in X, \forall\, q \in C(\varepsilon). \quad (5.4)$$

Statement (5.4) is equivalent to

$$(y^* \circ f)(x) \geq (y^* \circ f)(x_0) - \varepsilon\tau_C(y^*) + (y^* \circ T)(x - x_0), \quad \forall\, x \in X,$$

so $y^* \circ T \in \partial_{\varepsilon\tau_C(y^*)}(y^* \circ f)(x_0)$. Hence, from Theorem 5.2.5 we see that $T \in \partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0)$, concluding the proof. □

**Remark 5.2.17.** Theorem 5.2.16 is based on [110, Theorem 3.1] and extends the single-valued version of this result. Specifically, [110, Theorem 3.1] provides a sufficient condition for the existence of Benson $(C_q,\varepsilon)$-proper subgradients of a map $f$, for $q \in D\backslash\{0\}$, and $X, Y$ Banach spaces.

The following theorem is based on the single-valued version of [110, Theorem 3.2] and extends it to a general set $C \in \mathcal{F}_Y$.

**Theorem 5.2.18.** Let $x_0 \in \operatorname{dom} f$, $\varepsilon > 0$ and $C \in \mathcal{F}_Y$. Suppose that $C$ is a convex and compact set with $0 \notin C$ and $D \subset \operatorname{cone} C$. If there exists a closed and convex set $A \subset X \times Y$ such that

$(a)$ $\operatorname{gr} f - (x_0, f(x_0)) \subset A$,

$(b)$ $A \cap (-(\{0\} \times C(\varepsilon))) = \emptyset$,

then $\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) \neq \emptyset$.

*Proof.* Since $A$ is closed and convex, $\{0\} \times C(\varepsilon)$ is convex and compact and condition $(b)$ holds, by [64, Theorem 3.20] there exists $(x^*, y^*) \in X^* \times Y^* \backslash \{(0,0)\}$ such that

$$\langle (x^*, y^*), a \rangle > -\langle y^*, q \rangle, \quad \forall\, a \in A, \forall\, q \in C(\varepsilon). \tag{5.5}$$

By condition $(a)$ and (5.5) we have that

$$\langle x^*, x - x_0 \rangle + \langle y^*, f(x) - f(x_0) \rangle > -\langle y^*, q \rangle, \quad \forall\, x \in X, \forall\, q \in C(\varepsilon).$$

Reasoning as in the proof of Theorem 5.2.16, we obtain that $y^* \in D^{s+} \cap C^{\tau+}$ and there exists $T \in \mathcal{L}(X, Y)$ such that $y^* \circ T \in \partial_{\varepsilon \tau_C(y^*)}(y^* \circ f)(x_0)$. Then, by Theorem 5.2.5 it follows that $T \in \partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0)$ and the proof is complete. $\square$

## 5.3 $(C, \varepsilon)$-strong efficiency

The following new concept of approximate strong solution of problem $(\mathcal{P}_S)$ is motivated by the $(C, \varepsilon)$-efficiency notion (see Definition 1.2.10) and it generalizes the strong $\varepsilon$-efficiency concept with respect to a vector (see Definition 1.2.14).

**Definition 5.3.1.** Let $\varepsilon \geq 0$ and a nonempty set $C \subset D$. A point $x_0 \in S$ is a $(C, \varepsilon)$-strong efficient solution of $(\mathcal{P}_S)$, denoted by $x_0 \in \mathrm{SE}(f, S, C, \varepsilon)$, if

$$f(x_0) - q \leq_D f(x), \quad \forall\, q \in C(\varepsilon), \quad \forall\, x \in S. \tag{5.6}$$

It is clear from Definitions 1.2.14 and 5.3.1 that $x_0$ is a $(C, \varepsilon)$-strong efficient solution of $(\mathcal{P}_S)$ if and only if $x_0$ is a strong $\varepsilon$-efficient solution of $(\mathcal{P}_S)$ with respect to $q$, for each $q \in C(\varepsilon)$, i.e.,

$$\begin{aligned} x_0 \in \mathrm{SE}(f, S, C, \varepsilon) &\Longleftrightarrow f(S_0) \subset \bigcap_{q \in C(\varepsilon)} (f(x_0) - q + D) \\ &\Longleftrightarrow f(S_0) - f(x_0) + C(\varepsilon) \subset D. \end{aligned} \tag{5.7}$$

Moreover, $\mathrm{SE}(f, S, C, \varepsilon) \subset \mathrm{dom}\, f$ and then statement (5.6) can be checked only for points $x \in S_0$. In the following proposition we show the main properties of this new kind of approximate strong solution of problem $(\mathcal{P}_S)$. Several of them are consequence of the following previous lemma.

**Lemma 5.3.2.** Consider $\emptyset \neq F \subset Y$ and $\emptyset \neq C_1, C_2 \subset Y$. If $C_2 \subset \mathrm{cl}(C_1 + D)$, then

$$F + C_1 \subset D \Rightarrow F + \mathrm{co}\, C_2 \subset D,$$

and if $C_2 \subset D$, then

$$F + C_2 \subset D \Rightarrow F + \mathrm{shw}(\mathrm{co}\, C_2) \subset D.$$

*Proof.* Suppose that $C_2 \subset \mathrm{cl}(C_1 + D)$ and let $y \in F$ and $q \in C_2$. There exist two nets $(q_i) \subset C_1$ and $(d_i) \subset D$ such that $q_i + d_i \to q$. As $F + C_1 \subset D$ and $D$ is closed we deduce that

$$y + q = \lim((y + q_i) + d_i) \in \mathrm{cl}(D + D) = D$$

and then we have

$$F + C_2 \subset D. \tag{5.8}$$

Consider $q \in \mathrm{co}\, C_2$. There exist $q_j \in C_2$ and $\alpha_j \in [0, 1]$, $1 \leq j \leq k$, $k \in \mathbb{N}$, $k \geq 1$, such that $\sum_{j=1}^{k} \alpha_j = 1$ and $q = \sum_{j=1}^{k} \alpha_j q_j$. Thus, by (5.8) we have

$$F + q \subset \sum_{j=1}^{k} \alpha_j (F + q_j) \subset D$$

and then we deduce that

$$F + C_2 \subset D \Rightarrow F + \mathrm{co}\, C_2 \subset D. \tag{5.9}$$

Suppose that $C_2 \subset D$, $F + C_2 \subset D$ and let $q \in \mathrm{shw}(\mathrm{co}\, C_2)$. There exist $\alpha \geq 1$ and $d \in \mathrm{co}\, C_2 \subset D$ such that $q = \alpha d$ and by (5.9) it follows that

$$F + q = F + d + (\alpha - 1)d \subset D,$$

i.e., $F + \mathrm{shw}(\mathrm{co}\, C_2) \subset D$, which finishes the proof. $\square$

Recall that an element $\bar{y} \in Y$ is the infimum of a nonempty set $F \subset Y$ if the following two properties hold:

$(a)$ $\bar{y} \leq_D y$, for all $y \in F$.

$(b)$ If for some $y' \in Y$ one has $y' \leq_D y$ for all $y \in F$, then $y' \leq_D \bar{y}$.

If there exists an infimum of $F$, it is unique and we denote it by $\mathrm{Inf}_D F$.

**Proposition 5.3.3.** Consider $\varepsilon \geq 0$ and a nonempty set $C \subset D$. It follows that

$(a)$ $\mathrm{SE}(f,S) \subset \mathrm{SE}(f,S,C,\varepsilon) = \bigcap_{\delta>\varepsilon} \mathrm{SE}(f,S,C,\delta)$.

$(b)$ $\mathrm{SE}(f,S,C,\varepsilon) = \mathrm{SE}(f,S)$, if $0 \in \mathrm{cl}\, C(\varepsilon)$.

$(c)$ Consider a nonempty set $G \subset D$ and $\delta \geq 0$ satisfying $G(\delta) \subset \mathrm{cl}(C(\varepsilon) + D)$. Then $\mathrm{SE}(f,S,C,\varepsilon) \subset \mathrm{SE}(f,S,G,\delta)$. In particular,

$$\mathrm{SE}(f,S,C,\varepsilon) \subset \mathrm{SE}(f,S,C+C',\varepsilon), \quad \forall\, C' \subset D, C' \neq \emptyset, \tag{5.10}$$

$$\mathrm{SE}(f,S,C,\varepsilon) = \mathrm{SE}(f,S,C+C',\varepsilon), \quad \forall\, C' \subset D,\ 0 \in \mathrm{cl}\, C'. \tag{5.11}$$

$(d)$ $\mathrm{SE}(f,S,C,\varepsilon) = \mathrm{SE}(f,S,\mathrm{co}\, C,\varepsilon) = \mathrm{SE}(f,S,\mathrm{shw}\, C,\varepsilon) = \mathrm{SE}(f,S,\mathrm{cl}\, C,\varepsilon)$.

$(e)$ If $\bar{q} = \mathrm{Inf}_D C$, then $\mathrm{SE}(f,S,C,\varepsilon) = \mathrm{SE}(f,S,\bar{q},\varepsilon)$.

$(f)$ If $\bigcap_{c\in C}(-c+D) = D$, then $\mathrm{SE}(f,S,C,\varepsilon) = \mathrm{SE}(f,S)$.

*Proof.* $(a)$ Let $x_0 \in \mathrm{SE}(f,S)$. By applying Lemma 5.3.2 to $F = f(S_0) - f(x_0)$, $C_1 = \{0\}$ and $C_2 = C(\varepsilon)$ we deduce $f(S_0) - f(x_0) + C(\varepsilon) \subset D$, i.e., $x_0 \in \mathrm{SE}(f,S,C,\varepsilon)$ by (5.7).

Analogously, let $x_0 \in \mathrm{SE}(f,S,C,\varepsilon)$ and $\delta > \varepsilon$. It is clear that $C(\delta) \subset C(\varepsilon) + D$. Then, by applying Lemma 5.3.2 to $F = f(S_0) - f(x_0)$, $C_1 = C(\varepsilon)$ and $C_2 = C(\delta)$ we deduce that $x_0 \in \mathrm{SE}(f,S,C,\delta)$.

Reciprocally, let $x_0 \in \bigcap_{\delta>\varepsilon} \mathrm{SE}(f,S,C,\delta)$. As

$$C(\varepsilon) \subset \mathrm{cl}\left(\bigcup_{\delta>\varepsilon} C(\delta)\right),$$

by applying Lemma 5.3.2 to $F = f(S_0) - f(x_0)$, $C_1 = \bigcup_{\delta>\varepsilon} C(\delta)$ and $C_2 = C(\varepsilon)$ we deduce that $x_0 \in \mathrm{SE}(f,S,C,\varepsilon)$ by (5.7).

$(b)$ Suppose that $0 \in \mathrm{cl}\, C(\varepsilon)$ and consider $x_0 \in \mathrm{SE}(f,S,C,\varepsilon)$. Then by applying Lemma 5.3.2 to $F = f(S_0) - f(x_0)$, $C_1 = C(\varepsilon)$ and $C_2 = \{0\}$ we deduce that $f(S_0) - f(x_0) \subset D$, i.e., $x_0 \in \mathrm{SE}(f,S)$. The reciprocal inclusion follows by part $(a)$.

$(c)$ The inclusion

$$\mathrm{SE}(f,S,C,\varepsilon) \subset \mathrm{SE}(f,S,G,\delta) \tag{5.12}$$

is a direct consequence of Lemma 5.3.2 and (5.7). Moreover, statement (5.10) follows by applying (5.12) to $G = C + C'$ and $\delta = \varepsilon$, and statement (5.11) follows from (5.10) and by applying (5.12) to $G = C$, $C + C'$ instead of $C$ and $\delta = \varepsilon$.

$(d)$ By part $(c)$ we see that

$$\begin{aligned}
\mathrm{SE}(f,S,\mathrm{co}\, C,\varepsilon) &\subset \mathrm{SE}(f,S,C,\varepsilon),\\
\mathrm{SE}(f,S,\mathrm{shw}\, C,\varepsilon) &\subset \mathrm{SE}(f,S,C,\varepsilon),\\
\mathrm{SE}(f,S,\mathrm{cl}\, C,\varepsilon) &\subset \mathrm{SE}(f,S,C,\varepsilon).
\end{aligned}$$

The reciprocal inclusions are consequence of applying Lemma 5.3.2 to $F = f(S_0) - f(x_0)$ and $C_1 = C_2 = C(\varepsilon)$ (for the first two inclusions) and $C_1 = C(\varepsilon)$, $C_2 = (\mathrm{cl}\, C)(\varepsilon) = \mathrm{cl}\, C(\varepsilon)$ (for the last inclusion).

$(e)$ The result is clear for $\varepsilon = 0$ by part $(b)$. Suppose that $\varepsilon > 0$ and let $x_0 \in \mathrm{SE}(f,S,\bar{q},\varepsilon)$. Then $x_0 \in S_0$ and $f(x_0) - \varepsilon\bar{q} \leq_D f(x)$, for all $x \in S_0$. Since $\bar{q} = \mathrm{Inf}_D C$, it follows that

$$f(x_0) - f(x) \leq_D \varepsilon\bar{q} \leq_D \varepsilon c, \quad \forall\, c \in C, \quad \forall\, x \in S_0.$$

Hence, $x_0 \in \mathrm{SE}(f,S,C,\varepsilon)$. Now, take $x_0 \in \mathrm{SE}(f,S,C,\varepsilon)$. Then, $x_0 \in S_0$ and

$$f(x_0) - f(x) \leq_D \varepsilon c, \quad \forall\, x \in S_0, \quad \forall\, c \in C.$$

Therefore, as $\bar{q} = \mathrm{Inf}_D C$, for each fixed $x \in S_0$ it follows that $f(x_0) - f(x) \leq_D \varepsilon\bar{q}$. This inequality holds for every $x \in S_0$ and so $x_0 \in \mathrm{SE}(f,S,\bar{q},\varepsilon)$.

(*f*) The result is clear for $\varepsilon = 0$ by part (*b*). Suppose that $\varepsilon > 0$ and let $x_0 \in \mathrm{SE}(f, S, C, \varepsilon)$. By (5.7) we have that

$$f(S_0) - f(x_0) \subset \bigcap_{q \in C(\varepsilon)} (-q + D) = \bigcap_{c \in C}(-\varepsilon c + D) = \varepsilon \bigcap_{c \in C}(-c + D) = D.$$

Hence, $x_0 \in \mathrm{SE}(f, S)$ and so $\mathrm{SE}(f, S, C, \varepsilon) \subset \mathrm{SE}(f, S)$. The reciprocal inclusion follows by part (*a*) and the proof is complete. □

**Remark 5.3.4.** By Proposition 5.3.3(*c*),(*d*) it follows that

$$\mathrm{SE}(f, S, C, \varepsilon) = \mathrm{SE}(f, S, \operatorname{co} C + D, \varepsilon)$$

i.e., we can deal with $(C, \varepsilon)$-strong efficient solutions of problem $(\mathcal{P}_S)$ by assuming that $C$ is convex and $C = C + D$, which implies that $C$ is coradiant.

Next we characterize the $(C, \varepsilon)$-strong efficient solutions of problem $(\mathcal{P}_S)$ through approximate solutions of associated scalar optimization problems.

**Theorem 5.3.5.** Consider $\varepsilon \geq 0$ and a nonempty set $C \subset D$. Then,

$$\mathrm{SE}(f, S, C, \varepsilon) = \bigcap_{\mu \in D^+ \backslash \{0\}} \varepsilon\tau_C(\mu)\text{-}\mathrm{argmin}_S(\mu \circ f).$$

*Proof.* By the bipolar theorem (see for instance [83, Proposition 2.1(1)]) we deduce that

$$\begin{aligned}
& x_0 \in \bigcap_{\mu \in D^+ \backslash \{0\}} \varepsilon\tau_C(\mu)\text{-}\mathrm{argmin}_S(\mu \circ f) \\
& \Longleftrightarrow \langle \mu, f(x_0)\rangle \leq \langle \mu, f(x)\rangle + \tau_{C(\varepsilon)}(\mu), \quad \forall\, x \in S_0,\, \forall\, \mu \in D^+ \backslash \{0\} \\
& \Longleftrightarrow \langle \mu, f(x_0)\rangle \leq \langle \mu, f(x)\rangle + \langle \mu, q\rangle, \quad \forall\, q \in C(\varepsilon),\, \forall\, x \in S_0,\, \forall\, \mu \in D^+ \backslash \{0\} \\
& \Longleftrightarrow \langle \mu, f(x) - f(x_0) + q\rangle \geq 0, \quad \forall\, q \in C(\varepsilon),\, \forall\, x \in S_0,\, \forall\, \mu \in D^+ \backslash \{0\} \\
& \Longleftrightarrow f(x) - f(x_0) + q \in D, \quad \forall\, q \in C(\varepsilon),\, \forall\, x \in S_0 \\
& \Longleftrightarrow x_0 \in \mathrm{SE}(f, S, C, \varepsilon),
\end{aligned}$$

and the proof is complete. □

## 5.4 $(C,\varepsilon)$-strong subdifferential

In this section, we introduce and study a new notion of strong $\varepsilon$-subdifferential for extended vector-valued mappings. After that, we characterize it through $\varepsilon$-subgradients of associated scalar mappings.

**Definition 5.4.1.** Let $\varepsilon \geq 0$ and let $C \subset D$ be a nonempty set. The $(C,\varepsilon)$-strong subdifferential of $f$ at a point $x_0 \in \operatorname{dom} f$ is defined as follows:

$$\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) := \{T \in \mathcal{L}(X,Y) : x_0 \in \mathrm{SE}(f-T, X, C, \varepsilon)\}.$$

In the sequel, the usual (exact) strong subdifferential of $f$ at $x_0 \in \operatorname{dom} f$ is denoted by $\partial^{\mathrm{s}} f(x_0)$, i.e. (see, for instance, [86]),

$$\begin{aligned}\partial^{\mathrm{s}} f(x_0) &= \{T \in \mathcal{L}(X,Y) : f(x) \geq_D f(x_0) + T(x-x_0),\ \forall\, x \in X\}\\ &= \{T \in \mathcal{L}(X,Y) : x_0 \in \mathrm{SE}(f-T, X)\}.\end{aligned}$$

**Remark 5.4.2.** Definition 5.4.1 reduces to the $\varepsilon$-subdifferential notion introduced by Kutatelazde [70] by taking a singleton $C = \{q\}$, $q \in D$ (see also [83]).

As a direct consequence of Proposition 5.3.3, we obtain the following properties.

**Proposition 5.4.3.** Let $x_0 \in \operatorname{dom} f$, $\varepsilon \geq 0$ and a nonempty set $C \subset D$. It follows that:

$(a)$ $\partial^{\mathrm{s}} f(x_0) \subset \partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) = \bigcap_{\delta > \varepsilon} \partial^{\mathrm{s}}_{C,\delta} f(x_0)$.

$(b)$ $\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) = \partial^{\mathrm{s}} f(x_0)$, if $0 \in \operatorname{cl} C(\varepsilon)$.

$(c)$ Consider a nonempty set $G \subset D$ and $\delta \geq 0$ satisfying $G(\delta) \subset \operatorname{cl}(C(\varepsilon) + D)$. Then $\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) \subset \partial^{\mathrm{s}}_{G,\delta} f(x_0)$. In particular,

$$\begin{aligned}\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) &\subset \partial^{\mathrm{s}}_{C+C',\varepsilon} f(x_0), \quad \forall\, C' \subset D, C' \neq \emptyset,\\ \partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) &= \partial^{\mathrm{s}}_{C+C',\varepsilon} f(x_0), \quad \forall\, C' \subset D,\ 0 \in \operatorname{cl} C'.\end{aligned}$$

$(d)$ $\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) = \partial^{\mathrm{s}}_{\operatorname{co} C,\varepsilon} f(x_0) = \partial^{\mathrm{s}}_{\operatorname{shw} C,\varepsilon} f(x_0) = \partial^{\mathrm{s}}_{\operatorname{cl} C,\varepsilon} f(x_0)$.

$(e)$ If $\bar{q} = \mathrm{Inf}_D C$, then $\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) = \partial^{\mathrm{s}}_{\{\bar{q}\},\varepsilon} f(x_0)$.

$(f)$ If $\bigcap_{c \in C}(-c + D) = D$, then $\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) = \partial^{\mathrm{s}} f(x_0)$.

In the following result we characterize the $(C, \varepsilon)$-strong subdifferential of $f$ at a point $x_0 \in \mathrm{dom}\, f$ in terms of approximate subgradients of scalar mappings.

**Theorem 5.4.4.** Let $x_0 \in \mathrm{dom}\, f$, $\varepsilon \geq 0$ and a nonempty set $C \subset D$. Then,

$$\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) = \bigcap_{\mu \in D^+ \backslash \{0\}} \{T \in \mathcal{L}(X, Y) : \mu \circ T \in \partial_{\varepsilon \tau_C(\mu)}(\mu \circ f)(x_0)\}.$$

*Proof.* Let $A \in \mathcal{L}(X, Y)$. From Definition 5.4.1 and Theorem 5.3.5 we have that $A \in \partial^{\mathrm{s}}_{C,\varepsilon} f(x_0)$ if and only if

$$x_0 \in \bigcap_{\mu \in D^+ \backslash \{0\}} \varepsilon \tau_C(\mu)\text{-}\mathrm{argmin}_X(\mu \circ (f - A)).$$

Hence, by applying (1.7) it follows that

$$A \in \partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) \Longleftrightarrow A \in \bigcap_{\mu \in D^+ \backslash \{0\}} \{T \in \mathcal{L}(X, Y) : \mu \circ T \in \partial_{\varepsilon \tau_C(\mu)}(\mu \circ f)(x_0)\}$$

and the proof is complete. □

**Remark 5.4.5.** $(a)$ Let $C \subset D$ and $q \in D$. By Proposition 5.4.3$(b)$ and Theorem 5.4.4 we have

$$\partial^{\mathrm{s}} f(x_0) = \partial^{\mathrm{s}}_{C,0} f(x_0) = \bigcap_{\mu \in D^+ \backslash \{0\}} \{T \in \mathcal{L}(X, Y) : \mu \circ T \in \partial(\mu \circ f)(x_0)\},$$

$$\partial^{\mathrm{s}}_{\{q\},1} f(x_0) = \bigcap_{\mu \in D^+ \backslash \{0\}} \{T \in \mathcal{L}(X, Y) : \mu \circ T \in \partial_{\langle \mu, q \rangle}(\mu \circ f)(x_0)\}.$$

Therefore, Theorem 5.4.4 generalizes the first parts of [83, Theorem 3.2] and [86, Theorem 3.2].

$(b)$ Let $Y = \mathbb{R}$, $D = \mathbb{R}_+$, $x_0 \in \mathrm{dom}\, f$, $\varepsilon \geq 0$ and a nonempty set $C \subset \mathbb{R}_+$. It follows that

$$\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) = \partial_{\varepsilon \inf C} f(x_0), \tag{5.13}$$

where $\inf C := \mathrm{Inf}_{\mathbb{R}_+} C$ denotes the infimum of $C$. Indeed, given $T \in \mathcal{L}(X,Y)$, it follows that

$$\begin{aligned} T \in \partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) &\iff x_0 \in \mathrm{SE}(f-T, X, C, \varepsilon) \text{ by Definition 5.4.1} \\ &\iff x_0 \in \mathrm{SE}(f-T, X, \{\inf C\}, \varepsilon) \text{ by Proposition 5.3.3}(e) \\ &\iff x_0 \in \varepsilon \inf C\text{-argmin}_X(f-T) \text{ by Definition 5.3.1} \\ &\iff T \in \partial_{\varepsilon \inf C} f(x_0), \text{ by (1.7).} \end{aligned}$$

By statement (5.13) we see that the $(C,\varepsilon)$-strong subdifferential reduces to the $\varepsilon$-subdifferential of a scalar mapping when $\inf C = 1$.

Next theorem provides an easy calculus rule to obtain the $(C,\varepsilon)$-strong subdifferential in the Pareto case.

**Theorem 5.4.6.** Let $Y = \mathbb{R}^n$, $D = \mathbb{R}^n_+$, $x_0 \in \mathrm{dom}\, f$, $\varepsilon \geq 0$ and a nonempty set $C \subset \mathbb{R}^n_+$. It follows that

$$\partial^{\mathrm{s}}_{C,\varepsilon} f(x_0) = \prod_{i=1}^{n} \partial_{\varepsilon \tau_C(\bar{e}_i)} f_i(x_0),$$

where $\{\bar{e}_i\}_{1 \leq i \leq n}$ is the canonical base of $\mathbb{R}^n$.

*Proof.* Let $T = (T_1, T_2, \ldots, T_n) \in \partial^{\mathrm{s}}_{C,\varepsilon} f(x_0)$. From Theorem 5.4.4 we have that

$$\mu \circ T \in \partial_{\varepsilon \tau_C(\mu)}(\mu \circ f)(x_0), \quad \forall\, \mu \in \mathbb{R}^n_+ \backslash \{0\}.$$

Then, for $\mu = \bar{e}_i$ it follows that

$$T_i \in \partial_{\varepsilon \tau_C(\bar{e}_i)} f_i(x_0), \quad \forall\, i = 1, 2, \ldots, n,$$

and so $T \in \prod_{i=1}^{n} \partial_{\varepsilon \tau_C(\bar{e}_i)} f_i(x_0)$.

For checking the reciprocal inclusion, let us consider $T \in \prod_{i=1}^{n} \partial_{\varepsilon \tau_C(\bar{e}_i)} f_i(x_0)$ and $\mu \in \mathbb{R}^n_+ \backslash \{0\}$. Then, for each $i = 1, 2, \ldots, n$, $T_i \in \partial_{\varepsilon \tau_C(\bar{e}_i)} f_i(x_0)$ and $\mu_i \geq 0$, and so

$$\mu_i f_i(x) \geq \mu_i f_i(x_0) - \mu_i \varepsilon \tau_C(\bar{e}_i) + \mu_i T_i(x - x_0), \ \forall\, x \in X,\ i = 1, 2, \ldots, n. \quad (5.14)$$

By adding in (5.14) we obtain that

$$\langle \mu, f(x)\rangle \geq \langle \mu, f(x_0)\rangle - \varepsilon \sum_{i=1}^{n} \mu_i \tau_C(\bar{e}_i) + \langle \mu, T(x-x_0)\rangle, \quad \forall\, x \in X. \tag{5.15}$$

Moreover,

$$\sum_{i=1}^{n} \mu_i \tau_C(\bar{e}_i) = \inf_{c\in C}\{\mu_1 c_1\} + \cdots + \inf_{c\in C}\{\mu_n c_n\} \leq \inf_{c\in C}\{\langle \mu, c\rangle\} = \tau_C(\mu). \tag{5.16}$$

Hence, from (5.15) and (5.16) we deduce that

$$\langle \mu, f(x)\rangle \geq \langle \mu, f(x_0)\rangle - \varepsilon\tau_C(\mu) + \langle \mu, T(x-x_0)\rangle, \quad \forall\, x \in X,$$

so $\mu \circ T \in \partial_{\varepsilon\tau_C(\mu)}(\mu \circ f)(x_0)$. Applying Theorem 5.4.4 we conclude that $T \in \partial^{\mathrm{s}}_{C,\varepsilon} f(x_0)$, and the proof is finished. □

Theorem 5.4.6 reduces to [83, statement (6)] and [86, statement (3.6)] by considering $\varepsilon = 1$ and the sets $C = \{q\}$ and $C = D$, respectively.

## 5.5 Regularity conditions

In this section, we introduce and study a new regularity condition for extended vector-valued mappings, which is defined in terms of $(C,\varepsilon)$-strong subdifferentials. For this aim we assume that $C \in \mathcal{F}_Y$ and we denote

$$C^{s\tau+} := \{y^* \in Y^* : \tau_C(y^*) > 0\}.$$

**Definition 5.5.1.** Let $x_0 \in \operatorname{dom} f$ and $\varepsilon \geq 0$. We say that $f$ is regular $(C,\varepsilon)$-subdifferentiable at $x_0$ if $\varepsilon = 0$ and

$$\partial(\mu \circ f)(x_0) = \mu \circ \partial^{\mathrm{s}} f(x_0), \quad \forall\, \mu \in D^{s+}, \tag{5.17}$$

or $\varepsilon > 0$ and

$$\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu)=1}} \mu \circ \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0), \quad \forall\, \mu \in D^{s+} \cap C^{s\tau+}.$$

When statement (5.17) holds, we say that $f$ is p-regular subdifferentiable at $x_0$.

**Remark 5.5.2.** For all $C' \subset D$, $\varepsilon \geq 0$ and for all $A \in \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0)$, by Theorem 5.4.4 we have that

$$\mu \circ A \in \partial_{\varepsilon \tau_{C'}(\mu)}(\mu \circ f)(x_0), \quad \forall\, \mu \in D^+ \backslash \{0\}. \tag{5.18}$$

Hence, $f$ is regular $(C,\varepsilon)$-subdifferentiable at $x_0$ if and only if $\varepsilon = 0$ and

$$\partial(\mu \circ f)(x_0) \subset \mu \circ \partial^{\mathrm{s}} f(x_0), \quad \forall\, \mu \in D^{s+},$$

or $\varepsilon > 0$ and

$$\partial_\varepsilon(\mu \circ f)(x_0) \subset \bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu)=1}} \mu \circ \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0), \quad \forall\, \mu \in D^{s+} \cap C^{s\tau+}.$$

For each $\mu \in D^{s+} \cap C^{s\tau+}$ and $\varepsilon > 0$, the set

$$\bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu)=1}} \mu \circ \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0)$$

can be rewritten in different ways and from them, several equivalent formulations of the regular $(C,\varepsilon)$-subdifferentiablity are obtained, as it is showed in the following results.

**Theorem 5.5.3.** Let $x_0 \in \operatorname{dom} f$ and $\varepsilon > 0$. The following statements are equivalent:

$(a)$ $f$ is regular $(C,\varepsilon)$-subdifferentiable at $x_0$.

$(b)$ $\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu)=\tau_C(\mu)}} \mu \circ \partial^{\mathrm{s}}_{C',\frac{\varepsilon}{\tau_C(\mu)}} f(x_0), \quad \forall\, \mu \in D^{s+} \cap C^{s\tau+}.$

$(c)$ $\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu)=\varepsilon}} \mu \circ \partial^{\mathrm{s}}_{C',1} f(x_0), \quad \forall\, \mu \in D^{s+} \cap C^{s\tau+}.$

*Proof.* The result is a consequence of the equalities

$$\bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu)=\tau_C(\mu)}} \mu \circ \partial^{\mathrm{s}}_{C',\frac{\varepsilon}{\tau_C(\mu)}} f(x_0) = \bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu)=\varepsilon}} \mu \circ \partial^{\mathrm{s}}_{C',1} f(x_0) \tag{5.19}$$

$$= \bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu)=1}} \mu \circ \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0), \tag{5.20}$$

where $\mu$ is an arbitrary element of $D^{s+} \cap C^{s\tau+}$. Let us prove the equalities (5.19) and (5.20).

Consider $C' \subset D$ such that $\tau_{C'}(\mu) = \tau_C(\mu)$. It is obvious that

$$\partial^{\mathrm{s}}_{C', \frac{\varepsilon}{\tau_C(\mu)}} f(x_0) = \partial^{\mathrm{s}}_{C'\left(\frac{\varepsilon}{\tau_C(\mu)}\right), 1} f(x_0)$$

and by (1.9), $C'' := C'\left(\frac{\varepsilon}{\tau_C(\mu)}\right) \subset D$ satisfies $\tau_{C''}(\mu) = \frac{\varepsilon}{\tau_C(\mu)} \tau_{C'}(\mu) = \varepsilon$. Therefore,

$$\bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu) = \tau_C(\mu)}} \mu \circ \partial^{\mathrm{s}}_{C', \frac{\varepsilon}{\tau_C(\mu)}} f(x_0) \subset \bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu) = \varepsilon}} \mu \circ \partial^{\mathrm{s}}_{C', 1} f(x_0).$$

Consider $C' \subset D$ such that $\tau_{C'}(\mu) = \varepsilon$. It is clear that

$$\partial^{\mathrm{s}}_{C', 1} f(x_0) = \partial^{\mathrm{s}}_{C'(1/\varepsilon), \varepsilon} f(x_0)$$

and by (1.9), $C'' := C'(1/\varepsilon) \subset D$ satisfies $\tau_{C''}(\mu) = (1/\varepsilon)\tau_{C'}(\mu) = 1$. Thus, we see that

$$\bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu) = \varepsilon}} \mu \circ \partial^{\mathrm{s}}_{C', 1} f(x_0) \subset \bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu) = 1}} \mu \circ \partial^{\mathrm{s}}_{C', \varepsilon} f(x_0).$$

Finally, let $C' \subset D$ such that $\tau_{C'}(\mu) = 1$. It follows that

$$\partial^{\mathrm{s}}_{C', \varepsilon} f(x_0) = \partial^{\mathrm{s}}_{C'(\tau_C(\mu)), \frac{\varepsilon}{\tau_C(\mu)}} f(x_0)$$

and by (1.9), $C'' := C'(\tau_C(\mu)) \subset D$ satisfies $\tau_{C''}(\mu) = \tau_C(\mu)\tau_{C'}(\mu) = \tau_C(\mu)$. Therefore,

$$\bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu) = 1}} \mu \circ \partial^{\mathrm{s}}_{C', \varepsilon} f(x_0) \subset \bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu) = \tau_C(\mu)}} \mu \circ \partial^{\mathrm{s}}_{C', \frac{\varepsilon}{\tau_C(\mu)}} f(x_0)$$

and the proof is complete. □

For certain vector optimization problems it is possible to give a unified formulation of the regular $(C, \varepsilon)$-subdifferentiability notion, as it is showed in the following corollary.

**Corollary 5.5.4.** Let $x_0 \in \operatorname{dom} f$ and $\varepsilon \geq 0$. Assume that

$$0 \notin \operatorname{cl} C, \quad \operatorname{argmin}_{\operatorname{cl} C} \mu \neq \emptyset, \quad \forall\, \mu \in D^{s+}. \tag{5.21}$$

Then $D^{s+} \subset C^{s\tau+}$ and $f$ is regular $(C,\varepsilon)$-subdifferentiable at $x_0$ if and only if

$$\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu)=1}} \mu \circ \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0), \quad \forall\, \mu \in D^{s+}. \tag{5.22}$$

*Proof.* If $\varepsilon = 0$, by Proposition 5.4.3$(b)$ we deduce that $\partial^{\mathrm{s}}_{C',0} f(x_0) = \partial^s f(x_0)$, for all $C' \subset D$. Then statement (5.22) says that $f$ is p-regular subdifferentiable at $x_0$.

On the other hand, if $\varepsilon > 0$ then statement (5.22) reduces to the notion of regular $(C,\varepsilon)$-subdifferentiability of $f$ at $x_0$, since $D^{s+} \subset C^{s\tau+}$. Indeed, for each $\mu \in D^{s+}$, as $\operatorname{argmin}_{\operatorname{cl} C}\mu \neq \emptyset$, there exists $q \in \operatorname{cl} C$ such that

$$\tau_C(\mu) = \inf\{\langle \mu, d\rangle : d \in C\} = \inf\{\langle \mu, d\rangle : d \in \operatorname{cl} C\} = \langle \mu, q\rangle > 0, \tag{5.23}$$

since $q \in \operatorname{cl} C \subset D\backslash\{0\}$ and $\mu \in D^{s+}$. $\square$

**Remark 5.5.5.** In order to apply Corollary 5.5.4, let us observe that the assumption

$$\operatorname{argmin}_{\operatorname{cl} C}\mu \neq \emptyset, \quad \forall\, \mu \in D^{s+}$$

is satisfied if some of the following conditions holds (for part $(e)$, see for instance [67, Theorem 7.3.7], where the coercivity condition can be checked via [8, Lemma 2.7]):

$(a)$ $C$ is compact.

$(b)$ $C = P + Q$, with $P \subset D$ compact and $Q \subset D$.

$(c)$ $D$ has a compact base.

$(d)$ $Y$ is finite dimensional.

$(e)$ $Y$ is reflexive, $C$ is convex and $\operatorname{int} D^+ \neq \emptyset$.

In the following important result, we show that if $\varepsilon > 0$, then the regular $(C,\varepsilon)$-subdifferentiability condition can be checked by considering convex sets $C' \subset D$ such that $C' = C' + D$. This class of sets will be denoted in the sequel by $\Lambda_D$, i.e.,

$$\Lambda_D := \{C' \subset D : C' \neq \emptyset, C' = C' + D, C' \text{ is convex}\}.$$

Let us observe that each $C' \in \Lambda_D$ is a coradiant set.

**Theorem 5.5.6.** Let $x_0 \in \operatorname{dom} f$ and $\varepsilon > 0$. The following statements are equivalent:

$(a)$ $f$ is regular $(C, \varepsilon)$-subdifferentiable at $x_0$.

$(b)$ $\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{C' \in \Lambda_D \\ \tau_{C'}(\mu)=1}} \mu \circ \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0), \quad \forall\, \mu \in D^{s+} \cap C^{s\tau+}.$

$(c)$ $\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{C' \in \Lambda_D \\ \tau_{C'}(\mu)=\tau_C(\mu)}} \mu \circ \partial^{\mathrm{s}}_{C',\frac{\varepsilon}{\tau_C(\mu)}} f(x_0), \quad \forall\, \mu \in D^{s+} \cap C^{s\tau+}.$

$(d)$ $\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{C' \in \Lambda_D \\ \tau_{C'}(\mu)=\varepsilon}} \mu \circ \partial^{\mathrm{s}}_{C',1} f(x_0), \quad \forall\, \mu \in D^{s+} \cap C^{s\tau+}.$

*Proof.* Fix $\mu \in D^{s+} \cap C^{s\tau+}$. Clearly,

$$\bigcup_{\substack{C' \in \Lambda_D \\ \tau_{C'}(\mu)=1}} \mu \circ \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0) \subset \bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu)=1}} \mu \circ \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0).$$

Now, let $C' \subset D$ such that $\tau_{C'}(\mu) = 1$. By applying Proposition 5.4.3($c$),($d$), we know that

$$\partial^{\mathrm{s}}_{C',\varepsilon} f(x_0) = \partial^{\mathrm{s}}_{\operatorname{co} C',\varepsilon} f(x_0) = \partial^{\mathrm{s}}_{\operatorname{co} C'+D,\varepsilon} f(x_0). \tag{5.24}$$

Observe that $\operatorname{co} C' + D \in \Lambda_D$ and by the well-known properties of the support function (see for instance [120]) and by (1.9) it follows that

$$\tau_{\operatorname{co} C'+D}(\mu) = \tau_{\operatorname{co} C'}(\mu) = -\sigma_{-\operatorname{co} C'}(\mu) = -\sigma_{-C'}(\mu) = \tau_{C'}(\mu) = 1. \tag{5.25}$$

Hence, from statements (5.24) and (5.25) we deduce that

$$\bigcup_{\substack{C' \subset D \\ \tau_{C'}(\mu)=1}} \mu \circ \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0) \subset \bigcup_{\substack{C' \in \Lambda_D \\ \tau_{C'}(\mu)=1}} \mu \circ \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0)$$

and so parts $(a)$ and $(b)$ are equivalent.

Moreover, for each $C \in \Lambda_D$ and $\alpha > 0$ it is obvious that $\alpha C \in \Lambda_D$. Then, by using the same reasonings as in the proof of Theorem 5.5.3 it is easy to check that

$$\bigcup_{\substack{C'\in\Lambda_D\\ \tau_{C'}(\mu)=1}} \mu\circ\partial^{\mathrm{s}}_{C',\varepsilon} f(x_0) = \bigcup_{\substack{C'\in\Lambda_D\\ \tau_{C'}(\mu)=\tau_C(\mu)}} \mu\circ\partial^{\mathrm{s}}_{C',\frac{\varepsilon}{\tau_C(\mu)}} f(x_0)$$
$$= \bigcup_{\substack{C'\in\Lambda_D\\ \tau_{C'}(\mu)=\varepsilon}} \mu\circ\partial^{\mathrm{s}}_{C',1} f(x_0), \quad \forall\,\mu\in D^{s+}\cap C^{s\tau+}$$

and so parts $(b)$, $(c)$ and $(d)$ are equivalent, which finishes the proof. □

It is obvious that the regular $(C,\varepsilon)$-subdifferentiability condition reduces to the p-regular subdifferentiability due to El Maghri and Laghdir (see [86, pg. 1977]) when $\varepsilon = 0$. In the sequel, we relate the notion of p-regular $\varepsilon$-subdifferentiability introduced by El Maghri in [83] with the regular $(C,\varepsilon)$-subdifferentiability condition.

**Definition 5.5.7.** [83, Definition 3.1(ii) and $\sigma = p$] Let $x_0 \in \operatorname{dom} f$ and $\varepsilon \geq 0$. It is said that $f$ is p-regular $\varepsilon$-subdifferentiable at $x_0$ if $\varepsilon = 0$ and

$$\partial(\mu\circ f)(x_0) = \mu\circ\partial^{\mathrm{s}} f(x_0), \quad \forall\,\mu\in D^{s+}$$

or $\varepsilon > 0$ and

$$\partial_\varepsilon(\mu\circ f)(x_0) = \bigcup_{\substack{q\in D\\ \langle\mu,q\rangle=\varepsilon}} \mu\circ\partial^{\mathrm{s}}_{\{q\},1} f(x_0), \quad \forall\,\mu\in D^{s+}.$$

**Proposition 5.5.8.** Let $x_0 \in \operatorname{dom} f$ and $\varepsilon \geq 0$. If $f$ is p-regular $\varepsilon$-subdifferentiable at $x_0$ then $f$ is regular $(C,\varepsilon)$-subdifferentiable at $x_0$.

*Proof.* Suppose that $\varepsilon > 0$, since the result is obvious when $\varepsilon = 0$. Then,

$$\partial_\varepsilon(\mu\circ f)(x_0) = \bigcup_{\substack{q\in D\\ \langle\mu,q\rangle=\varepsilon}} \mu\circ\partial^{\mathrm{s}}_{\{q\},1} f(x_0)$$
$$\subset \bigcup_{\substack{C'\subset D\\ \tau_{C'}(\mu)=\varepsilon}} \mu\circ\partial^{\mathrm{s}}_{C',1} f(x_0), \quad \forall\,\mu\in D^{s+}$$

and by (5.19)-(5.20) and Remark 5.5.2 we deduce that $f$ is regular $(C,\varepsilon)$-subdifferentiable at $x_0$. □

By Proposition 5.5.8 we see that, in general, the concept of regular $(C,\varepsilon)$-subdifferentiability is weaker than the notion of p-regular $\varepsilon$-subdifferentiability. Next we show that both notions are the same if

$$\operatorname{argmin}_{\operatorname{cl} C'}\mu \neq \emptyset, \quad \forall\, \mu \in D^{s+}, \quad \forall\, C' \in \Lambda_D,\ 0 \notin \operatorname{cl} C'. \tag{5.26}$$

**Proposition 5.5.9.** Let $x_0 \in \operatorname{dom} f$ and $\varepsilon \geq 0$. Assume that conditions (5.21) and (5.26) are fulfilled. If $f$ is regular $(C,\varepsilon)$-subdifferentiable at $x_0$ then $f$ is p-regular $\varepsilon$-subdifferentiable at $x_0$.

*Proof.* Suppose that $\varepsilon > 0$, since the result is obvious for $\varepsilon = 0$. Then, by Corollary 5.5.4 and Theorem 5.5.6 we have that

$$\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{C' \in \Lambda_D \\ \tau_{C'}(\mu)=\varepsilon}} \mu \circ \partial^{\mathrm{s}}_{C',1} f(x_0), \quad \forall\, \mu \in D^{s+}. \tag{5.27}$$

Consider $C' \in \Lambda_D$ such that $\tau_{C'}(\mu) = \varepsilon$. Then $0 \notin \operatorname{cl} C'$ and by using the same reasoning as in (5.23) we deduce that there exists $q' \in \operatorname{cl} C'$ such that $\tau_{C'}(\mu) = \langle \mu, q'\rangle$. By Proposition 5.4.3($c$) we deduce that $\partial^{\mathrm{s}}_{C',1} f(x_0) \subset \partial^{\mathrm{s}}_{\{q'\},1} f(x_0)$ and by (5.27) we obtain that

$$\partial_\varepsilon(\mu \circ f)(x_0) \quad \subset \quad \bigcup_{\substack{q \in D \\ \langle \mu, q\rangle=\varepsilon}} \mu \circ \partial^{\mathrm{s}}_{\{q\},1} f(x_0), \quad \forall\, \mu \in D^{s+}.$$

By (5.18) we conclude that $f$ is p-regular $\varepsilon$-subdifferentiable at $x_0$, which finishes the proof. □

In the next result we show that the regular $(C,\varepsilon)$-subdifferentiability notion reduces in the Pareto context to the well-known calculus rule on the $\varepsilon$-subdifferential of a sum of convex mappings.

**Proposition 5.5.10.** Let $Y = \mathbb{R}^n$, $D = \mathbb{R}^n_+$, $x_0 \in \operatorname{dom} f$, $\varepsilon \geq 0$ and suppose that $0 \notin \operatorname{cl} C$. Then $f$ is regular $(C,\varepsilon)$-subdifferentiable at $x_0$ if and only if for all $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \operatorname{int} \mathbb{R}^n_+$ one has

$$\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)\in\mathbb{R}^n_+ \\ \varepsilon_1+\varepsilon_2+\cdots+\varepsilon_n=\varepsilon}} \sum_{i=1}^{n} \partial_{\varepsilon_i}(\mu_i f_i)(x_0). \tag{5.28}$$

*Proof.* The case $\varepsilon = 0$ is clear by applying Proposition 5.4.3$(b)$ and Theorem 5.4.6 to an arbitrary nonempty set $C \subset \mathbb{R}^n_+$ and $\varepsilon = 0$. If $\varepsilon > 0$, by Remark 5.5.5$(d)$ and Propositions 5.5.8 and 5.5.9 we deduce that $f$ is regular $(C, \varepsilon)$-subdifferentiable at $x_0$ if and only if

$$\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{q \in \mathbb{R}^n_+ \\ \langle \mu, q\rangle = \varepsilon}} \mu \circ \partial^{\mathrm{s}}_{\{q\},1} f(x_0), \quad \forall\, \mu \in \operatorname{int} \mathbb{R}^n_+. \tag{5.29}$$

By applying Theorem 5.4.6 it follows that for all $\mu = (\mu_1, \mu_2, \dots, \mu_n) \in \operatorname{int} \mathbb{R}^n_+$

$$\mu \circ \partial^{\mathrm{s}}_{\{q\},1} f(x_0) = \sum_{i=1}^{n} \mu_i \partial_{q_i} f_i(x_0) = \sum_{i=1}^{n} \partial_{\varepsilon_i}(\mu_i f_i)(x_0),$$

where $\varepsilon_i = \mu_i q_i$ for all $i = 1, 2, \dots, n$. Therefore, by (5.29) and (5.30) we obtain that

$$\partial_\varepsilon(\mu \circ f)(x_0) = \bigcup_{\substack{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \mathbb{R}^n_+ \\ \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n = \varepsilon}} \sum_{i=1}^{n} \partial_{\varepsilon_i}(\mu_i f_i)(x_0),\ \forall\, \mu = (\mu_1, \mu_2, \dots, \mu_n) \in \operatorname{int} \mathbb{R}^n_+,$$

which finishes the proof. □

**Remark 5.5.11.** The previous proposition can be used, in connection with some constraint qualifications, to check the regular $(C, \varepsilon)$-subdifferentiability of objective mappings of Pareto problems. For example, it is well-known (see for instance [120]) that calculus rule (5.28) is fulfilled under the Moreau-Rockafellar constraint qualification (MRCQ in short form): $X$ is a Hausdorff locally convex space, all objectives $f_i$ are convex and there exists $\overline{x} \in \bigcap_{i=1}^{n} \operatorname{dom} f_i$ such that $n - 1$ objectives $f_i$ are continuous at $\overline{x}$. Then, (MRCQ) implies that the objective mappings of Pareto problems are regular $(C, \varepsilon)$-subdifferentiable at every point in the effective domain.

## 5.6 Moreau-Rockafellar type theorems

In this section, we provide some calculus rules for the Benson $(C, \varepsilon)$-proper subdifferential of the sum of two vector-valued mappings. In particular, we obtain two Moreau-Rockafellar type theorems, and as a consequence, two formulae that reveal the gap between the strong $(C,\varepsilon)$-subdifferential and the Benson $(C,\varepsilon)$-proper subdifferential.

For an arbitrary set $C \in \mathcal{H}_Y$ and $\varepsilon \geq 0$, we denote

$$S_{C(\varepsilon)} := \{(C_1, C_2) \in \mathcal{H}_Y \times 2^D : C(\varepsilon) \subset C_1 + C_2\}$$

and $S_C := S_{C(1)}$.

**Theorem 5.6.1.** Let $\varepsilon \geq 0$, $C \in \mathcal{H}_Y$ and consider two proper mappings $f_1, f_2 : X \to \overline{Y}$ and $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$. It follows that

$$\partial^{\mathrm{Be}}_{C,\varepsilon}(f_1 + f_2)(x_0) \supset \bigcup_{\substack{(C_1,C_2)\in S_{C(\varepsilon)} \\ C_1 + D \subset \operatorname{cl} C_1}} \{\partial^{\mathrm{Be}}_{C_1,1} f_1(x_0) + \partial^{\mathrm{s}}_{C_2,1} f_2(x_0)\}.$$

Moreover, if $D$ has a compact base, then

$$\partial^{\mathrm{Be}}_{C,\varepsilon}(f_1 + f_2)(x_0) \supset \bigcup_{(C_1,C_2)\in S_{C(\varepsilon)}} \{\partial^{\mathrm{Be}}_{C_1,1} f_1(x_0) + \partial^{\mathrm{s}}_{C_2,1} f_2(x_0)\}.$$

*Proof.* Consider $(C_1, C_2) \in S_{C(\varepsilon)}$, with $C_1 + D \subset \operatorname{cl} C_1$, and let $A \in \partial^{\mathrm{Be}}_{C_1,1} f_1(x_0)$, $B \in \partial^{s}_{C_2,1} f_2(x_0)$. Suppose by contradiction that $A + B \notin \partial^{\mathrm{Be}}_{C,\varepsilon}(f_1 + f_2)(x_0)$. Then $x_0 \notin \mathrm{Be}(f_1 + f_2 - (A + B), X, C, \varepsilon)$, i.e., there exists $d_0 \in D\backslash\{0\}$ such that

$$-d_0 \in \operatorname{cl} \operatorname{cone}((f_1 + f_2)_{A+B}(\operatorname{dom}(f_1 + f_2)) + C(\varepsilon) - (f_1 + f_2)_{A+B}(x_0)).$$

Hence, there exist nets $(\alpha_i) \subset \mathbb{R}_+$, $(x_i) \subset \operatorname{dom}(f_1 + f_2)$, $(d_i) \subset C(\varepsilon)$ such that

$$\alpha_i((f_1 - A)(x_i) + d_i - (f_1 - A)(x_0) + (f_2 - B)(x_i) - (f_2 - B)(x_0)) \to -d_0.$$

As $C(\varepsilon) \subset C_1 + C_2$ there exist nets $(u_i) \subset C_1$ and $(v_i) \subset C_2$ such that $d_i = u_i + v_i$ for all $i$, and so

$$\alpha_i((f_1)_A(x_i) + u_i - (f_1)_A(x_0) + (f_2)_B(x_i) + v_i - (f_2)_B(x_0)) \to -d_0. \tag{5.30}$$

Since $B \in \partial^{s}_{C_2,1} f_2(x_0)$, it follows that $x_0 \in \mathrm{SE}(f_2 - B, X, C_2, 1)$ and then

$$(f_2 - B)(x_i) + v_i - (f_2 - B)(x_0) \in D, \quad \forall\, i.$$

By denoting $y_i := u_i + (f_2 - B)(x_i) + v_i - (f_2 - B)(x_0) \in C_1 + D$ we deduce from (5.30) that

$$\alpha_i((f_1 - A)(x_i) + y_i - (f_1 - A)(x_0)) \to -d_0. \tag{5.31}$$

If $C_1 + D \subset \operatorname{cl} C_1$, then $y_i \in \operatorname{cl} C_1$ for all $i$ and (5.31) is a contradiction, since by Proposition 5.2.3($b$) with $C' = \{0\}$ it follows that $A \in \partial^{\mathrm{Be}}_{C_1,1} f_1(x_0) = \partial^{\mathrm{Be}}_{\operatorname{cl} C_1,1} f_1(x_0)$ and so $x_0 \in \mathrm{Be}(f_1 - A, X, \operatorname{cl} C_1, 1)$.

On the other hand, if $D$ has a compact base, by Proposition 2.2.3($e$) we have that

$$\mathrm{Be}(f_1 - A, X, C_1 + D, 1) = \mathrm{Be}(f_1 - A, X, C_1, 1).$$

Therefore, by (5.31) we see that $x_0 \notin \mathrm{Be}(f_1 - A, X, C_1, 1)$, i.e, $A \notin \partial^{\mathrm{Be}}_{C_1,1} f_1(x_0)$, which is a contradiction. □

**Remark 5.6.2.** Let $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and $\varepsilon \geq 0$. Suppose that $\operatorname{int} D^+ \neq \emptyset$ and $(f_1)_T - (f_1)_T(x_0)$ is nearly $(C_q, \varepsilon)$-subconvexlike on $X$, for all $T \in \mathcal{L}(X, Y)$ and for all $q \in Y \backslash (-D \backslash \{0\})$. Let $q, q_1, q_2 \in Y$ be such that $q, q_1 \notin -D \backslash \{0\}$, $q_2 \in D$ and $q = q_1 + q_2$. By applying the first part of Theorem 5.6.1 to $C = C_q$, $\varepsilon = 1$, $C_1 = C_{q_1}$ and $C_2 = \{q_2\}$ we obtain that

$$\partial^{\mathrm{Be}}_{C_{q_1},1} f_1(x_0) + \partial^{\mathrm{s}}_{\{q_2\},1} f_2(x_0) \subset \partial^{\mathrm{Be}}_{C_q,\varepsilon} (f_1 + f_2)(x_0)$$

and by Remark 5.2.7($b$) we see that the first part of Theorem 5.6.1 reduces to the first part of [83, Theorem 4.1, $\sigma = p$]. Analogously, by applying the first part of Theorem 5.6.1 to $C = C_1 = D$, $C_2 = \{0\}$ and $\varepsilon = 1$ it follows by the definitions that

$$\partial^{\mathrm{Be}} f_1(x_0) + \partial^{\mathrm{s}} f_2(x_0) \subset \partial^{\mathrm{Be}} (f_1 + f_2)(x_0)$$

and then the first part of Theorem 5.6.1 reduces to the first part of [86, Theorem 4.1, $\sigma = p$].

The following notion is motivated by [83, Definition 2.1(ii)].

**Definition 5.6.3.** We say that $f$ is star $D$-continuous at a point $x_0 \in \operatorname{dom} f$ if $\mu \circ f$ is continuous at $x_0$, for all $\mu \in D^{s+}$.

**Theorem 5.6.4.** Assume that $\operatorname{int} D^+ \neq \emptyset$. Let $\varepsilon \geq 0$, $C \in \mathcal{F}_Y$, $f_1, f_2 : X \to \overline{Y}$ and $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$. Suppose that $f_1$ and $f_2$ are $D$-convex on $X$, $(f_1 + f_2)_T - (f_1 + f_2)_T(x_0)$ is nearly $(C, \varepsilon)$-subconvexlike on $X$ for all $T \in \mathcal{L}(X, Y)$, $f_2$

is p-regular $\varepsilon'$-subdifferentiable at $x_0$ for all $\varepsilon' \geq 0$, and the following qualification condition (MRQC0) holds:

(MRQC0) There exists $\bar{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ such that $f_1$ or $f_2$ is star $D$-continuous at $\bar{x}$.

Then,

$$\partial^{\mathrm{Be}}_{C,\varepsilon}(f_1 + f_2)(x_0) \subset \bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \bigcup_{\substack{(C_1,q) \in \mathcal{F}_Y \times D,\, C(\varepsilon) = C_1 + \{q\} \\ \tau_{C_1}(\mu) \geq 0}} \{\partial^{\mathrm{Be}}_{C_1,1} f_1(x_0) + \partial^{\mathrm{s}}_{\{q\},1} f_2(x_0)\}. \tag{5.32}$$

If additionally $C + D = C$ or $D$ has a compact base, then

$$\partial^{\mathrm{Be}}_{C,\varepsilon}(f_1 + f_2)(x_0) = \bigcup_{\substack{(C_1,q) \in \mathcal{F}_Y \times D \\ C(\varepsilon) = C_1 + \{q\}}} \{\partial^{\mathrm{Be}}_{C_1,1} f_1(x_0) + \partial^{\mathrm{s}}_{\{q\},1} f_2(x_0)\}.$$

*Proof.* In order to prove inclusion (5.32), let us consider $T \in \partial^{\mathrm{Be}}_{C,\varepsilon}(f_1 + f_2)(x_0)$. Since $(f_1 + f_2)_T - (f_1 + f_2)_T(x_0)$ is nearly $(C,\varepsilon)$-subconvexlike on $X$, by Theorem 5.2.4 there exists $\mu \in D^{s+} \cap C^{\tau+}$ such that

$$\mu \circ T \in \partial_{\varepsilon\tau_C(\mu)}(\mu \circ (f_1 + f_2))(x_0) = \partial_{\varepsilon\tau_C(\mu)}(\mu \circ f_1 + \mu \circ f_2)(x_0).$$

As condition (MRQC0) holds, we can apply the well-known $\varepsilon$-subdifferential sum rule for proper extended real-valued convex mappings (see, e.g., [120, Theorem 2.8.7]) obtaining that

$$\partial_{\varepsilon\tau_C(\mu)}(\mu \circ f_1 + \mu \circ f_2)(x_0) = \bigcup_{\substack{\varepsilon_1,\varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon\tau_C(\mu)}} \{\partial_{\varepsilon_1}(\mu \circ f_1)(x_0) + \partial_{\varepsilon_2}(\mu \circ f_2)(x_0)\}.$$

Hence, there exist $\bar{\varepsilon}_1, \bar{\varepsilon}_2 \geq 0$, with $\bar{\varepsilon}_1 + \bar{\varepsilon}_2 = \varepsilon\tau_C(\mu)$, $x_1^* \in \partial_{\bar{\varepsilon}_1}(\mu \circ f_1)(x_0)$ and $x_2^* \in \partial_{\bar{\varepsilon}_2}(\mu \circ f_2)(x_0)$ such that $\mu \circ T = x_1^* + x_2^*$.

Suppose that $\bar{\varepsilon}_2 > 0$. As $f_2$ is p-regular $\bar{\varepsilon}_2$-subdifferentiable at $x_0$, there exists $q \in D$ such that $\langle \mu, q \rangle = \bar{\varepsilon}_2$ and $x_2^* \in \mu \circ \partial^{\mathrm{s}}_{\{q\},1} f_2(x_0)$. Hence, there exists $B \in \partial^{\mathrm{s}}_{\{q\},1} f_2(x_0)$ such that $\mu \circ (T - B) = x_1^* \in \partial_{\bar{\varepsilon}_1}(\mu \circ f_1)(x_0)$. By defining $C_1 := C(\varepsilon) - \{q\}$ we obtain $C(\varepsilon) = C_1 + \{q\}$ and $\tau_{C_1}(\mu) = \varepsilon\tau_C(\mu) - \langle \mu, q \rangle = \bar{\varepsilon}_1$.

Thus, $\mu \in D^{s+} \cap C_1^{\tau+}$ and $C_1 \in \mathcal{F}_Y$. Moreover, by Theorem 5.2.5 we see that $T - B \in \partial^{\mathrm{Be}}_{C_1,1} f_1(x_0)$ and so

$$T = (T - B) + B \in \partial^{\mathrm{Be}}_{C_1,1} f_1(x_0) + \partial^{\mathrm{s}}_{\{q\},1} f_2(x_0).$$

If $\bar{\varepsilon}_2 = 0$, by applying the p-regularity condition and Proposition 5.4.3($b$) we deduce that $x_2^* \in \mu \circ \partial^{\mathrm{s}} f_2(x_0) = \mu \circ \partial^{\mathrm{s}}_{\{q\},1} f_2(x_0)$, where $q = 0$. From here we obtain the same conclusions as in the case $\bar{\varepsilon}_2 > 0$ by following the same reasonings, and so the proof of inclusion (5.32) is complete.

On the other hand, it is clear that

$$\begin{aligned} &\bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \bigcup_{\substack{(C_1,q) \in \mathcal{F}_Y \times D,\, C(\varepsilon) = C_1 + \{q\} \\ \tau_{C_1}(\mu) \geq 0}} \{\partial^{\mathrm{Be}}_{C_1,1} f_1(x_0) + \partial^{\mathrm{s}}_{\{q\},1} f_2(x_0)\} \\ &\subset \bigcup_{\substack{(C_1,q) \in \mathcal{F}_Y \times D \\ C(\varepsilon) = C_1 + \{q\}}} \{\partial^{\mathrm{Be}}_{C_1,1} f_1(x_0) + \partial^{\mathrm{s}}_{\{q\},1} f_2(x_0)\}. \end{aligned} \tag{5.33}$$

If $C(\varepsilon) = C_1 + \{q\}$ and $C + D = C$, then it is easy to prove that $C_1 + D \subset \operatorname{cl} C_1$. Therefore, by Theorem 5.6.1 we deduce that (5.33) is included in $\partial^{\mathrm{Be}}_{C,\varepsilon}(f_1 + f_2)(x_0)$, which finishes the proof. □

By using the same reasonings as in Remark 5.6.2 we see that Theorem 5.6.4 generalizes [83, Theorem 4.1, $\sigma = p$] and the second part of [86, Theorem 4.1, $\sigma = p$].

Next, we prove other Moreau-Rockafellar type results based on regular $(C,\varepsilon)$-subdifferentiable mappings.

For an arbitrary set $C \in \mathcal{F}_Y$, $\varepsilon \geq 0$ and $x_0 \in \operatorname{dom} f$, we define the set-valued mapping $\Gamma_{C,\varepsilon} f(x_0) : D^{s+} \cap C^{\tau+} \rightrightarrows \mathcal{L}(X,Y)$ as follows:

$$\Gamma_{C,\varepsilon} f(x_0)(\mu) := \{T \in \mathcal{L}(X,Y) : \mu \circ T \in \partial_{\varepsilon \tau_C(\mu)}(\mu \circ f)(x_0)\}.$$

If $\operatorname{cl}\operatorname{cone} C = D$, then we denote $\Gamma f(x_0)(\mu) := \Gamma_{C,0} f(x_0)(\mu)$.

**Remark 5.6.5.** By Theorem 5.2.5 it is clear that

$$\operatorname{Im} \Gamma_{C,\varepsilon} f(x_0) := \bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \Gamma_{C,\varepsilon} f(x_0)(\mu) \subset \partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0)$$

and by Corollary 5.2.6, if $\operatorname{int} D^+ \neq \emptyset$ and $f_T - f_T(x_0)$ is nearly $(C, \varepsilon)$-subconvexlike on $X$, for all $T \in \mathcal{L}(X, Y)$, then

$$\operatorname{Im} \Gamma_{C,\varepsilon} f(x_0) = \partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0).$$

Moreover, if $\operatorname{cl} \operatorname{cone} C = D$, then $\operatorname{Im} \Gamma f(x_0) = \partial^{\mathrm{Be}} f(x_0)$.

For each $C \in \mathcal{F}_Y$ and $\mu \in D^{s+} \cap C^{\tau+}$ we denote

$$S_C(\mu) := \{(C_1, C_2) \in \mathcal{F}_Y \times 2^D : \tau_{C_1}(\mu) \geq 0, \tau_C(\mu) \geq \tau_{C_1+C_2}(\mu)\}.$$

**Proposition 5.6.6.** Let $f_1, f_2 : X \to \overline{Y}$, $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, $\varepsilon \geq 0$ and $C \in \mathcal{F}_Y$. Then,

$$\partial^{\mathrm{Be}}_{C,\varepsilon}(f_1 + f_2)(x_0) \supset \bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \geq 0 \\ (C_1, C_2) \in \mathcal{F}_Y \times 2^D \\ (C_1(\varepsilon_1), C_2(\varepsilon_2)) \in S_{C(\varepsilon)}(\mu)}} \{\Gamma_{C_1,\varepsilon_1} f_1(x_0)(\mu) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} f_2(x_0)\}.$$

*Proof.* Consider $\mu \in D^{s+} \cap C^{\tau+}$, $\varepsilon_1, \varepsilon_2 \geq 0$, $C_1 \in \mathcal{F}_Y$ and $C_2 \subset D$ such that

$$(C_1(\varepsilon_1), C_2(\varepsilon_2)) \in S_{C(\varepsilon)}(\mu), \tag{5.34}$$

and let $A \in \Gamma_{C_1,\varepsilon_1} f_1(x_0)(\mu)$ and $B \in \partial^{\mathrm{s}}_{C_2,\varepsilon_2} f_2(x_0)$. By Theorem 5.4.4 it follows that $\mu \circ B \in \partial_{\varepsilon_2 \tau_{C_2}(\mu)}(\mu \circ f_2)(x_0)$ and then we deduce that

$$\mu \circ (A + B) \in \partial_{\varepsilon_1 \tau_{C_1}(\mu) + \varepsilon_2 \tau_{C_2}(\mu)}(\mu \circ (f_1 + f_2))(x_0). \tag{5.35}$$

Moreover, by (5.34) and (1.9) we deduce that

$$\varepsilon_1 \tau_{C_1}(\mu) + \varepsilon_2 \tau_{C_2}(\mu) = \tau_{C_1(\varepsilon_1) + C_2(\varepsilon_2)}(\mu) \leq \tau_{C(\varepsilon)}(\mu)$$

and by (5.35) we have that $\mu \circ (A + B) \in \partial_{\tau_{C(\varepsilon)}(\mu)}(\mu \circ (f_1 + f_2))(x_0)$. Then, by Theorem 5.2.5 we deduce that $A + B \in \partial^{\mathrm{Be}}_{C,\varepsilon}(f_1 + f_2)(x_0)$ and the proof is complete. □

In the sequel, we denote

$$\Omega := \{C \in \mathcal{F}_Y : C = C + D, C \text{ is convex}\}.$$

It is clear that $\Lambda_D = \Omega \cap 2^D$.

**Theorem 5.6.7.** Suppose that $\operatorname{int} D^+ \neq \emptyset$. Let $\varepsilon \geq 0$, $C \in \mathcal{F}_Y$, $f_1, f_2 : X \to \overline{Y}$ and $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$. Suppose that $f_1$ is $D$-convex on $X$. Then,

$$\partial^{\mathrm{Be}}_{C,\varepsilon}(f_1 + f_2)(x_0) \supset \bigcup_{\substack{\varepsilon_1,\varepsilon_2 \geq 0 \\ (C_1,C_2)\in\Omega\times 2^D \\ (C_1(\varepsilon_1),C_2(\varepsilon_2))\in S_{C(\varepsilon)}}} \{\partial^{\mathrm{Be}}_{C_1,\varepsilon_1} f_1(x_0) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} f_2(x_0)\}.$$

*Proof.* Let $\varepsilon_1, \varepsilon_2 \geq 0$, $C_1 \in \Omega$, $C_2 \subset D$ such that $(C_1(\varepsilon_1), C_2(\varepsilon_2)) \in S_{C(\varepsilon)}$ and let $A \in \partial^{\mathrm{Be}}_{C_1,\varepsilon_1} f_1(x_0)$ and $B \in \partial^{\mathrm{s}}_{C_2,\varepsilon_2} f_2(x_0)$. Since $C_1 \in \Omega$ and $f_1$ is $D$-convex on $X$, we know from Remark 1.2.23 that $(f_1)_T - (f_1)_T(x_0)$ is nearly $(C_1, \varepsilon_1)$-subconvexlike on $X$, for all $T \in \mathcal{L}(X, Y)$. Then, from Theorem 5.2.4 there exists $\mu \in D^{s+} \cap C_1^{\tau+}$ such that $\mu \circ A \in \partial_{\varepsilon_1 \tau_{C_1}(\mu)}(\mu \circ f_1)(x_0)$ and so

$$A + B \in \Gamma_{C_1,\varepsilon_1} f_1(x_0)(\mu) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} f_2(x_0).$$

Since $\mu \in C_1^{\tau+}$ and $(C_1(\varepsilon_1), C_2(\varepsilon_2)) \in S_{C(\varepsilon)}$ we see that

$$(C_1(\varepsilon_1), C_2(\varepsilon_2)) \in S_{C(\varepsilon)}(\mu)$$

and the result follows by Proposition 5.6.6. □

Given a set $C \in \mathcal{F}_Y$ and $\mu \in D^{s+} \cap C^{\tau+}$, we recall that

$$C^\tau_\mu = \{y \in Y : \langle \mu, y\rangle \geq \tau_C(\mu)\}$$

and we define

$$\Lambda_C(\mu) := \{C' \in \Lambda_D : \tau_{C'}(\mu) = \tau_C(\mu)\},$$
$$\Omega_C(\mu) := \{C' \in \Omega : \tau_{C'}(\mu) = \tau_C(\mu)\}.$$

It is clear that $\Omega_C(\mu) \cap 2^D = \Lambda_C(\mu)$. Moreover, given $\varepsilon \geq 0$ and $x_0 \in \operatorname{dom} f_1$, for all $C', C'' \in \Omega_C(\mu)$ it follows that

$$\Gamma_{C',\varepsilon} f_1(x_0)(\mu) = \Gamma_{C'',\varepsilon} f_1(x_0)(\mu). \tag{5.36}$$

**Theorem 5.6.8.** Assume that $\operatorname{int} D^+ \neq \emptyset$. Let $\varepsilon \geq 0$, $C \in \mathcal{F}_Y$, $f_1, f_2 : X \to \overline{Y}$ and $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$. Suppose that $f_1$ and $f_2$ are $D$-convex on $X$, $(f_1+f_2)_T-(f_1+f_2)_T(x_0)$ is nearly $(C, \varepsilon)$-subconvexlike on $X$ for all $T \in \mathcal{L}(X, Y)$,

$f_2$ is regular $(C,\varepsilon')$-subdifferentiable at $x_0$ for all $\varepsilon' \geq 0$ and the qualification condition (MRQC0) holds. Then,

$$\begin{aligned}
&\partial^{\mathrm{Be}}_{C,\varepsilon}(f_1+f_2)(x_0) \\
&\subset \bigcup_{\substack{\mu\in D^{s+}\cap C^{\tau+}\\ \varepsilon_1,\varepsilon_2\geq 0\\ \varepsilon_1+\varepsilon_2=\varepsilon}} \bigcup_{C_2\in\Lambda_C(\mu)} \bigcap_{C_1\in\Omega_C(\mu)} \{\Gamma_{C_1,\varepsilon_1}f_1(x_0)(\mu)+\partial^{\mathrm{s}}_{C_2,\varepsilon_2}f_2(x_0)\} && (5.37)\\
&= \bigcup_{\substack{\mu\in D^{s+}\cap C^{\tau+}\\ \varepsilon_1,\varepsilon_2\geq 0\\ \varepsilon_1+\varepsilon_2=\varepsilon}} \bigcup_{C_2\in\Lambda_C(\mu)} \{\Gamma_{C^\tau_\mu,\varepsilon_1}f_1(x_0)(\mu)+\partial^{\mathrm{s}}_{C_2,\varepsilon_2}f_2(x_0)\}. && (5.38)
\end{aligned}$$

*Proof.* Let $T \in \partial^{\mathrm{Be}}_{C,\varepsilon}(f_1+f_2)(x_0)$. By reasoning as in the proof of Theorem 5.6.4 we deduce that there exist $\mu \in D^{s+}\cap C^{\tau+}$, $\bar\varepsilon_1,\bar\varepsilon_2\geq 0$, $x_1^*\in\partial_{\bar\varepsilon_1}(\mu\circ f_1)(x_0)$ and $x_2^*\in\partial_{\bar\varepsilon_2}(\mu\circ f_2)(x_0)$ such that $\bar\varepsilon_1+\bar\varepsilon_2=\varepsilon\tau_C(\mu)$ and $\mu\circ T = x_1^*+x_2^*$.

Suppose that $\mu\in C^{s\tau+}$. If $\bar\varepsilon_2>0$, since $f_2$ is $(C,\bar\varepsilon_2)$-regular subdifferentiable at $x_0$, by Theorem 5.5.6$(c)$, there exists $C_2\in\Lambda_C(\mu)$ satisfying

$$x_2^*\in\mu\circ\partial^{\mathrm{s}}_{C_2,\frac{\bar\varepsilon_2}{\tau_C(\mu)}}f_2(x_0).$$

Hence, there exists $B\in\partial^{\mathrm{s}}_{C_2,\frac{\bar\varepsilon_2}{\tau_C(\mu)}}f_2(x_0)$ such that $\mu\circ(T-B)=x_1^*$. Let $C_1\in\Omega_C(\mu)$ be arbitrary. It follows that $T-B\in\Gamma_{C_1,\frac{\bar\varepsilon_1}{\tau_C(\mu)}}f_1(x_0)(\mu)$ and so

$$T=(T-B)+B\in\Gamma_{C_1,\varepsilon_1}f_1(x_0)(\mu)+\partial^{\mathrm{s}}_{C_2,\varepsilon_2}f_2(x_0),$$

where $\varepsilon_1:=\frac{\bar\varepsilon_1}{\tau_C(\mu)}\geq 0$ and $\varepsilon_2:=\frac{\bar\varepsilon_2}{\tau_C(\mu)}\geq 0$ satisfy $\varepsilon_1+\varepsilon_2=\varepsilon$, as we want to prove.

If $\bar\varepsilon_2=0$, by the regular $(C,0)$-subdifferentiability property of $f_2$ at $x_0$ we see that $x_2^*\in\mu\circ\partial^{\mathrm{s}}f_2(x_0)$. By Proposition 5.4.3$(b)$ it follows that $\mu\circ\partial^{\mathrm{s}}f_2(x_0)=\mu\circ\partial^{\mathrm{s}}_{C_2,0}f_2(x_0)$, where

$$C_2=\{q\in D:\langle\mu,q\rangle\geq\tau_C(\mu)\}=C^\tau_\mu\cap D\in\Lambda_C(\mu).$$

Then, reasoning in the same way as above we deduce that

$$T\in\Gamma_{C_1,\varepsilon_1}f_1(x_0)(\mu)+\partial^{\mathrm{s}}_{C_2,\varepsilon_2}f_2(x_0)$$

and $\varepsilon_1=\frac{\bar\varepsilon_1}{\tau_C(\mu)}=\varepsilon$ and $\varepsilon_2=0$.

On the other hand, if $\tau_C(\mu)=0$, then $\bar\varepsilon_1=\bar\varepsilon_2=0$. By the regular $(C,0)$-subdifferentiability property of $f_2$ at $x_0$ and Proposition 5.4.3$(b)$ we obtain that

$x_2^* \in \mu \circ \partial^{\mathrm{s}} f_2(x_0) = \mu \circ \partial^{\mathrm{s}}_{D,0} f_2(x_0)$. Thus, there exists $B \in \partial^{\mathrm{s}}_{D,0} f_2(x_0)$ such that $\mu \circ (T - B) = x_1^* \in \partial(\mu \circ f_1)(x_0)$ and we deduce that $T - B \in \Gamma_{C_1,\varepsilon} f_1(x_0)(\mu)$ for all $C_1 \in \Omega_C(\mu)$. Therefore,

$$T = (T - B) + B \in \Gamma_{C_1,\varepsilon_1} f_1(x_0)(\mu) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} f_2(x_0),$$

where $C_2 := D \in \Lambda_C(\mu)$, $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = 0$ and $C_1 \in \Omega_C(\mu)$ is arbitrary, and the proof of inclusion (5.37) is complete.

Finally, equality (5.38) is clear by statement (5.36), because $C^\tau_\mu \in \Omega_C(\mu)$, for all $\mu \in D^{s+} \cap C^{\tau+}$. □

**Theorem 5.6.9.** Let $\varepsilon \geq 0$, $C \in \mathcal{F}_Y$, $f_1, f_2 : X \to \overline{Y}$ and $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$. Then,

$$\bigcup_{\substack{\mu \in D^{s+} \cap C^{\tau+} \\ \varepsilon_1,\varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 \leq \varepsilon}} \bigcup_{C_2 \in \Lambda_C(\mu)} \{\Gamma_{C^\tau_\mu,\varepsilon_1} f_1(x_0)(\mu) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} f_2(x_0)\} \subset \partial^{\mathrm{Be}}_{C,\varepsilon}(f_1 + f_2).$$

*Proof.* For each $\mu \in D^{s+} \cap C^{\tau+}$ it is easy to check that

$$(C^\tau_\mu(\varepsilon_1), C_2(\varepsilon_2)) \in S_{C(\varepsilon)}(\mu), \quad \forall\, \varepsilon_1, \varepsilon_2 \geq 0, \varepsilon_1 + \varepsilon_2 \leq \varepsilon, \quad \forall\, C_2 \in \Lambda_C(\mu),$$

and so the result is a direct consequence of Proposition 5.6.6. □

By Theorems 5.6.8 and 5.6.9 we deduce the following Moreau-Rockafellar theorem for Benson $(C, \varepsilon)$-proper subdifferentials.

**Corollary 5.6.10.** Assume that $\operatorname{int} D^+ \neq \emptyset$. Let $\varepsilon \geq 0$, $C \in \mathcal{F}_Y$, $f_1, f_2 : X \to \overline{Y}$, $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$. Suppose that $f_1$ and $f_2$ are $D$-convex on $X$, $(f_1 + f_2)_T - (f_1 + f_2)_T(x_0)$ is nearly $(C, \varepsilon)$-subconvexlike on $X$ for all $T \in \mathcal{L}(X, Y)$, $f_2$ is regular $(C, \varepsilon')$-subdifferentiable at $x_0$ for all $\varepsilon' \geq 0$ and the qualification condition (MRQC0) holds. Then,

$$\partial^{\mathrm{Be}}_{C,\varepsilon}(f_1 + f_2)(x_0) = \bigcup_{\substack{\mu \in D^{s+} \cap C^{\tau+} \\ \varepsilon_1,\varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 \leq \varepsilon}} \bigcup_{C_2 \in \Lambda_C(\mu)} \{\Gamma_{C^\tau_\mu,\varepsilon_1} f_1(x_0)(\mu) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} f_2(x_0)\}.$$

**Remark 5.6.11.** If $C \in \Omega$ and $f_1, f_2$ are $D$-convex on $X$, then by Remark 1.2.23 we see that $(f_1 + f_2)_T - (f_1 + f_2)_T(x_0)$ is nearly $(C, \varepsilon)$-subconvexlike on $X$, for all $x_0 \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for all $T \in \mathcal{L}(X,Y)$. Then the nearly $(C,\varepsilon)$-subconvexlikeness assumption of Theorems 5.6.4 and 5.6.8 and Corollary 5.6.10 on the mapping $(f_1+f_2)_T - (f_1+f_2)_T(x_0)$ can be removed when $C \in \Omega$.

In the sequel we focus on the gap between the $(C,\varepsilon)$-strong subdifferential and the Benson $(C,\varepsilon)$-proper subdifferential, which is obtained for regular $(C,\varepsilon)$-subdifferentiable and p-regular $\varepsilon$-subdifferentiable mappings. For $\mu \in D^{s+}$ and $C \in \mathcal{F}_Y$, we denote

$$\begin{aligned} \mathrm{K}(\mu) &:= \{T \in \mathcal{L}(X,Y) : \mu \circ T = 0\}, \\ \mathrm{K}(C) &:= \bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \mathrm{K}(\mu). \end{aligned}$$

**Theorem 5.6.12.** Suppose that $\operatorname{int} D^+ \neq \emptyset$. Let $\varepsilon \geq 0$, $C \in \mathcal{F}_Y$ and $x_0 \in \operatorname{dom} f$. Assume that $f_T - f_T(x_0)$ is nearly $(C,\varepsilon)$-subconvexlike on $X$, for all $T \in \mathcal{L}(X,Y)$ and $f$ is $D$-convex on $X$.

$(a)$ If $f$ is regular $(C,\varepsilon')$-subdifferentiable at $x_0$, for all $\varepsilon' \geq 0$, then,

$$\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) = \bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \left\{ \mathrm{K}(\mu) + \bigcup_{C' \in \Lambda_C(\mu)} \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0) \right\}.$$

$(b)$ If $C$ is convex, $C + D = C$ or $D$ has a compact base, and $f$ is p-regular $\varepsilon'$-subdifferentiable at $x_0$, for all $\varepsilon' \geq 0$, then,

$$\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) = \bigcup_{\substack{q \in D \\ C(\varepsilon) - \{q\} \in \mathcal{F}_Y}} \bigcup_{\substack{\mu \in D^{s+} \\ \langle \mu, q \rangle \leq \varepsilon \tau_C(\mu)}} \left\{ \mathrm{K}(\mu) + \partial^{\mathrm{s}}_{\{q\},1} f(x_0) \right\}. \tag{5.39}$$

*Proof.* $(a)$ By applying Corollary 5.6.10 to the mappings $f_1 \equiv 0$ and $f_2 \equiv f$ we obtain that

$$\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) = \bigcup_{\substack{\mu \in D^{s+} \cap C^{\tau+} \\ \varepsilon_1, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 \leq \varepsilon}} \bigcup_{C_2 \in \Lambda_C(\mu)} \{\Gamma_{C^{\tau}_{\mu}, \varepsilon_1} 0(x_0)(\mu) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} f(x_0)\}.$$

It is easy to see that $\Gamma_{C^\tau_\mu,\varepsilon_1}0(x_0)(\mu) = \mathrm{K}(\mu)$, for all $\varepsilon_1 \geq 0$. Thus, it follows that

$$\begin{aligned}
\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) &= \bigcup_{\substack{\mu\in D^{s+}\cap C^{\tau+}\\ 0\leq\varepsilon_2\leq\varepsilon}} \bigcup_{C_2\in\Lambda_C(\mu)} \{\mathrm{K}(\mu) + \partial^{\mathrm{s}}_{C_2,\varepsilon_2} f(x_0)\}\\
&= \bigcup_{\mu\in D^{s+}\cap C^{\tau+}} \left\{\mathrm{K}(\mu) + \bigcup_{\substack{0\leq\varepsilon_2\leq\varepsilon\\ C_2\in\Lambda_C(\mu)}} \partial^{\mathrm{s}}_{C_2,\varepsilon_2} f(x_0)\right\}\\
&= \bigcup_{\mu\in D^{s+}\cap C^{\tau+}} \left\{\mathrm{K}(\mu) + \bigcup_{C'\in\Lambda_C(\mu)} \partial^{\mathrm{s}}_{C',\varepsilon} f(x_0)\right\},
\end{aligned}$$

where the last equality follows by applying Proposition 5.4.3($a$).

($b$) For a nonempty and convex set $C' \in \mathcal{F}_Y$, it is easy to check via Corollary 5.2.6 that

$$\partial^{\mathrm{Be}}_{C',\varepsilon} 0(x_0) = \mathrm{K}(C'), \quad \forall\, \varepsilon \geq 0.$$

Then, by applying Theorem 5.6.4 to $f_1 \equiv 0$ and $f_2 \equiv f$ it follows that

$$\begin{aligned}
\partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0) &= \bigcup_{\substack{(C_1,q)\in\mathcal{F}_Y\times D\\ C(\varepsilon)=C_1+\{q\}}} \left\{\mathrm{K}(C_1) + \partial^{\mathrm{s}}_{\{q\},1} f(x_0)\right\}\\
&= \bigcup_{\substack{q\in D\\ C(\varepsilon)-\{q\}\in\mathcal{F}_Y}} \left\{\mathrm{K}(C(\varepsilon)-\{q\}) + \partial^{\mathrm{s}}_{\{q\},1} f(x_0)\right\}\\
&= \bigcup_{\substack{q\in D\\ C(\varepsilon)-\{q\}\in\mathcal{F}_Y}} \bigcup_{\substack{\mu\in D^{s+}\\ \langle\mu,q\rangle\leq\varepsilon\tau_C(\mu)}} \left\{\mathrm{K}(\mu) + \partial^{\mathrm{s}}_{\{q\},1} f(x_0)\right\}.
\end{aligned}$$

□

**Remark 5.6.13.** Part ($b$) of Theorem 5.6.12 reduces to the following formula by considering $\bar{q} \in Y\backslash(-D\backslash\{0\})$, $C = C_{\bar{q}}$ and $\varepsilon = 1$:

$$\partial^{\mathrm{Be}}_{C_{\bar{q}},1} f(x_0) = \bigcup_{q\in D\backslash(\bar{q}+D\backslash\{0\})} \bigcup_{\substack{\mu\in D^{s+}\\ \langle\mu,q\rangle\leq\langle\mu,\bar{q}\rangle}} \left\{\mathrm{K}(\mu) + \partial^{\mathrm{s}}_{\{q\},1} f(x_0)\right\}. \tag{5.40}$$

Indeed, by Example 2.2.5 it is easy to check that

$$C_{\bar{q}} - \{q\} = C_{\bar{q}-q} \in \mathcal{F}_Y \iff \bar{q} - q \notin -D\backslash\{0\}$$

and formula (5.40) is obtained by statement (5.39). In view of Remark 5.2.7($b$), it can be checked that part ($b$) of Theorem 5.6.12 does not reduce to [83, Theorem

4.2] in this case. Next, we give a counterexample that shows that [83, Theorem 4.2] is not correct.

**Example 5.6.14.** Consider $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and the mapping $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (0, t)$, for all $t \in \mathbb{R}$. Let $D = \mathbb{R}^2_+$, $\bar{q} = (1, -1)$ and $x_0 = 0$. Clearly, $f$ is $\mathbb{R}^2_+$-convex and p-regular $\varepsilon$-subdifferentiable at 0, for all $\varepsilon \geq 0$. Moreover, $-\bar{q} \notin \mathbb{R}^2_+\backslash\{(0,0)\}$. Hence, applying [83, Theorem 4.2] we obtain that

$$\partial^{\text{Be}}_{C_{\bar{q}},1} f(0) = \bigcup_{\substack{q \in \mathbb{R}^2_+ \\ q \notin \bar{q} + \mathbb{R}^2_+ \backslash \{(0,0)\}}} \left\{ \partial^{\text{s}}_{\{q\},1} f(0) + Z_p(\mathbb{R}, \mathbb{R}^2) \right\}, \tag{5.41}$$

where $Z_p(\mathbb{R}, \mathbb{R}^2) = \{A \in \mathcal{L}(\mathbb{R}, \mathbb{R}^2) : \exists \mu \in \operatorname{int} \mathbb{R}^2_+, \mu \circ A = 0\}$. By Theorem 5.4.6 we deduce that $\partial^{\text{s}}_{\{q\},1} f(0) = \{(0,1)\}$ and with easy calculus we have that

$$Z_p(\mathbb{R}, \mathbb{R}^2) = \{(a_1, a_2) \in \mathbb{R}^2 : a_1 a_2 < 0\} \cup \{(0,0)\}.$$

Therefore, by statement (5.41) it follows that

$$\partial^{\text{Be}}_{C_{\bar{q}},1} f(0) = \{(a_1, a_2) \in \mathbb{R}^2 : a_1(a_2 - 1) < 0\} \cup \{(0,1)\}. \tag{5.42}$$

Now, we determinate $\partial^{\text{Be}}_{C_{\bar{q}},1} f(0)$ applying formula (5.40). Denoting $\mu = (\mu_1, \mu_2)$ it is clear that

$$\text{K}(\mu) = \{(a_1, a_2) \in \mathbb{R}^2 : a_1\mu_1 + a_2\mu_2 = 0\}.$$

Hence,

$$\begin{aligned}
\partial^{\text{Be}}_{C_{\bar{q}},1} f(0) &= \bigcup_{\substack{q_1 \in [0,1) \\ q_2 \in [0,+\infty)}} \bigcup_{\substack{\mu_1,\mu_2 > 0 \\ \frac{\mu_2}{\mu_1} \leq \frac{1-q_1}{1+q_2}}} \left\{(a_1, a_2) \in \mathbb{R}^2 : a_1\mu_1 + (a_2 - 1)\mu_2 = 0\right\} \\
&= \bigcup_{\substack{\mu_1,\mu_2 > 0 \\ \mu_1 \geq \mu_2}} \left\{(a_1, a_2) \in \mathbb{R}^2 : a_1\mu_1 + (a_2 - 1)\mu_2 = 0\right\} \\
&= \left\{(a_1, a_2) \in \mathbb{R}^2 : a_1 < 0, a_1 + a_2 \geq 1\right\} \cup \\
&\quad \left\{(a_1, a_2) \in \mathbb{R}^2 : a_1 > 0, a_1 + a_2 \leq 1\right\} \cup \{(0,1)\}.
\end{aligned} \tag{5.43}$$

Since from Remark 1.2.23, $f_T - f_T(0)$ is nearly $(C_{\bar{q}}, 1)$-subconvexlike on $\mathbb{R}$ for all $T \in \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$, we can also apply Corollary 5.2.6 to calculate $\partial^{\text{Be}}_{C_{\bar{q}},1} f(0)$, and it can be checked that we obtain the same result as in statement (5.43). As it is observed, statements (5.42) and (5.43) do not coincide, which proves that [83, Theorem 4.2] is incorrect.

Next, as another application of Theorem 5.6.12, we present an example in which we determinate the approximate Benson proper subdifferential of a map at a point, for more general sets $C \in \mathcal{F}_Y$.

**Example 5.6.15.** Let $X = Y = \mathbb{R}^2$ and consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x_1, x_2) = (x_1 + x_2, (x_1 - x_2)^2).$$

Let $D = \mathbb{R}^2_+$, $\varepsilon \geq 0$, $x_0 = (0,0)$ and $C \in \mathcal{F}_Y$ a convex set such that $C + \mathbb{R}^2_+ = C$ and $0 \notin \operatorname{cl} C$. It is easy to check that $f$ is $\mathbb{R}^2_+$-convex on $\mathbb{R}^2$ and by Remark 1.2.23 we have that $f_T - f_T(0,0)$ is nearly $(C, \varepsilon)$-subconvexlike on $\mathbb{R}^2$, for all $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$. Moreover, from Remarks 5.5.11 and 5.5.5 and Proposition 5.5.9 we deduce that $f$ is p-regular $\varepsilon'$-subdifferentiable at $(0,0)$, for all $\varepsilon' \geq 0$. Hence, by Theorem 5.6.12$(b)$ we know that

$$\partial^{\mathrm{Be}}_{C,\varepsilon} f(0,0) = \bigcup_{\substack{q \in \mathbb{R}^2_+ \\ C(\varepsilon) - \{q\} \in \mathcal{F}_Y}} \ \bigcup_{\substack{\mu \in \operatorname{int} \mathbb{R}^2_+ \\ \langle \mu, q \rangle \leq \varepsilon \tau_C(\mu)}} \left\{ \mathrm{K}(\mu) + \partial^{\mathrm{s}}_{\{q\},1} f(0,0) \right\}.$$

For each $\mu = (\mu_1, \mu_2) \in \operatorname{int} \mathbb{R}^2_+$ we have that

$$\mathrm{K}(\mu) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\frac{\mu_1}{\mu_2}\alpha & -\frac{\mu_1}{\mu_2}\beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}. \tag{5.44}$$

Applying Theorem 5.4.6, for $q = (q_1, q_2) \in \mathbb{R}^2_+$ it follows that $\partial^{\mathrm{s}}_{\{q\},1} f(0,0) = \partial_{q_1} f_1(0,0) \times \partial_{q_2} f_2(0,0)$, where $f = (f_1, f_2)$. For $\delta \geq 0$, it is easy to check that

$$\partial_\delta f_1(0,0) = \{(1,1)\},$$
$$\partial_\delta f_2(0,0) = \{(w_1, w_2) \in \mathbb{R}^2 : w_1 = -w_2, -2\sqrt{\delta} \leq w_2 \leq 2\sqrt{\delta}\}.$$

Therefore,

$$\partial^{\mathrm{s}}_{\{q\},1} f(0,0) = \left\{ \begin{pmatrix} 1 & 1 \\ w & -w \end{pmatrix} : -2\sqrt{q_2} \leq w \leq 2\sqrt{q_2} \right\}. \tag{5.45}$$

Taking into account statements (5.44) and (5.45) we obtain that

$$\begin{aligned}
&\partial^{\mathrm{Be}}_{C,\varepsilon} f(0,0)\\
&= \bigcup_{\substack{q\in\mathbb{R}^2_+\\ C(\varepsilon)-\{q\}\in\mathcal{F}_Y}} \ \bigcup_{\substack{\mu\in\operatorname{int}\mathbb{R}^2_+\\ \langle\mu,q\rangle\leq\varepsilon\tau_C(\mu)}} \left\{ \begin{pmatrix} \alpha+1 & \beta+1\\ w-\frac{\mu_1}{\mu_2}\alpha & -w-\frac{\mu_1}{\mu_2}\beta \end{pmatrix} : \begin{array}{c} \alpha,\beta\in\mathbb{R}\\ -2\sqrt{q_2}\leq w\leq 2\sqrt{q_2} \end{array} \right\}\\
&= \bigcup_{\mu\in\operatorname{int}\mathbb{R}^2_+\cap C^{\tau+}} \left\{ \begin{pmatrix} \alpha+1 & \beta+1\\ w-\frac{\mu_1}{\mu_2}\alpha & -w-\frac{\mu_1}{\mu_2}\beta \end{pmatrix} : \begin{array}{c} \alpha,\beta\in\mathbb{R}\\ -2\sqrt{\delta_\mu}\leq w\leq 2\sqrt{\delta_\mu} \end{array} \right\},
\end{aligned}$$

where $\delta_\mu := \frac{\varepsilon\tau_C(\mu)}{\mu_2}$. Inclusion "$\subset$" in last equality is obtained by taking into account that $q_2 \leq \delta_\mu$, for all $\mu \in \operatorname{int}\mathbb{R}^2_+$ and for all $q \in \mathbb{R}^2_+$ such that $\langle\mu, q\rangle \leq \varepsilon\tau_C(\mu)$. The reverse inclusion follows by considering for each $\mu \in \operatorname{int}\mathbb{R}^2_+ \cap C^{\tau+}$ the point $q_\mu = (0, \delta_\mu)$.

# Capítulo 6

# Proper $\varepsilon$-subdifferentials of vector mappings: Chain rules

## 6.1 Introduction

In order to complete the study of the Benson $(C, \varepsilon)$-proper subdifferential and its calculus rules initiated in Chapter 5, next we establish chain rules for obtaining the Benson $(C, \varepsilon)$-proper subdifferential of the composition of two vector mappings under generalized convexity assumptions. For this aim, we also use the $(C, \varepsilon)$-strong subdifferential, introduced in Section 5.4.

Specifically, we derive two chain rules. The first one is obtained by the regularity condition introduced by El Maghri in [83] (see Definition 5.5.7), and the second one is stated via the regular $(C, \varepsilon)$-subdifferentiability condition introduced in Definition 5.5.1.

In [84], El Maghri obtained calculus rules for the composition of vector mappings using the approximate proper subdifferential stated in Definition 1.2.26. Several results of this chapter, which are collected in [46], extend them.

## 6.2 Chain rules

In this section, we derive formulae for the Benson $(Q, \varepsilon)$-proper subdifferential of the composition of a proper vector mapping $f_1 : X \to \overline{Y}$ with another nondecreasing proper vector mapping $f_2 : Y \to \overline{Z}$, where $\emptyset \neq Q \subset Z$, and the ordering cone $K \subset Z$ is assumed to be nontrivial, pointed, closed and convex. Moreover, we assume that $f_2(+\infty_Y) = +\infty_Z$ and we consider that the topology of $Z^*$ is compatible with the dual pairing.

As a particular case, we obtain calculus rules for the Benson $(Q, \varepsilon)$-proper subdifferential of the mappings $A \circ f_1$, for $A \in \mathcal{L}_+(Y, Z)$ and $f_2 \circ B$, for $B \in \mathcal{L}(X, Y)$ (recall that $\mathcal{L}_+(Y, Z)$ denotes the subset of $\mathcal{L}(Y, Z)$ of the linear mappings $T$ such that $T(D) \subset K$).

We recall that $f_2$ is $(D, K)$-nondecreasing on a nonempty set $F \subset Y$, if

$$\forall y_1, y_2 \in F \cap \operatorname{dom} f_2,\ y_1 \leq_D y_2 \Longrightarrow f_2(y_1) \leq_K f_2(y_2).$$

We denote

$$\mathcal{H}_Z := \{Q \subset Z : Q \neq \emptyset, \operatorname{cl}(Q(0)) \cap (-K\backslash\{0\}) = \emptyset\},$$
$$\mathcal{F}_Z := \{Q \in \mathcal{H}_Z : Q \neq \emptyset, K^{s+} \cap Q^{\tau+} \neq \emptyset\}.$$

For $\varepsilon \geq 0$ and $Q \in \mathcal{H}_Z$, we define

$$S_{Q(\varepsilon)} := \{(Q_1, Q_2) \in \mathcal{H}_Z \times 2^K : Q(\varepsilon) \subset Q_1 + Q_2\}.$$

**Theorem 6.2.1.** Let $\varepsilon \geq 0$, $Q \in \mathcal{H}_Z$ and $x_0 \in \operatorname{dom}(f_2 \circ f_1)$. Then,

$$\partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2 \circ f_1)(x_0) \supset \bigcup_{\substack{(Q_1,Q_2)\in S_{Q(\varepsilon)} \\ Q_1+K\subset \operatorname{cl} Q_1}} \ \bigcup_{A\in\partial^{\mathrm{s}}_{Q_2,1} f_2(f_1(x_0))} \partial^{\mathrm{Be}}_{Q_1,1}(A \circ f_1)(x_0).$$

Moreover, if $K$ has a compact base, then

$$\partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2 \circ f_1)(x_0) \supset \bigcup_{(Q_1,Q_2)\in S_{Q(\varepsilon)}} \ \bigcup_{A\in\partial^{\mathrm{s}}_{Q_2,1} f_2(f_1(x_0))} \partial^{\mathrm{Be}}_{Q_1,1}(A \circ f_1)(x_0). \tag{6.1}$$

*Proof.* Let $(Q_1, Q_2) \in S_{Q(\varepsilon)}$, with $Q_1 + K \subset \operatorname{cl} Q_1$, $A \in \partial^{\mathrm{s}}_{Q_2,1} f_2(f_1(x_0))$ and $B \in \partial^{\mathrm{Be}}_{Q_1,1}(A \circ f_1)(x_0)$. Suppose by contradiction that $B \notin \partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2 \circ f_1)(x_0)$, i.e., $x_0 \notin \operatorname{Be}((f_2 \circ f_1) - B, X, Q, \varepsilon)$. Thus, there exist $k_0 \in K\backslash\{0\}$ and nets $(\alpha_i) \subset \mathbb{R}_+$, $(x_i) \subset \operatorname{dom}(f_2 \circ f_1)$ and $(q_i) \subset Q(\varepsilon)$ such that

$$\alpha_i((f_2 \circ f_1)(x_i) - B(x_i) + q_i - (f_2 \circ f_1)(x_0) + B(x_0)) \to -k_0. \tag{6.2}$$

Since $Q(\varepsilon) \subset Q_1 + Q_2$, there exist nets $(q_i^1) \subset Q_1$ and $(q_i^2) \subset Q_2$ such that $q_i = q_i^1 + q_i^2$, for each $i$. On the other hand, as $A \in \partial^s_{Q_2,1} f_2(f_1(x_0))$, it follows that

$$(f_2 - A)(y) + u - (f_2 - A)(f_1(x_0)) \in K, \ \forall y \in \operatorname{dom} f_2, \forall u \in Q_2.$$

Hence, in particular we have that

$$(f_2 - A)(f_1(x_i)) + q_i^2 - (f_2 - A)(f_1(x_0)) \in K, \ \forall i,$$

so there exists $(k_i) \subset K$ such that

$$(f_2 \circ f_1)(x_i) + q_i^2 - (f_2 \circ f_1)(x_0) = k_i + (A \circ f_1)(x_i) - (A \circ f_1)(x_0). \tag{6.3}$$

Thus, taking into account statement (6.3) we deduce that (6.2) is equivalent to

$$\alpha_i((A \circ f_1)(x_i) - B(x_i) + q_i^1 + k_i - (A \circ f_1)(x_0) + B(x_0)) \to -k_0,$$

which implies that $x_0 \notin \operatorname{Be}((A \circ f_1) - B, X, Q_1, 1)$, since $Q_1 + K \subset \operatorname{cl} Q_1$. Therefore, $B \notin \partial^{\text{Be}}_{Q_1,1}(A \circ f_1)(x_0)$ and we reach a contradiction.

Finally, suppose that $K$ has a compact base. Reasoning in the same way as above and applying Proposition 5.2.3($e$) it follows that $B \notin \partial^{\text{Be}}_{Q_1+K,1}(A \circ f_1)(x_0) = \partial^{\text{Be}}_{Q_1,1}(A \circ f_1)(x_0)$, obtaining again a contradiction, and (6.1) is proved. □

**Remark 6.2.2.** Let $q \notin -K\backslash\{0\}$ and suppose that $\operatorname{int} K^+ \neq \emptyset$ and $(f_2 \circ f_1)_T - (f_2 \circ f_1)_T(x_0)$ is nearly $(q + K, 1)$-subconvexlike on $X$, for all $T \in \mathcal{L}(X, Z)$. Let $q_1 \notin -K\backslash\{0\}$ and $q_2 \in K$ and such that $q_1 + q_2 = q$. Thus, taking into account Remark 5.2.7($b$), if we consider $\varepsilon = 1$ and we define $Q := q + K$, $Q_1 := q_1 + K$ and $Q_2 := \{q_2\}$, then the first part of Theorem 6.2.1 reduces to the first part of [84, Theorem 3.1, $\sigma = p$, $F \equiv 0$].

Moreover, if we consider that $Q = Q_1 = K$ and $Q_2 = \{0\}$, then the first part of Theorem 6.2.1 reduces to the first part of [86, Theorem 4.2, $\sigma = p$].

Next result provides a calculus rule for the $\varepsilon$-subdifferential of the post-composition of the mapping $f_1$ with a scalar mapping $h : Y \to \overline{\mathbb{R}}$. Part ($a$) was stated in [120, Theorem 2.8.10] and part ($b$) was proved in [56, Theorem 2.2].

**Theorem 6.2.3.** Let $\varepsilon \geq 0$, $h : Y \to \overline{\mathbb{R}}$ and $x_0 \in \operatorname{dom}(h \circ f_1)$. Suppose that $f_1$ is $D$-convex on $X$, $h$ is convex on $Y$ and the following constraint qualification holds:

(MRQC1) There exists $\bar{x} \in \operatorname{dom}(h \circ f_1)$ such that $h$ is continuous at $f_1(\bar{x})$.

$(a)$ If $h$ is $(D, \mathbb{R}_+)$-nondecreasing on $\mathrm{Im} f_1 + D$ then

$$\partial_\varepsilon (h \circ f_1)(x_0) = \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \bigcup_{y^* \in \partial_{\varepsilon_2} h(f_1(x_0)) \cap D^+} \partial_{\varepsilon_1} (y^* \circ f_1)(x_0).$$

$(b)$ If $T \in \mathcal{L}(X, Y)$ then

$$\partial_\varepsilon (h \circ T)(x_0) = T^* \partial_\varepsilon h(T(x_0)),$$

where $T^*$ denotes the adjoint of $T$.

Next theorem will be useful in the following, since it gives a sufficient condition for the mapping $(f_2 \circ f_1)_B - (f_2 \circ f_1)_B(x_0)$ to be nearly $(Q, \varepsilon)$-subconvexlike on $X$, for all $B \in \mathcal{L}(X, Z)$.

**Theorem 6.2.4.** Let $\varepsilon \geq 0$, $\emptyset \neq Q \subset Z$ and $x_0 \in \operatorname{dom}(f_2 \circ f_1)$. If $f_1$ is $D$-convex on $X$, $f_2$ is $K$-convex on $Y$ and $(D, K)$-nondecreasing on $\mathrm{Im} f_1 + D$, and $Q$ is convex and verifies that $Q + K = Q$, then $(f_2 \circ f_1)_B - (f_2 \circ f_1)_B(x_0)$ is nearly $(Q, \varepsilon)$-subconvexlike on $X$, for all $B \in \mathcal{L}(X, Z)$.

*Proof.* Let $B \in \mathcal{L}(X, Z)$. We are going to show that the set

$$E := ((f_2 \circ f_1) - B)(X) + Q(\varepsilon) - ((f_2 \circ f_1) - B)(x_0)$$

is convex. Let $y_1, y_2 \in E$ and $\alpha \in (0, 1)$. There exist $x_i \in X$ and $q_i \in Q(\varepsilon)$, $i = 1, 2$, such that

$$y_i = (f_2 \circ f_1)(x_i) - B(x_i) + q_i - (f_2 \circ f_1)(x_0) + B(x_0), \; i = 1, 2.$$

On the one hand, since $f_2$ is $K$-convex on $Y$ it follows that

$$f_2(\alpha f_1(x_1) + (1 - \alpha) f_1(x_2)) \leq_K \alpha f_2(f_1(x_1)) + (1 - \alpha) f_2(f_1(x_2)). \tag{6.4}$$

On the other hand, as $f_1$ is $D$-convex on $X$ and $f_2$ is $(D,K)$-nondecreasing on $\mathrm{Im} f_1 + D$ we have that

$$f_2(f_1(\alpha x_1 + (1-\alpha)x_2)) \leq_K f_2(\alpha f_1(x_1) + (1-\alpha)f_1(x_2)). \tag{6.5}$$

Thus, by inequalities (6.4) and (6.5) we deduce that

$$f_2(f_1(\alpha x_1 + (1-\alpha)x_2)) \leq_K \alpha f_2(f_1(x_1)) + (1-\alpha)f_2(f_1(x_2)),$$

which is equivalent to

$$\alpha(f_2 \circ f_1)(x_1) + (1-\alpha)(f_2 \circ f_1)(x_2) \in (f_2 \circ f_1)(\alpha x_1 + (1-\alpha)x_2) + K. \tag{6.6}$$

Taking into account (6.6), it follows that

$$\begin{aligned} \alpha y_1 + (1-\alpha)y_2 =& \alpha(f_2 \circ f_1)(x_1) + (1-\alpha)(f_2 \circ f_1)(x_2) - B(\alpha x_1 + (1-\alpha)x_2) + \\ & \alpha q_1 + (1-\alpha)q_2 - ((f_2 \circ f_1) - B)(x_0) \\ \subset& (f_2 \circ f_1)(\alpha x_1 + (1-\alpha)x_2) - B(\alpha x_1 + (1-\alpha)x_2) + Q(\varepsilon) + K \\ & - ((f_2 \circ f_1) - B)(x_0) \\ \subset& ((f_2 \circ f_1) - B)(X) + Q(\varepsilon) - ((f_2 \circ f_1) - B)(x_0) = E, \end{aligned}$$

so $E$ is convex, and then $\mathrm{cl}\,\mathrm{cone}(E)$ is convex too, as we want to prove. □

For $\lambda \in K^{s+}$, we define the set $\mathcal{L}_D(\lambda) \subset \mathcal{L}(Y,Z)$ by

$$\mathcal{L}_D(\lambda) := \{A \in \mathcal{L}(Y,Z) : \lambda \circ A \in D^+\}.$$

**Theorem 6.2.5.** Suppose that $\mathrm{int}\, K^+ \neq \emptyset$. Let $\varepsilon \geq 0$, $Q \in \mathcal{F}_Z$ and $x_0 \in \mathrm{dom}(f_2 \circ f_1)$. Suppose that $(f_2 \circ f_1)_B - (f_2 \circ f_1)_B(x_0)$ is nearly $(Q,\varepsilon)$-subconvexlike on $X$ for all $B \in \mathcal{L}(X,Z)$, $f_1$ is $D$-convex on $X$, $f_2$ is $K$-convex on $Y$, $(D,K)$-nondecreasing on $\mathrm{Im} f_1 + D$ and p-regular $\bar{\varepsilon}$-subdifferentiable at $f_1(x_0)$, for all $\bar{\varepsilon} \geq 0$, and the following constraint qualification holds:

(MRQC2) There exists $\bar{x} \in \mathrm{dom}(f_2 \circ f_1)$ such that $f_2$ is star $K$-continuous at $f_1(\bar{x})$.

Then,

$$\partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2\circ f_1)(x_0)\subset \bigcup_{\lambda\in K^{s+}\cap Q^{\tau+}}\ \bigcup_{\substack{(q,\hat{Q})\in K\times\mathcal{F}_Z\\ Q(\varepsilon)=\hat{Q}+\{q\}\\ \tau_{\hat{Q}}(\lambda)\geq 0}}\ \bigcup_{A\in\partial^{\mathrm{s}}_{\{q\},1}f_2(f_1(x_0))\cap\mathcal{L}_D(\lambda)}\partial^{\mathrm{Be}}_{\hat{Q},1}(A\circ f_1)(x_0).$$

Moreover, if either $Q+K=Q$ or $K$ has a compact base, then

$$\partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2\circ f_1)(x_0)=\bigcup_{\substack{(q,\hat{Q})\in K\times\mathcal{F}_Z\\ Q(\varepsilon)=\hat{Q}+\{q\}}}\ \bigcup_{A\in\partial^{\mathrm{s}}_{\{q\},1}f_2(f_1(x_0))}\partial^{\mathrm{Be}}_{\hat{Q},1}(A\circ f_1)(x_0). \tag{6.7}$$

*Proof.* Let $B\in\partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2\circ f_1)(x_0)$. By Theorem 5.2.4 there exists $\lambda\in K^{s+}\cap Q^{\tau+}$ such that

$$\lambda\circ B\in\partial_{\varepsilon\tau_Q(\lambda)}((\lambda\circ f_2)\circ f_1)(x_0).$$

By the hypotheses, it is clear that $\lambda\circ f_2$ is convex on $Y$, $(D,\mathbb{R}_+)$-nondecreasing on $\mathrm{Im} f_1+D$, and there exists a point $\bar{x}\in\mathrm{dom}(f_2\circ f_1)$ such that $\lambda\circ f_2$ is continuous at $f_1(\bar{x})$. Hence, applying Theorem 6.2.3$(a)$ for $h=\lambda\circ f_2$, there exist $\delta_1,\delta_2\geq 0$, with $\delta_1+\delta_2=\varepsilon\tau_Q(\lambda)$, and $y^*\in\partial_{\delta_2}(\lambda\circ f_2)(f_1(x_0))\cap D^+$ such that $\lambda\circ B\in\partial_{\delta_1}(y^*\circ f_1)(x_0)$. If $\delta_2>0$, since $f_2$ is p-regular $\delta_2$-subdifferentiable at $f_1(x_0)$, there exist $q\in K$, with $\langle\lambda,q\rangle=\delta_2$ and $A\in\partial^{\mathrm{s}}_{\{q\},1}f_2(f_1(x_0))$ such that $y^*=\lambda\circ A$. Thus, it is clear that $A\in\mathcal{L}_D(\lambda)$. Consider the set $\hat{Q}:=Q(\varepsilon)-\{q\}$. It follows that $\tau_{\hat{Q}}(\lambda)=\varepsilon\tau_Q(\lambda)-\delta_2=\delta_1$. Hence, we obtain that

$$\lambda\circ B\in\partial_{\delta_1}(\lambda\circ(A\circ f_1))(x_0)=\partial_{\tau_{\hat{Q}}(\lambda)}(\lambda\circ(A\circ f_1))(x_0),$$

and by Theorem 5.2.5 we have that $B\in\partial^{\mathrm{Be}}_{\hat{Q},1}(A\circ f_1)(x_0)$, as we want to prove. If $\delta_2=0$, there exists $A\in\partial^{\mathrm{s}}f_2(f_1(x_0))=\partial^{\mathrm{s}}_{\{0\},1}f_2(f_1(x_0))$ such that $y^*=\lambda\circ A$, and considering $q=0$ and $\hat{Q}=Q(\varepsilon)$ the result follows.

Suppose additionally that $Q+K=Q$ and let $(q,\hat{Q})\in K\times\mathcal{F}_Z$ such that $Q(\varepsilon)=\hat{Q}+\{q\}$. Then $\hat{Q}+K=Q(\varepsilon)+K-\{q\}\subset\mathrm{cl}\,Q(\varepsilon)-\{q\}=\mathrm{cl}\,\hat{Q}$ and by Theorem 6.2.1 we have that

$$\bigcup_{\substack{(q,\hat{Q})\in K\times\mathcal{F}_Z\\ Q(\varepsilon)=\hat{Q}+\{q\}}}\ \bigcup_{A\in\partial^{\mathrm{s}}_{\{q\},1}f_2(f_1(x_0))}\partial^{\mathrm{Be}}_{\hat{Q},1}(A\circ f_1)(x_0)\subset\partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2\circ f_1)(x_0).$$

Also, it follows that

$$\partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2\circ f_1)(x_0)\subset \bigcup_{\lambda\in K^{s+}\cap Q^{\tau+}}\ \bigcup_{\substack{(q,\hat{Q})\in K\times\mathcal{F}_Z\\ Q(\varepsilon)=\hat{Q}+\{q\}\\ \tau_{\hat{Q}}(\lambda)\geq 0}}\ \bigcup_{A\in\partial^{\mathrm{s}}_{\{q\},1}f_2(f_1(x_0))\cap\mathcal{L}_D(\lambda)}\partial^{\mathrm{Be}}_{\hat{Q},1}(A\circ f_1)(x_0)$$

$$\subset \bigcup_{\substack{(q,\hat{Q})\in K\times\mathcal{F}_Z\\ Q(\varepsilon)=\hat{Q}+\{q\}}}\ \bigcup_{A\in\partial^{\mathrm{s}}_{\{q\},1}f_2(f_1(x_0))}\partial^{\mathrm{Be}}_{\hat{Q},1}(A\circ f_1)(x_0)$$

so (6.7) is proved. The same conclusion is obtained applying Theorem 6.2.1 when $K$ has a compact base. □

**Remark 6.2.6.** ($a$) Observe that if we suppose that $Q$ is convex and $Q+K=Q$ in Theorem 6.2.5, then by Theorem 6.2.4 we can avoid the hypothesis of nearly $(Q,\varepsilon)$-subconvexlikeness of $(f_2\circ f_1)_B-(f_2\circ f_1)_B(x_0)$ for all $B\in\mathcal{L}(X,Z)$.

($b$) Theorem 6.2.5 reduces to [84, Theorem 3.1] considering that $Q=q+K$, for $q\notin -K\backslash\{0\}$, and reduces to [86, Theorem 4.2], when $Q=K$.

Now, we are going to prove another chain rule for the calculus of $\partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2\circ f_1)(x_0)$, considering the regular $(Q,\bar{\varepsilon})$-subdifferentiability assumption of the mapping $f_2$ at $f_1(x_0)$.

Given $\varepsilon\geq 0$, $Q\in\mathcal{F}_Z$, a proper mapping $h: X\to\overline{Z}$ and $x_0\in\operatorname{dom}h$, recall that the set-valued mapping $\Gamma_{Q,\varepsilon}h(x_0): K^{s+}\cap Q^{\tau+}\rightrightarrows\mathcal{L}(X,Z)$ is

$$\Gamma_{Q,\varepsilon}h(x_0)(\lambda)=\{T\in\mathcal{L}(X,Z):\lambda\circ T\in\partial_{\varepsilon\tau_Q(\lambda)}(\lambda\circ h)(x_0)\},$$

and we denote $\Gamma h(x_0)(\lambda):=\Gamma_{Q,0}h(x_0)(\lambda)$, whenever $\operatorname{cl}\operatorname{cone}Q=K$.

Observe that under the hypotheses of Corollary 5.2.6 referred to the set $Q$ and the mapping $h$, we have that

$$\partial^{\mathrm{Be}}_{Q,\varepsilon}h(x_0)=\bigcup_{\lambda\in K^{s+}\cap Q^{\tau+}}\Gamma_{Q,\varepsilon}h(x_0)(\lambda).$$

Next, for the convenience of the reader we recall the sets $\Lambda_K$, $\Lambda_Q(\lambda)$ and $S_Q(\lambda)$, with $\lambda\in K^{s+}\cap Q^{\tau+}$:

$$\Lambda_K=\{Q'\subset K: Q'\neq\emptyset, Q'=Q'+K, Q'\text{ is convex}\},$$

$$\Lambda_Q(\lambda)=\{Q'\in\Lambda_K:\tau_{Q'}(\lambda)=\tau_Q(\lambda)\},$$

$$S_Q(\lambda)=\{(Q_1,Q_2)\in\mathcal{F}_Z\times 2^K,\tau_{Q_1}(\lambda)\geq 0,\tau_Q(\lambda)\geq\tau_{Q_1+Q_2}(\lambda)\}.$$

Next result is obtained without any additional assumption.

**Theorem 6.2.7.** Let $\varepsilon \geq 0$, $Q \in \mathcal{F}_Z$ and $x_0 \in \operatorname{dom}(f_2 \circ f_1)$. It follows that

$$\partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2 \circ f_1)(x_0) \supset \bigcup_{\lambda \in K^{s+} \cap Q^{\tau+}} \bigcup_{\substack{\varepsilon_1,\varepsilon_2 \geq 0 \\ (Q_1,Q_2) \in \mathcal{F}_Z \times 2^K \\ (Q_1(\varepsilon_1),Q_2(\varepsilon_2)) \in S_{Q(\varepsilon)}(\lambda)}} \bigcup_{A \in \partial^{\mathrm{s}}_{Q_2,\varepsilon_2} f_2(f_1(x_0))} \Gamma_{Q_1,\varepsilon_1}(A \circ f_1)(x_0)(\lambda).$$

*Proof.* Let $\lambda \in K^{s+} \cap Q^{\tau+}$, $\varepsilon_1, \varepsilon_2 \geq 0$ with $\varepsilon_1 + \varepsilon_2 = \varepsilon$, $(Q_1, Q_2) \in \mathcal{F}_Z \times 2^K$ with $(Q_1(\varepsilon_1), Q_2(\varepsilon_2)) \in S_{Q(\varepsilon)}(\lambda)$, $A \in \partial^{\mathrm{s}}_{Q_2,\varepsilon_2} f_2(f_1(x_0))$ and $B \in \Gamma_{Q_1,\varepsilon_1}(A \circ f_1)(x_0)(\lambda)$. Then,

$$\lambda \circ B \in \partial_{\varepsilon_1 \tau_{Q_1}(\lambda)}(\lambda \circ (A \circ f_1))(x_0),$$

i.e.,

$$(\lambda \circ A \circ f_1)(x) \geq (\lambda \circ A \circ f_1)(x_0) - \varepsilon_1 \tau_{Q_1}(\lambda) + \langle \lambda \circ B, x - x_0 \rangle, \ \forall x \in \operatorname{dom} f_1. \tag{6.8}$$

On the other hand, as $A \in \partial^{\mathrm{s}}_{Q_2,\varepsilon_2} f_2(f_1(x_0))$, by Theorem 5.4.4 it follows that

$$\lambda \circ A \in \partial_{\varepsilon_2 \tau_{Q_2}(\lambda)}(\lambda \circ f_2)(f_1(x_0)),$$

i.e.,

$$(\lambda \circ f_2)(y) \geq (\lambda \circ f_2)(f_1(x_0)) - \varepsilon_2 \tau_{Q_2}(\lambda) + \langle \lambda \circ A, y - f_1(x_0) \rangle, \ \forall y \in \operatorname{dom} f_2.$$

In particular, from the inequality above we have for all $x \in \operatorname{dom}(f_2 \circ f_1)$ that

$$(\lambda \circ f_2 \circ f_1)(x) \geq (\lambda \circ f_2 \circ f_1)(x_0) - \varepsilon_2 \tau_{Q_2}(\lambda) + \langle \lambda \circ A, f_1(x) - f_1(x_0) \rangle. \tag{6.9}$$

Thus, adding in (6.8) and (6.9) and taking into account that

$$\varepsilon_1 \tau_{Q_1}(\lambda) + \varepsilon_2 \tau_{Q_2}(\lambda) = \tau_{Q_1(\varepsilon_1)+Q_2(\varepsilon_2)}(\lambda) \leq \varepsilon \tau_Q(\lambda),$$

we obtain for all $x \in \operatorname{dom}(f_2 \circ f_1)$ that

$$\begin{aligned} \lambda \circ (f_2 \circ f_1)(x) &\geq \lambda \circ (f_2 \circ f_1)(x_0) - \varepsilon_1 \tau_{Q_1}(\lambda) - \varepsilon_2 \tau_{Q_2}(\lambda) + \langle \lambda \circ B, x - x_0 \rangle \\ &\geq \lambda \circ (f_2 \circ f_1)(x_0) - \varepsilon \tau_Q(\lambda) + \langle \lambda \circ B, x - x_0 \rangle. \end{aligned}$$

Hence, it follows that $\lambda \circ B \in \partial_{\varepsilon \tau_Q(\lambda)}(\lambda \circ (f_2 \circ f_1))(x_0)$, and by Theorem 5.2.5 we conclude that $B \in \partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2 \circ f_1)(x_0)$, as we want to prove. □

**Theorem 6.2.8.** Suppose that $\operatorname{int} K^+ \neq \emptyset$. Let $\varepsilon \geq 0$, $Q \in \mathcal{F}_Z$ and $x_0 \in \operatorname{dom}(f_2 \circ f_1)$. Suppose that $(f_2 \circ f_1)_B - (f_2 \circ f_1)_B(x_0)$ is nearly $(Q, \varepsilon)$-subconvexlike on $X$ for all $B \in \mathcal{L}(X, Z)$, $f_1$ is $D$-convex on $X$, $f_2$ is $K$-convex on $Y$, $(D, K)$-nondecreasing on $\operatorname{Im} f_1 + D$ and regular $(Q, \bar{\varepsilon})$-subdifferentiable at $f_1(x_0)$, for all $\bar{\varepsilon} \geq 0$, and (MRQC2) holds. Then,

$$\partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2 \circ f_1)(x_0) \subset \bigcup_{\substack{\lambda \in K^{s+} \cap Q^{\tau+} \\ \varepsilon_2, \varepsilon_1 \geq 0 \\ \varepsilon_2 + \varepsilon_1 = \varepsilon}} \; \bigcup_{Q' \in \Lambda_Q(\lambda)} \; \bigcup_{A \in \partial^{\mathrm{s}}_{Q', \varepsilon_2} f_2(f_1(x_0)) \cap \mathcal{L}_D(\lambda)} \Gamma_{Q, \varepsilon_1}(A \circ f_1)(x_0)(\lambda).$$

*Proof.* Let $B \in \partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2 \circ f_1)(x_0)$. Then, by Theorem 5.2.4 there exists $\lambda \in K^{s+} \cap Q^{\tau+}$ such that

$$\lambda \circ B \in \partial_{\varepsilon \tau_Q(\lambda)}((\lambda \circ f_2) \circ f_1)(x_0).$$

From the hypotheses, we can apply Theorem 6.2.3$(a)$ for $h = \lambda \circ f_2$, from which there exist $\delta_1, \delta_2 \geq 0$, with $\delta_1 + \delta_2 = \varepsilon \tau_Q(\lambda)$, and $y^* \in \partial_{\delta_2}(\lambda \circ f_2)(f_1(x_0)) \cap D^+$ such that $\lambda \circ B \in \partial_{\delta_1}(y^* \circ f_1)(x_0)$. We analyze the following cases.

If $\tau_Q(\lambda) > 0$ and $\delta_2 > 0$, by the regular $(Q, \delta_2)$-subdifferentiability of $f_2$ at $f_1(x_0)$ and Theorem 5.5.6$(c)$ it follows that

$$\partial_{\delta_2}(\lambda \circ f_2)(f_1(x_0)) = \bigcup_{Q' \in \Lambda_Q(\lambda)} \lambda \circ \partial^{\mathrm{s}}_{Q', \frac{\delta_2}{\tau_Q(\lambda)}} f_2(f_1(x_0)),$$

so there exists $Q' \in \Lambda_Q(\lambda)$ and $A \in \partial^{\mathrm{s}}_{Q', \frac{\delta_2}{\tau_Q(\lambda)}} f_2(f_1(x_0))$ such that $y^* = \lambda \circ A$. Thus, $A \in \mathcal{L}_D(\lambda)$ and $B \in \Gamma_{Q, \frac{\delta_1}{\tau_Q(\lambda)}}(A \circ f_1)(x_0)(\lambda)$. By denoting $\varepsilon_1 := \frac{\delta_1}{\tau_Q(\lambda)}$ and $\varepsilon_2 := \frac{\delta_2}{\tau_Q(\lambda)}$, we have that $\varepsilon_1 + \varepsilon_2 = \varepsilon$, $A \in \partial^{\mathrm{s}}_{Q', \varepsilon_2} f_2(f_1(x_0)) \cap \mathcal{L}_D(\lambda)$ and $B \in \Gamma_{Q, \varepsilon_1}(A \circ f_1)(x_0)(\lambda)$, as we want to prove.

If $\tau_Q(\lambda) > 0$ and $\delta_2 = 0$, since $f_2$ is regular $(Q, 0)$-subdifferentiable at $f_1(x_0)$, it follows that $\partial_{\delta_2}(\lambda \circ f_2)(f_1(x_0)) = \lambda \circ \partial^{\mathrm{s}} f_2(f_1(x_0))$. By Proposition 5.4.3$(b)$ we deduce that $\partial^{\mathrm{s}} f_2(f_1(x_0)) = \partial^{\mathrm{s}}_{Q', 0} f_2(f_1(x_0))$, where $Q' = q + K$, with $q \in K$ such that $\langle \lambda, q \rangle = \tau_Q(\lambda)$. It is clear that $Q' \in \Lambda_Q(\lambda)$ and reasoning in the same way as above we have that $A \in \partial^{\mathrm{s}}_{Q', \varepsilon_2} f_2(f_1(x_0)) \cap \mathcal{L}_D(\lambda)$ and $B \in \Gamma_{Q, \varepsilon_1}(A \circ f_1)(x_0)(\lambda)$, with $\varepsilon_1 := \varepsilon$ and $\varepsilon_2 := 0$.

Finally, if $\tau_Q(\lambda) = 0$, then $\delta_1 = \delta_2 = 0$ and, again, by the regular $(Q, 0)$-subdifferentiability of $f_2$ at $f_1(x_0)$ we conclude that $A \in \partial^{\mathrm{s}}_{K, \varepsilon_2} f_2(f_1(x_0)) \cap \mathcal{L}_D(\lambda)$ and $B \in \Gamma_{Q, \varepsilon_1}(A \circ f_1)(x_0)(\lambda)$, with $\varepsilon_2 = 0$ and $\varepsilon_1 = \varepsilon$. □

As a direct consequence of Theorems 6.2.7 and 6.2.8 we obtain the following chain rule.

**Corollary 6.2.9.** With the same hypotheses of Theorem 6.2.8 we have that

$$\partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2 \circ f_1)(x_0) = \bigcup_{\substack{\lambda \in K^{s+} \cap Q^{\tau+} \\ \varepsilon_1, \varepsilon_2 \geq 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \ \bigcup_{Q' \in \Lambda_Q(\lambda)} \ \bigcup_{A \in \partial^{\mathrm{s}}_{Q',\varepsilon_2} f_2(f_1(x_0)) \cap \mathcal{L}_D(\lambda)} \Gamma_{Q,\varepsilon_1}(A \circ f_1)(x_0)(\lambda).$$

For the exact case, we obtain the following theorem.

**Theorem 6.2.10.** Let $x_0 \in \mathrm{dom}(f_2 \circ f_1)$. Suppose that $\mathrm{int}\, K^+ \neq \emptyset$, $f_1$ is $D$-convex on $X$, $f_2$ is $K$-convex on $Y$, $(D, K)$-nondecreasing on $\mathrm{Im} f_1 + D$ and p-regular subdifferentiable at $f_1(x_0)$, and (MRQC2) holds. Then,

$$\partial^{\mathrm{Be}}(f_2 \circ f_1)(x_0) = \bigcup_{\lambda \in K^{s+}} \ \bigcup_{A \in \partial^{\mathrm{s}} f_2(f_1(x_0)) \cap \mathcal{L}_D(\lambda)} \Gamma(A \circ f_1)(x_0)(\lambda).$$

**Remark 6.2.11.** Observe that $\partial^{\mathrm{Be}}(f_2 \circ f_1)(x_0) = \partial^{\mathrm{Be}}_{K,0}(f_2 \circ f_1)(x_0)$. Moreover, by Theorem 6.2.4 $(f_2 \circ f_1)_B - (f_2 \circ f_1)_B(x_0)$ is nearly $(K, 0)$-subconvexlike on $X$, for all $B \in \mathcal{L}(X, Z)$. Because of this, we can avoid the assumption of nearly subconvexlikeness in Theorem 6.2.10.

In the following result, we obtain formulae for the calculus of the Benson $(Q, \varepsilon)$-proper subdifferential of the mapping $f_2 \circ T$, for $T \in \mathcal{L}(X, Y)$.

For $\lambda \in K^{s+}$, recall that $K(\lambda) \subset \mathcal{L}(X, Z)$ is the set

$$K(\lambda) := \{B \in \mathcal{L}(X, Z) : \lambda \circ B = 0\}.$$

**Theorem 6.2.12.** Suppose that $\mathrm{int}\, K^+ \neq \emptyset$. Let $\varepsilon \geq 0$, $Q \in \mathcal{F}_Z$, $T \in \mathcal{L}(X, Y)$ and $x_0 \in \mathrm{dom}(f_2 \circ T)$. Suppose that $(f_2 \circ T)_B - (f_2 \circ T)_B(x_0)$ is nearly $(Q, \varepsilon)$-subconvexlike on $X$ for all $B \in \mathcal{L}(X, Z)$, $f_2$ is $K$-convex on $Y$ and (MRQC2) holds for $f_1 = T$.

$(a)$ If $f_2$ is regular $(Q, \bar{\varepsilon})$-subdifferentiable at $T(x_0)$, for all $\bar{\varepsilon} \geq 0$, then

$$\partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2 \circ T)(x_0) = \bigcup_{\lambda \in K^{s+} \cap Q^{\tau+}} \ \bigcup_{Q' \in \Lambda_Q(\lambda)} \left\{ \left( \partial^{\mathrm{s}}_{Q',\varepsilon} f_2(T(x_0)) \right) \circ T + K(\lambda) \right\}.$$

$(b)$ If $f_2$ is p-regular $\bar{\varepsilon}$-subdifferentiable at $T(x_0)$, for all $\bar{\varepsilon} \geq 0$ and either $Q+K = Q$ or $K$ has a compact base, then

$$\partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2 \circ T)(x_0) = \bigcup_{\lambda \in K^{s+} \cap Q^{\tau+}} \bigcup_{\substack{q \in K, \\ \langle \lambda, q \rangle = \varepsilon \tau_Q(\lambda)}} \left\{ \left( \partial^{\mathrm{s}}_{\{q\},1} f_2(T(x_0)) \right) \circ T + K(\lambda) \right\}.$$

*Proof.* $(a)$ Inclusion"$\supset$": Let $\lambda \in K^{s+} \cap Q^{\tau+}$, $Q' \in \Lambda_Q(\lambda)$, $A \in \partial^{\mathrm{s}}_{Q',\varepsilon} f_2(T(x_0))$, $\overline{B} \in K(\lambda)$ and let $B = (A \circ T) + \overline{B}$. Then, it follows that $\lambda \circ (B - (A \circ T)) = 0$. On the other hand, since $A \circ T \in \mathcal{L}(X,Z)$, it follows for all $\varepsilon \geq 0$ that

$$\begin{aligned} \Gamma_{Q,\varepsilon}(A \circ T)(x_0)(\lambda) &= \{\overline{T} \in \mathcal{L}(X,Z) : \lambda \circ \overline{T} \in \partial_{\varepsilon \tau_Q(\lambda)}(\lambda \circ A \circ T)(x_0)\} \\ &= \{\overline{T} \in \mathcal{L}(X,Z) : \lambda \circ \overline{T} = \lambda \circ A \circ T\} \\ &= \{\overline{T} \in \mathcal{L}(X,Z) : \lambda \circ (\overline{T} - (A \circ T)) = 0\}. \end{aligned} \tag{6.10}$$

Thus, we have in particular that $B \in \Gamma_{Q,0}(A \circ T)(x_0)(\lambda)$ and applying Theorem 6.2.7 we deduce that $B \in \partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2 \circ T)(x_0)$.

Inclusion "$\subset$": Let $B \in \partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2 \circ T)(x_0)$. By Theorems 5.2.4 and 6.2.3$(b)$ there exists $\lambda \in K^{s+} \cap Q^{\tau+}$ such that

$$\lambda \circ B \in \partial_{\varepsilon \tau_Q(\lambda)}((\lambda \circ f_2) \circ T)(x_0) = T^* \partial_{\varepsilon \tau_Q(\lambda)}(\lambda \circ f_2)(T(x_0)).$$

Hence, there exists $y^* \in \partial_{\varepsilon \tau_Q(\lambda)}(\lambda \circ f_2)(T(x_0))$ such that $\lambda \circ B = y^* \circ T$. If $\tau_Q(\lambda) > 0$ and $\varepsilon > 0$, by the regular $(Q, \varepsilon \tau_Q(\lambda))$-subdifferentiability of $f_2$ at $T(x_0)$ and Theorem 5.5.6$(c)$ it follows that

$$\partial_{\varepsilon \tau_Q(\lambda)}(\lambda \circ f_2)(T(x_0)) = \bigcup_{Q' \in \Lambda_Q(\lambda)} \lambda \circ \partial^{\mathrm{s}}_{Q',\varepsilon} f_2(T(x_0)),$$

so there exist $Q' \in \Lambda_Q(\lambda)$ and $A \in \partial^{\mathrm{s}}_{Q',\varepsilon} f_2(T(x_0))$ such that $y^* = \lambda \circ A$. Therefore, $\lambda \circ B = \lambda \circ A \circ T$, which implies that $B - (A \circ T) \in K(\lambda)$ and then

$$B \in (A \circ T) + K(\lambda) \subset \left( \partial^{\mathrm{s}}_{Q',\varepsilon} f_2(T(x_0)) \right) \circ T + K(\lambda).$$

If $\tau_Q(\lambda) = 0$ or $\varepsilon = 0$, since $f_2$ is regular $(Q,0)$-subdifferentiable at $T(x_0)$ we have that $\partial(\lambda \circ f_2)(T(x_0)) = \lambda \circ \partial^{\mathrm{s}}_0 f_2(T(x_0))$, and following an analogous reasoning as above we obtain the result.

(*b*) Inclusion "$\supset$": Let $\lambda \in K^{s+} \cap Q^{\tau+}$, $q \in K$ such that $\langle \lambda, q \rangle = \varepsilon\tau_Q(\lambda)$, $A \in \partial^{\mathrm{s}}_{\{q\},1} f_2(T(x_0))$, $\overline{B} \in K(\lambda)$ and let $B = A \circ T + \overline{B}$. Consider the set $\hat{Q} := Q(\varepsilon) - \{q\}$. Since $\tau_{\hat{Q}}(\lambda) = \varepsilon\tau_Q(\lambda) - \langle \lambda, q \rangle = 0$, it follows that $\lambda \in \hat{Q}^{\tau+}$. Thus, $\hat{Q} \in \mathcal{F}_Z$ and by (6.10) and Theorem 5.2.5 we deduce that

$$B \in \Gamma_{\hat{Q},1}(A \circ T)(x_0)(\lambda) \subset \partial^{\mathrm{Be}}_{\hat{Q},1}(A \circ T)(x_0).$$

If $Q + K = Q$, we have that $\hat{Q} + K \subset \operatorname{cl} \hat{Q}$, and applying Theorem 6.2.1 we finish. If $K$ has compact base, we obtain the result applying again Theorem 6.2.1.

Inclusion "$\subset$" follows reasoning in a similar way as in part (*a*), but taking into account the p-regular $\varepsilon\tau_Q(\lambda)$-subdifferentiability of $f_2$ at $T(x_0)$. □

In the particular case when $Y = \mathbb{R}^n$, $D = \mathbb{R}^n_+$, $Z = \mathbb{R}^p$ and $K = \mathbb{R}^p_+$, we obtain the following result, which facilitates the calculus of $\partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2 \circ T)(x_0)$.

**Theorem 6.2.13.** Let $\varepsilon \geq 0$, $Q \in \mathcal{F}_{\mathbb{R}^p}$, $T \in \mathcal{L}(X, \mathbb{R}^n)$ and $x_0 \in \operatorname{dom}(f_2 \circ T)$. Suppose that $(f_2 \circ T)_B - (f_2 \circ T)_B(x_0)$ is nearly $(Q, \varepsilon)$-subconvexlike on $X$ for all $B \in \mathcal{L}(X, \mathbb{R}^p)$, $f_2$ is $\mathbb{R}^p_+$-convex on $\mathbb{R}^n$ and p-regular $\bar{\varepsilon}$-subdifferentiable at $T(x_0)$, for all $\bar{\varepsilon} \geq 0$, and (MRQC2) holds for $f = T$. Then,

$$\partial^{\mathrm{Be}}_{Q,\varepsilon}(f_2 \circ T)(x_0) = \bigcup_{\lambda \in \operatorname{int} \mathbb{R}^p_+ \cap Q^{\tau+}} \; \bigcup_{\substack{q \in \mathbb{R}^p_+ \\ \langle \lambda, q \rangle = \varepsilon\tau_Q(\lambda)}} \left\{ \prod_{j=1}^{p} T^* \partial_{q_j}(f_2)_j(T(x_0)) + K(\lambda) \right\}.$$

*Proof.* The proof follows directly by Theorem 6.2.12 and Theorem 5.4.6. □

In the following result, we state chain rules for the Benson $(Q, \varepsilon)$-proper subdifferential of the map $T \circ f_1$, where in this case $T \in \mathcal{L}_+(Y, Z)$.

Let us denote

$$\mathcal{L}_{0+}(Y, Z) := \{T \in \mathcal{L}_+(Y, Z) : T(D \backslash \{0\}) \subset K \backslash \{0\}\}.$$

**Theorem 6.2.14.** Suppose that $\operatorname{int} K^+ \neq \emptyset$. Let $\varepsilon \geq 0$, $Q \in \mathcal{F}_Z$, $T \in \mathcal{L}_{0+}(Y, Z)$ and $x_0 \in \operatorname{dom} f_1$. Suppose that $(T \circ f_1)_B - (T \circ f_1)_B(x_0)$ is nearly $(Q, \varepsilon)$-subconvexlike on $X$ for all $B \in \mathcal{L}(X, Z)$.

($a$) If $f_1$ is regular $(C,\bar{\varepsilon})$-subdifferentiable at $x_0$, for a nonempty set $C \subset D$, such that $D^{s+} \subset C^{s\tau+}$, and for all $\bar{\varepsilon} \geq 0$, then,

$$\partial^{\mathrm{Be}}_{Q,\varepsilon}(T \circ f_1)(x_0) = \bigcup_{\lambda \in K^{s+} \cap Q^{\tau+}} \bigcup_{\substack{C' \in \Lambda_D \\ \tau_{C'}(\lambda \circ T)=1}} \left\{T \circ \partial^{\mathrm{s}}_{C',\varepsilon\tau_Q(\lambda)} f_1(x_0) + K(\lambda)\right\}. \quad (6.11)$$

($b$) If $f_1$ is p-regular $\bar{\varepsilon}$-subdifferentiable at $x_0$, for all $\bar{\varepsilon} \geq 0$, it follows that

$$\partial^{\mathrm{Be}}_{Q,\varepsilon}(T \circ f_1)(x_0) = \bigcup_{\lambda \in K^{s+} \cap Q^{\tau+}} \bigcup_{\substack{q \in D \\ \langle \lambda \circ T, q\rangle = \varepsilon\tau_Q(\lambda)}} \left\{T \circ \partial^{\mathrm{s}}_{\{q\},1} f_1(x_0) + K(\lambda)\right\}. \quad (6.12)$$

*Proof.* ($a$) Inclusion "$\supset$": Let $\lambda \in K^{s+} \cap Q^{\tau+}$, $C' \in \Lambda_D$ with $\tau_{C'}(\lambda \circ T) = 1$, $A \in \partial^{\mathrm{s}}_{C',\varepsilon\tau_Q(\lambda)} f_1(x_0)$, $\overline{B} \in K(\lambda)$ and let $B = T \circ A + \overline{B}$. Then, it follows that $\lambda \circ B = (\lambda \circ T) \circ A$. Since $\lambda \circ T \in D^{+}\backslash\{0\}$, by Theorem 5.4.4 we have that

$$\lambda \circ B = (\lambda \circ T) \circ A \in \partial_{\varepsilon\tau_Q(\lambda)\cdot\tau_{C'}(\lambda \circ T)}(\lambda \circ T \circ f_1)(x_0) = \partial_{\varepsilon\tau_Q(\lambda)}(\lambda \circ (T \circ f_1))(x_0),$$

and by Theorem 5.2.5 we deduce that $B \in \partial^{\mathrm{Be}}_{Q,\varepsilon}(T \circ f_1)(x_0)$.

Inclusion "$\subset$": Let $B \in \partial^{\mathrm{Be}}_{Q,\varepsilon}(T \circ f_1)(x_0)$. By Theorem 5.2.4 there exists $\lambda \in K^{s+} \cap Q^{\tau+}$ such that $\lambda \circ B \in \partial_{\varepsilon\tau_Q(\lambda)}((\lambda \circ T) \circ f_1)(x_0)$. Since $\lambda \in K^{s+}$ and $T \in \mathcal{L}_{0+}(Y,Z)$ it follows that $\lambda \circ T \in D^{s+}$. Thus, it is clear that $\lambda \circ T \in D^{s+} \cap C^{s\tau+}$. If $\varepsilon\tau_Q(\lambda) > 0$, by the regular $(C,\varepsilon\tau_Q(\lambda))$-subdifferentiability of $f_1$ at $x_0$ and Theorem 5.5.6($b$), there exists $C' \in \Lambda_D$, with $\tau_{C'}(\lambda \circ T) = 1$, and $A \in \partial^{\mathrm{s}}_{C',\varepsilon\tau_Q(\lambda)} f_1(x_0)$ such that $\lambda \circ B = \lambda \circ T \circ A$, which implies that $B - T \circ A \in K(\lambda)$. Therefore, $B \in T \circ \partial^{\mathrm{s}}_{C',\varepsilon\tau_Q(\lambda)} f_1(x_0) + K(\lambda)$, and inclusion (6.11) is proved. If $\varepsilon\tau_Q(\lambda) = 0$, applying a similar reasoning as above and taking into account that $f_1$ is regular $(C,0)$-subdifferentiable at $x_0$ we have that $B \in T \circ \partial^{\mathrm{s}} f_1(x_0) + K(\lambda)$. Thus, considering $q \in D\backslash\{0\}$ such that $\langle \lambda \circ T, q\rangle = 1$ and $C' := q + D \in \Lambda_D$, by Proposition 5.4.3($b$) it follows that $\partial^{\mathrm{s}} f_1(x_0) = \partial^{\mathrm{s}}_{C',0} f_1(x_0)$, and part ($a$) is proved.

($b$) It follows reasoning in analogous form as in part ($a$), but considering the assumption of p-regular $\varepsilon\tau_Q(\lambda)$-subdifferentiability of $f_1$ at $x_0$ instead of the regular $(C,\varepsilon\tau_Q(\lambda))$-subdifferentiability of $f_1$ at $x_0$. □

**Remark 6.2.15.** ($a$) If $C \subset D$, then assumption $D^{s+} \subset C^{s\tau+}$ is satisfied if some of conditions ($a$)-($e$) of Remark 5.5.5 holds.

($b$) If we consider $Q = \bar{q} + K$, for $\bar{q} \notin -K\backslash\{0\}$ and $\varepsilon = 1$, equality (6.12) reduces to

$$\partial^{\text{Be}}_{Q,1}(T \circ f_1)(x_0) = \bigcup_{\substack{\lambda \in K^{s+} \\ \langle \lambda, \bar{q} \rangle \geq 0}} \bigcup_{\substack{q \in D \\ \langle \lambda \circ T, q \rangle = \langle \lambda, \bar{q} \rangle}} \{T \circ \partial^{\text{s}}_{\{q\},1} f_1(x_0) + K(\lambda)\}, \tag{6.13}$$

which does not coincide with [84, Proposition 3.2]. In the following example we show that [84, Proposition 3.2] is not correct.

**Example 6.2.16.** Consider $X = \mathbb{R}$, $Y = Z = \mathbb{R}^2$, $D = K = \mathbb{R}^2_+$ and let us define the mapping $f : \mathbb{R} \to \mathbb{R}^2$ by $f(x) = (0, x)$, for all $x \in \mathbb{R}$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the identity mapping on $\mathbb{R}^2$, $\bar{q} = (1, -1)$ and $x_0 = 0$. Clearly, $f$ is $\mathbb{R}^2_+$-convex on $\mathbb{R}$ and then, in particular, $(T \circ f)_B - (T \circ f)_B(0)$ is nearly $(K + \bar{q}, 1)$ subconvexlike on $\mathbb{R}$ for all $B \in \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$ (see Theorem 6.2.4). It is also clear that $f$ is p-regular $\varepsilon$-subdifferentiable at 0, for all $\varepsilon \geq 0$. By [84, Proposition 3.2] we have that

$$\partial^{\text{Be}}_{\bar{q}+K,1} f(0) = \bigcup_{\substack{q \in \mathbb{R}^2_+ \\ \bar{q} \not\leq_{\mathbb{R}^2_+ \backslash \{(0,0)\}} q \\ q \not\leq_{\mathbb{R}^2_+ \backslash \{(0,0)\}} \bar{q}}} \partial^{\text{s}}_{\{q\},1} f(0) + Z_p(\mathbb{R}, \mathbb{R}^2), \tag{6.14}$$

where $Z_p(\mathbb{R}, \mathbb{R}^2) = \{A \in \mathcal{L}(\mathbb{R}, \mathbb{R}^2) : \exists \lambda \in \operatorname{int} \mathbb{R}^2_+ : \lambda \circ A = 0\}$.

Applying Theorem 5.4.6 it follows that $\partial^{\text{s}}_{\{q\},1} f(0) = \{(0, 1)\}$, for all $q \in \mathbb{R}^2_+$ and it is easy to check that

$$Z_p(\mathbb{R}, \mathbb{R}^2) = \{(a_1, a_2) \in \mathbb{R}^2 : a_1 a_2 < 0\} \cup \{(0, 0)\}.$$

Thus, by formula (6.14) we have that

$$\partial^{\text{Be}}_{\bar{q}+K,1} f(0) = \{(a_1, a_2) \in \mathbb{R}^2 : a_1(a_2 - 1) < 0\} \cup \{(0, 1)\}. \tag{6.15}$$

Next, we determinate $\partial^{\text{Be}}_{\bar{q}+K,1} f(0)$ applying formula (6.13). It follows that

$$K(\lambda) = \{(a_1, a_2) \in \mathbb{R}^2 : \lambda_1 a_1 + \lambda_2 a_2 = 0\},$$

and therefore, by (6.13) we have that

$$\begin{aligned}\partial^{\mathrm{Be}}_{\bar{q}+K,1} f(0) &= \bigcup_{\substack{\lambda_1,\lambda_2>0\\ \lambda_1\geq\lambda_2}} \bigcup_{\substack{q\in\mathbb{R}^2_+\\ \langle\lambda\circ T,q\rangle=\langle\lambda,\bar{q}\rangle}} \{(a_1,a_2)\in\mathbb{R}^2 : \lambda_1 a_1+\lambda_2(a_2-1)=0\}\\ &= \bigcup_{\substack{\lambda_1,\lambda_2>0\\ \lambda_1\geq\lambda_2}} \{(a_1,a_2)\in\mathbb{R}^2 : \lambda_1 a_1+\lambda_2(a_2-1)=0\}\\ &= \{(a_1,a_2)\in\mathbb{R}^2 : a_1<0, a_1+a_2\geq 1\}\cup\\ &\quad \{(a_1,a_2)\in\mathbb{R}^2 : a_1>0, a_1+a_2\leq 1\}\cup\{(0,1)\}. \qquad (6.16)\end{aligned}$$

Moreover, if we determine $\partial^{\mathrm{Be}}_{\bar{q}+K,1} f(0)$ applying Corollary 5.2.6, we obtain the same result as in (6.16), which shows that [84, Proposition 3.2] is not correct.

In the particular case when $Y=\mathbb{R}^n$, $D=\mathbb{R}^n_+$, $Z=\mathbb{R}^p$ and $K=\mathbb{R}^p_+$, we obtain the following result.

**Theorem 6.2.17.** Let $\varepsilon\geq 0$, $Q\in\mathcal{F}_{\mathbb{R}^p}$, $T\in\mathcal{L}_{0+}(\mathbb{R}^n,\mathbb{R}^p)$ and $x_0\in\operatorname{dom} f_1$. Suppose that $(T\circ f_1)_B-(T\circ f_1)_B(x_0)$ is nearly $(Q,\varepsilon)$-subconvexlike on $X$, for all $B\in\mathcal{L}(X,\mathbb{R}^p)$ and $f_1$ is p-regular $\bar{\varepsilon}$-subdifferentiable at $x_0$, for all $\bar{\varepsilon}\geq 0$. Then,

$$\partial^{\mathrm{Be}}_{Q,\varepsilon}(T\circ f_1)(x_0) = \bigcup_{\lambda\in\operatorname{int}\mathbb{R}^p_+\cap Q^{\tau+}} \bigcup_{\substack{q\in\mathbb{R}^n_+\\ \langle\lambda\circ T,q\rangle=\varepsilon\tau_Q(\lambda)}} \left\{T\circ\left(\prod_{j=1}^{n}\partial_{q_j}(f_1)_j(x_0)\right)+K(\lambda)\right\}.$$

*Proof.* It is an immediate consequence of Theorem 6.2.14(*b*) and Theorem 5.4.6. □

# Capítulo 7

# Conclusiones y futuras líneas de desarrollo

En este último capítulo, se resumen los resultados más importantes de la memoria, para proporcionar una visión global del trabajo realizado. También se presentan nuevas líneas de investigación que han surgido a lo largo del estudio llevado a cabo, y cuyo desarrollo se considera de interés para completar el trabajo expuesto en este documento.

## 7.1 Conclusiones

El objetivo principal de esta tesis ha sido la definición y estudio detallado de una noción de eficiencia propia aproximada para problemas de optimización vectorial. Concretamente, se ha introducido el concepto de solución $(C, \varepsilon)$-propia Benson, donde $C$ es un conjunto del espacio objetivo y $\varepsilon$ es un escalar no negativo (véase Definición 2.2.1). Este concepto es la versión propia en el sentido de Benson de la noción de $(C, \varepsilon)$-eficiencia introducida por Gutiérrez, Jiménez y Novo [48, 49] y supone una generalización importante del concepto de eficiencia propia aproximada dado por Gao, Yang y Teo en [33], también basado en la clase de soluciones $(C, \varepsilon)$-eficientes. Cuando $\varepsilon$ es igual a cero y el cono generado por $C$ coincide con el cono de orden del espacio objetivo, la noción de solución $(C, \varepsilon)$-propia Benson se reduce al concepto clásico de solución propia en el sentido de

Benson (véase Definición 1.2.7).

Las soluciones $(C, \varepsilon)$-propias Benson satisfacen excelentes propiedades, estudiadas en el Capítulo 2 de esta memoria, entre las que destacan dos fundamentales. En primer lugar, se caracterizan a través de escalarización lineal, es decir, por medio de soluciones aproximadas de ciertos problemas de optimización escalares asociados al problema de optimización vectorial (véase Teoremas 2.2.6 y 2.2.8, y Corolario 2.2.10), y bajo una nueva condición de convexidad generalizada (véase Definición 1.2.21) que extiende el concepto de "nearly" convexidad. Esta caracterización translada el problema de obtener soluciones $(C, \varepsilon)$-propias Benson al cálculo de soluciones aproximadas de problemas escalares, más fáciles de determinar.

En segundo lugar, las soluciones $(C, \varepsilon)$-propias Benson tienen un buen comportamiento límite cuando el error de aproximación $\varepsilon$ tiende a cero, en el sentido de que el límite superior de Painlevé-Kuratowski del conjunto formado por estas soluciones cuando $\varepsilon$ tiende a cero está incluido en el conjunto de soluciones eficientes exactas del problema de optimización vectorial. La razón de este buen comportamiento límite reside en la selección de conjuntos $C$ adecuados, como se prueba en el Teorema 2.3.3. Esta propiedad se traduce en la posibilidad de determinar un conjunto de soluciones propias aproximadas de modo que todas ellas estén próximas al conjunto de soluciones eficientes exactas.

En el Capítulo 2, se demostró que el concepto de solución $(C, \varepsilon)$-propia Benson generaliza el de solución propia aproximada introducido por Gao, Yang y Teo (véase Definición 1.2.19), reduciéndose al mismo cuando el cono generado por $C$ coincide con el cono de orden del espacio objetivo, lo que implica que $C$ debe estar contenido en el cono de orden necesariamente. Sin embargo, los conjuntos $C$ para los cuales las soluciones $(C, \varepsilon)$-propias Benson tienen un buen comportamiento límite no están contenidos en el cono de orden, por lo que en el caso particular en el que la clase de soluciones $(C, \varepsilon)$-propias Benson coincide con la dada por Gao, Yang y Teo, los resultados establecidos en el Teorema 2.3.3 no pueden ser aplicados. Por este motivo, el concepto de solución $(C, \varepsilon)$-propia Benson generaliza significativamente la noción previa introducida por Gao, Yang y Teo.

Por otra parte, los Teoremas 2.2.6 y 2.2.8 mejoran la caracterización mediante escalarización lineal obtenida por Gao, Yang y Teo sobre sus soluciones aproximadas propias porque se utilizan hipótesis más débiles y, además, las condiciones necesarias y suficientes alcanzan la misma precisión (véase Notas 2.2.2, 2.2.7 y 2.2.9).

La noción de solución $(C, \varepsilon)$-propia Benson extiende los conceptos de solución propia aproximada introducidos, respectivamente, por Li y Wang [74] (para el caso multiobjetivo Pareto) y Rong [97] y, bajo condiciones de convexidad generalizada, también el definido por El Maghri [83] (véase Corolario 2.2.16). A diferencia de lo que ocurre con las soluciones $(C, \varepsilon)$-propias Benson, en estos tres conceptos el error se cuantifica por medio de un único vector, lo que, en la práctica, da lugar a conjuntos de soluciones aproximadas demasiado grandes, que no aproximan bien al conjunto eficiente. Este hecho se ha ilustrado en el Ejemplo 2.3.6.

Asumiendo que el conjunto factible está definido por una cono restricción, en la Sección 3.2 se han obtenido condiciones de optimalidad para soluciones $(C, \varepsilon)$-propias Benson por medio de una Lagrangiana escalar, considerando la hipótesis de convexidad generalizada descrita en la Definición 1.2.21. Los resultados de esta sección se han utilizado en la Sección 3.3, donde se han introducido, por un lado, una multifunción Lagrangiana, que generaliza las funciones Lagrangianas vectoriales más importantes de la literatura (véase Definición 3.3.1) y, por otro lado, el concepto de punto de silla $(C, \varepsilon)$-propio (Definición 3.3.3), que se basa en la noción de punto de silla dada por Li [72] para el caso exacto y que se define a través de soluciones $(C, \varepsilon)$-propias Benson del problema de optimización asociado con la multifunción Lagrangiana anterior.

En la Sección 3.3, se han obtenido condiciones necesarias y suficientes para soluciones $(C, \varepsilon)$-propias Benson de un problema de optimización vectorial por medio de puntos de silla $(C, \varepsilon)$-propios. En estos resultados, la condición de holgura complementaria es acotada con conjuntos $C$ adecuados (véase Lema 3.3.7 y Nota 3.3.8). Esta importante propiedad no se cumple para la mayoría de los conceptos de punto de silla aproximado existentes en la literatura para problemas de optimización vectorial, ya que éstos están basados en soluciones aproximadas respecto a un único vector, como ocurre, por ejemplo, con las nociones de punto de silla aproximado utilizadas, respectivamente, por Vályi [114] y Gupta y Mehra [41].

Merece destacarse que la versión exacta de algunos resultados obtenidos en la Sección 3.3, proporcionan nuevos teoremas de puntos de silla para soluciones propias exactas en el sentido de Benson más fuertes que otros resultados similares de la literatura. Es el caso del Teorema 3.3.11, y del Corolario 3.3.16 (véase Notas 3.3.12 y 3.3.17).

El principal objetivo del Capítulo 4 ha sido definir dos problemas duales de tipo Lagrangiano y derivar resultados de dualidad débil y fuerte que relacionen las soluciones maximales aproximadas del problema dual con soluciones $(C, \varepsilon)$-propias Benson del problema primal. Se supone de nuevo que el conjunto factible está definido por una cono restricción.

En la Sección 4.2 se ha definido un problema dual mediante una Lagrangiana escalar y el conjunto $C$, que se reduce al conocido problema dual introducido por Jahn [63,64] cuando el cono generado por $C$ coincide con el cono de orden. Se han obtenido teoremas de dualidad débil y fuerte que relacionan las soluciones maximales aproximadas del problema dual con las soluciones $(C, \varepsilon)$-propias Benson del problema primal, bajo condiciones de estabilidad y de convexidad generalizada, que extienden al caso aproximado resultados obtenidos por Jahn en [63,64] y Boţ, Grad y Wanka [13].

En la Sección 4.3 se ha introducido otro problema dual, motivado por la conocida formulación dual de Li [72], y cuya función objetivo está definida mediante soluciones $(C, \varepsilon)$-propias Benson del problema de optimización vectorial Lagrangiano asociado a la multifunción Lagrangiana de la Definición 3.3.1. Se han establecido teoremas de dualidad débil y fuerte para soluciones $(C, \varepsilon)$-propias Benson del problema primal, también bajo condiciones de estabilidad y de convexidad generalizada, obteniéndose novedosos resultados incluso para soluciones propias exactas. En particular, se ha caracterizado el conjunto dual objetivo en términos de soluciones aproximadas de problemas de optimización escalares

Lagrangianos y se han analizado las relaciones entre este problema dual y el introducido en la Sección 4.2 (véase Teoremas 4.3.11 y 4.3.14).

Los Capítulos 5 y 6 se han dedicado al estudio de un nuevo concepto de $\varepsilon$-subdiferencial vectorial propia, definida a partir de la clase de soluciones $(C, \varepsilon)$-propias Benson de un problema de optimización vectorial, que llamamos $(C, \varepsilon)$-subdiferencial Benson (véase Definición 5.2.1).

La $(C, \varepsilon)$-subdiferencial Benson hereda las buenas propiedades de las soluciones $(C, \varepsilon)$-propias Benson. Específicamente, sus elementos se caracterizan a través de $\varepsilon$-subgradientes de funciones escalares asociadas, asumiendo la condición de convexidad generalizada de la Definición 1.2.21 (véase Teoremas 5.2.4, 5.2.5 y Corolario 5.2.6), lo que facilita su cálculo. Además, extiende las nociones más importantes de $\varepsilon$-subdiferencial vectorial propia introducidas en la literatura. En concreto, generaliza la $\varepsilon$-subdiferencial introducida por Tuan [110] y, en problemas convexos, la definida por El Maghri [83] (véase Definiciones 1.2.25 y 1.2.26).

La $(C, \varepsilon)$-subdiferencial Benson destaca también por aproximar adecuadamente el conjunto de soluciones eficientes exactas de un problema de optimización vectorial mediante sucesiones minimizantes (véase Definición 5.2.9), como se demuestra en la Proposición 5.2.10. En este sentido, la $(C, \varepsilon)$-subdiferencial Benson mejora las introducidas por Tuan y El Maghri, donde puede suceder que el límite de una sucesión minimizante se encuentre tan alejado del conjunto eficiente como se quiera.

Por último, se han obtenido reglas de cálculo para evaluar la $(C, \varepsilon)$-subdiferencial Benson de la suma y la composición de funciones vectoriales. Para ello, se ha introducido la llamada $(C, \varepsilon)$-subdiferencial fuerte, que se estudia en la Sección 5.4, y una nueva condición de regularidad, analizada en la Sección 5.5.

La $(C, \varepsilon)$-subdiferencial fuerte se define a partir de una noción de solución aproximada fuerte introducida en la Sección 5.3, y extiende la $\varepsilon$-subdiferencial definida por Kutateladze en [70]. Dicha subdiferencial se caracteriza mediante $\varepsilon$-subgradientes de funciones escalares asociadas, sin ninguna hipótesis, y de esta caracterización resulta una fórmula sencilla cuando el espacio objetivo es finito-dimensional y el cono de orden es el ortante positivo.

La condición de regularidad definida en la Sección 5.5 se introduce por medio de la $(C, \varepsilon)$-subdiferencial fuerte, y generaliza la introducida por El Maghri en [86] y también la definida por Raffin [95] en el caso exacto. En la Sección 5.5 se han obtenido diversas formulaciones equivalentes para esta nueva condición de regularidad.

En la Sección 5.6 se han establecido dos reglas de cálculo de tipo Moreau-Rockafellar para la $(C, \varepsilon)$-subdiferencial Benson de la suma. La primera de ellas se obtiene suponiendo la condición de regularidad dada por El Maghri y se reduce a la regla de la suma obtenida por este autor en [83] cuando se aplica a conjuntos $C$ que son translaciones del cono de orden. La segunda regla se demuestra considerando la condición de regularidad introducida en la Sección 5.5 y bajo condiciones de convexidad generalizada. Al final de la sección se ha relacionado la $(C, \varepsilon)$-subdiferencial Benson con la $(C, \varepsilon)$-subdiferencial fuerte.

En el Capítulo 6 se han establecido reglas de cálculo para la $(C, \varepsilon)$-subdiferencial Benson de la composición de dos funciones vectoriales. Al igual que en la Sección 5.6, se han obtenido dos reglas de la cadena utilizando, respectivamente, la condición de regularidad dada por El Maghri y la introducida en la Sección 5.5. La primera de ellas extiende la regla de la cadena establecida por El Maghri en [84] en el mismo sentido que el mencionado para la regla de la suma. Como casos particulares, se han deducido reglas de la cadena para la $(C, \varepsilon)$-subdiferencial Benson cuando una de las dos funciones que intervienen es lineal, que dan lugar a fórmulas sencillas cuando el espacio objetivo es finito-dimensional y el cono de orden es el ortante positivo.

## 7.2 Futuras líneas de desarrollo

En esta sección se exponen las futuras líneas de investigación que han surgido como consecuencia del estudio realizado. En algunas de ellas se plantean cuestiones específicas, con el fin de completar el trabajo presentado en esta memoria. Otras podrían dar lugar a nuevas vías de investigación, relacionadas con el estudio desarrollado, pero con un carácter más independiente.

**Línea 1.** Los dos problemas duales analizados en el Capítulo 4 son de tipo Lagrangiano y pensamos que es interesante extender el estudio dual de las soluciones $(C, \varepsilon)$-propias Benson a otras formulaciones, como son los problemas duales conjugados, de tipo Wolfe y Mond-Weir (véase, por ejemplo [12–15, 65]).

Para definir el correspondiente problema dual conjugado, se utilizará la caracterización por escalarización lineal dada en los Teoremas 2.2.6 y 2.2.8. Análogamente, los problemas duales de tipo Wolfe y Mond-Weir se definirán utilizando la $(C, \varepsilon)$-subdiferencial Benson, y en su estudio será fundamental la caracterización de esta $\varepsilon$-subdiferencial por medio de subgradientes aproximados de funciones escalares asociadas (véase Teoremas 5.2.4 y 5.2.5).

**Línea 2.** El estudio de condiciones de optimalidad mediante reglas de multiplicadores de tipo Kuhn-Tucker y Fritz-John constituye una parte importante de la Optimización Matemática, ya que proporciona condiciones para determinar el conjunto de soluciones de un problema de optimización (véase, por ejemplo, [7, 18, 36, 37, 47, 64, 66, 75, 82, 88, 101]).

En programación multiobjetivo Pareto, son especialmente valiosas las condiciones necesarias y suficientes de tipo Kuhn-Tucker fuerte, donde los multiplicadores correspondientes a cada una de las funciones objetivo son estrictamente positivos, de modo que todas las componentes de la función objetivo tienen un papel efectivo en la determinación de las soluciones del problema de optimización. Pretendemos extender este tipo de condiciones a las soluciones $(C, \varepsilon)$-propias Benson.

En este sentido, la caracterización por escalarización dada en los Teoremas

2.2.6 y 2.2.8 constituye una herramienta fundamental para derivar este tipo de condiciones por medio de $\varepsilon$-subdiferenciales de funciones escalares en problemas convexos.

**<u>Línea 3</u>.** De la definición de $(C,\varepsilon)$-subdiferencial Benson se deduce inmediatamente que

$$x_0 \in \mathrm{Be}(f,S,C,\varepsilon) \Longleftrightarrow 0 \in \partial^{\mathrm{Be}}_{C,\varepsilon}(f+I_S)(x_0),$$

donde $I_S$ denota la función indicatriz del conjunto factible. Mediante esta relación y las reglas de cálculo para la $(C,\varepsilon)$-subdiferencial Benson de la suma presentadas en la Sección 5.6 se pretende obtener condiciones de optimalidad para soluciones $(C,\varepsilon)$-propias Benson mediante $(C,\varepsilon)$-subdiferenciales Benson y fuertes.

Tuan [110] y El Maghri [83] dedujeron condiciones de optimalidad para las soluciones $\varepsilon$-propias Benson (Definición 1.2.15) y $\varepsilon$-propias Henig (Definición 1.2.17), respectivamente, utilizando este procedimiento y las $\varepsilon$-subdiferenciales vectoriales introducidas por cada uno de ellos. Las condiciones de optimalidad para las soluciones $(C,\varepsilon)$-propias Benson deberían generalizar las obtenidas por Tuan y El Maghri.

**<u>Línea 4</u>.** Se pretende estudiar qué propiedades intrínsecas de la $\varepsilon$-subdiferencial de Brøndsted-Rockafellar se pueden extender a la $(C,\varepsilon)$-subdiferencial Benson. Concretamente, analizaremos la semicontinuidad inferior de la multifunción $x \to \partial^{\mathrm{Be}}_{C,\varepsilon} f(x)$ y trataremos de estimar la distancia Hausdorff de las subdiferenciales $\partial^{\mathrm{Be}}_{C,\varepsilon} f(x)$ y $\partial^{\mathrm{Be}}_{C,\delta} f(z)$ a partir de las distancias entre los puntos $x$ y $z$ y de los valores $\varepsilon$ y $\delta$.

**<u>Línea 5</u>.** Una herramienta fundamental en el desarrollo de reglas de cálculo y técnicas variacionales es el conocido lema de Brøndsted-Rockafellar (véase [17, Sección 3]), donde se demuestra que es posible aproximar los $\varepsilon$-subgradientes de una función escalar en un punto mediante subgradientes en puntos próximos. Debido a su importancia, se han derivado nuevas versiones, y merecen destacarse la obtenida por Borwein en [10, Teorema 1] y la dada por Thibault en [109, Teorema 1.3].

Consideramos interesante extender este tipo de resultados a la $(C, \varepsilon)$-subdiferencial Benson de una función vectorial, recurriendo a la caracterización por escalarización obtenida en el Corolario 5.2.6, y asumiendo hipótesis de convexidad generalizada y la semicontinuidad inferior de la función.

**Línea 6.** En [60, 61] se establecieron reglas de cálculo de la suma y la composición para la subdiferencial y la $\varepsilon$-subdiferencial del Análisis Convexo sin utilizar condiciones de cualificación. Por medio de estas fórmulas, varios autores han obtenido reglas de cálculo de carácter secuencial para las mismas operaciones y subdiferenciales (véase [24, 32, 109]). Por ejemplo, es posible deducir una regla secuencial de la cadena para la $\varepsilon$-subdiferencial, mediante las fórmulas de Hiriart-Urruty, Moussaoui, Seeger y Volle [60] y utilizando un resultado variacional dado por Combari, Marcellin y Thibault en [22], en espacios de Banach reflexivos y bajo hipótesis de convexidad y semicontinuidad inferior de las funciones implicadas, sin suponer condiciones de cualificación.

Estos resultados sugieren estudiar reglas de cálculo secuenciales que extiendan las anteriores a la $(C, \varepsilon)$-subdiferencial Benson, lo que constituye una nueva línea de trabajo.

**Línea 7.** La $\varepsilon$-subdiferencial juega también un papel fundamental en el estudio de problemas de optimización en los que la función objetivo es la diferencia de dos funciones convexas. Esta clase de problemas se conoce con el nombre de DC (del inglés, "Difference of Convex functions"), y el estudio de los mismos es interesante ya que existen importantes funciones no convexas que se pueden expresar como diferencia de dos que sí lo son (véase [62, 111]).

Hiriart-Urruty [57] caracterizó las soluciones de problemas DC escalares sin restricciones, en términos de $\varepsilon$-subdiferenciales.

Recientemente, El Maghri [85] ha caracterizado las soluciones eficientes aproximadas débiles y fuertes en el sentido de Kutateladze [70] de problemas DC vectoriales (con y sin restricciones) por medio de la $\varepsilon$-subdiferencial fuerte dada por Kutateladze (véase [70,83]) y una $\varepsilon$-subdiferencial vectorial débil introducida en [83].

Trataremos de extender los resultados anteriores a las soluciones $(C,\varepsilon)$-propias Benson de problemas DC vectoriales, con objeto de caracterizar esta clase de soluciones mediante la $(C,\varepsilon)$-subdiferencial Benson y la $(C,\varepsilon)$-subdiferencial fuerte.

**Línea 8.** Las buenas propiedades que satisfacen las soluciones $(C,\varepsilon)$-propias Benson se han obtenido mediante escalarización lineal, como resultado del teorema de separación estricta de conos dado por Jahn (véase [64, Teorema 3.22] y Lema 2.2.4), en el que se supone, en particular, que el interior topológico del polar del cono de orden es no vacío. De hecho, la mayoría de las técnicas de separación convexa se basan en la solidez de uno de los dos conjuntos que intervienen, como es el caso del conocido teorema de separación de Eidelheit (véase, por ejemplo, [64, Teorema 3.16]). Mas aún, en el teorema de separación no convexa mediante el funcional de Tammer (véase, por ejemplo, [106, Sección 3.1]) se asume también que uno de los dos conjuntos tiene interior no vacío.

Esta hipótesis de solidez de uno de los conjuntos en las técnicas de separación convexa y no convexa motiva introducir un concepto de eficiencia propia aproximada en el sentido de Henig y obtener, de este modo, resultados más fuertes de eficiencia propia aproximada mediante técnicas de escalarización lineal y no lineal. En esta línea ya se han obtenido algunos avances, que se exponen a continuación.

**Definición 7.1.** Sea $\varepsilon \geq 0$ y $C \in \mathcal{H}_Y$. Se dice que $x_0 \in S_0$ es una solución $(C,\varepsilon)$-eficiente propia Henig de $(\mathcal{P}_S)$, y se denota por $x_0 \in \mathrm{He}(f,S,C,\varepsilon)$, si existe un cono convexo $D' \subset Y$, $D' \neq Y$, tal que $D\backslash\{0\} \subset \operatorname{int} D'$, $C \cap (-\operatorname{int} D') = \emptyset$ y $x_0 \in \mathrm{AE}(f,S,C',\varepsilon)$, donde $C' := C + \operatorname{int} D'$.

Se cumple que $\mathrm{He}(f,S,C,\varepsilon) \subset \mathrm{Be}(f,S,C,\varepsilon)$. Además, esta nueva clase de soluciones se caracteriza mediante escalarización lineal.

**Teorema 7.2.** Sea $\varepsilon \geq 0$ y $C \in \mathcal{F}_Y$. Se tiene que

$$\bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \varepsilon\tau_C(\mu)\text{-}\operatorname{argmin}_S(\mu \circ f) \subset \mathrm{He}(f,S,C,\varepsilon).$$

**Teorema 7.3.** Sea $x_0 \in S_0$, $\varepsilon \geq 0$ y $C \in \mathcal{F}_Y$. Supóngase que $f - f(x_0)$ es nearly $(C, \varepsilon)$-subconvexlike sobre $S$. Si $x_0 \in \mathrm{He}(f, S, C, \varepsilon)$, entonces existe $\mu \in D^{s+} \cap C^{\tau+}$ tal que $x_0 \in \varepsilon\tau_C(\mu)\text{-argmin}_S(\mu \circ f)$.

**Corolario 7.4.** Sea $\varepsilon \geq 0$ y $C \in \mathcal{F}_Y$. Supóngase que $f - f(x)$ es nearly $(C, \varepsilon)$-subconvexlike sobre $S$ para todo $x \in S_0$. Se tiene que,

$$\mathrm{He}(f, S, C, \varepsilon) = \bigcup_{\mu \in D^{s+} \cap C^{\tau+}} \varepsilon\tau_C(\mu)\text{-argmin}_S(\mu \circ f).$$

Si adicionalmente $\mathrm{int}(D^+) \neq \emptyset$, entonces

$$\mathrm{He}(f, S, C, \varepsilon) = \mathrm{Be}(f, S, C, \varepsilon).$$

Del corolario anterior, se deduce que las soluciones $(C, \varepsilon)$-propias Henig se caracterizan mediante escalarización lineal bajo la condición de convexidad generalizada de la Definición 1.2.21, y sin suponer la solidez del cono polar estricto de $D$.

Por otra parte, la misma definición de solución $(C, \varepsilon)$-propia Henig permite caracterizar este tipo de soluciones mediante escalarización no lineal sin imponer condiciones de convexidad, aplicando el teorema de separación no convexa dado en [106]. Esta línea de trabajo es muy importante, ya que supone extender los principales resultados de esta memoria al caso no convexo y sin asumir la solidez del polar del cono de orden.

**<u>Línea 9</u>.** El concepto de solución $(C, \varepsilon)$-propia Henig motiva la definición de una nueva $\varepsilon$-subdiferencial propia para funciones vectoriales del siguiente modo:

**Definición 7.5.** Sea $x_0 \in \mathrm{dom}\, f$, $\varepsilon \geq 0$ y $C \in \mathcal{H}_Y$. Se define la $(C, \varepsilon)$-subdiferencial propia en el sentido de Henig de $f$ en $x_0$ como

$$\partial^{\mathrm{He}}_{C,\varepsilon} f(x_0) = \{T \in \mathcal{L}(X, Y) : x_0 \in \mathrm{He}(f - T, X, C, \varepsilon)\}.$$

Se cumple que $\partial^{\mathrm{He}}_{C,\varepsilon} f(x_0) \subset \partial^{\mathrm{Be}}_{C,\varepsilon} f(x_0)$, y es interesante estudiar qué propiedades satisface esta nueva $\varepsilon$-subdiferencial vectorial, así como obtener reglas de cálculo para la suma y la composición de funciones.

**Línea 10.** Los problemas de optimización con multifunciones han suscitado un creciente interés durante los últimos años (véase, por ejemplo [65,72,87,98,99, 104,110,116]). Sin embargo, las soluciones propias aproximadas de esta clase de problemas no han sido estudiadas en profundidad. Uno de los pocos trabajos en este campo es el debido a Tuan [110], y en él se analizan soluciones propias aproximadas en el sentido de Benson con respecto a un vector (Definición 1.2.15). Por ello, pretendemos extender los principales resultados obtenidos en esta memoria a los problemas de optimización con multifunciones.

# Bibliografía


[1] A. Abraham, L. Jain, R. Goldberg, Evolutionary multiobjective optimization, Springer-Verlag, London, 2005.

[2] M. Adán, V. Novo, Weak efficiency in vector optimization using a closure of algebraic type under cone-convexlikeness, Eur. J. Oper. Res. 149 (2003) 641–653.

[3] M. Adán, V. Novo, Proper efficiency in vector optimization on real linear spaces, J. Optim. Theory Appl. 121 (2004) 515–540.

[4] M. Adán, V. Novo, Duality and saddle-points for convex-like vector optimization problems on real linear spaces, TOP 13 (2005) 343–357.

[5] K. J. Arrow, E. W. Barankin, D. Blackwell, Admissible points of convex sets, Contributions to the Theory of Games 2, Ann. of Math. Studies 28, Princeton University Press, Princeton (1953) 87–91.

[6] H. P. Benson, An improved definition of proper efficiency for vector maximization with respect to cones, J. Math. Anal. Appl. 71 (1979) 232–241.

[7] G. Bigi, M. Papalardo, Regularity conditions in vector optimization, J. Optim. Theory Appl. 102 (1999) 83–96.

[8] S. Bolintinéanu (H. Bonnel), Vector variational principles; $\varepsilon$-efficiency and scalar stationarity, J. Convex Anal. 8 (2001) 71–85.

[9] J. Borwein, Proper efficient points for maximizations with respect to cones, SIAM J. Control Optim. 15 (1977) 57–63.

[10] J.M. Borwein, A note on $\varepsilon$-subgradients and maximal monotonicity, Pacific J. Math. 103 (1982) 307–314.

[11] J. M. Borwein, D. Zhuang, Super efficiency in vector optimization, Trans. Amer. Math. Soc. 338 (1993) no. 1 105–122.

[12] R. I. Boţ, S.-M. Grad, Duality for vector optimization problems via a general scalarization, Optimization 60 (2011) 1269–1290.

[13] R. I. Boţ, S.-M. Grad, G. Wanka, Duality in Vector Optimization, Springer-Verlag, Berlin, 2009.

[14] R. I. Boţ, G. Wanka, An analysis of some dual problems in multiobjective optimization (I), Optimization 53 (2004) 281–300.

[15] R. I. Boţ, G. Wanka, An analysis of some dual problems in multiobjective optimization (II), Optimization 53 (2004) 301–324.

[16] W. W. Breckner, M. Sekatzek, C. Tammer, Approximate saddle point assertions for a general class of approximation problems. In Approximation, Optimization and Mathematical Economics, Physica-Verlag, Heidelberg (2001) 71–80.

[17] A. Brøndsted, R. T. Rockafellar, On the subdifferentiability of convex functions, Proc. Amer. Math. Soc. 16 (1965) 605–611.

[18] R.S. Burachik, M.M. Rizvi, On weak and strong Kuhn-Tucker conditions for smooth multiobjective optimization, J. Optim. Theory Appl. 155 (2012) 477–491.

[19] G. Y. Chen, W. D. Rong, Characterizations of the Benson proper efficiency for nonconvex vector optimization, J. Optim. Theory Appl. 98 (1998) 365–384.

[20] G.Y. Chen, X. Huang, X. Yang, Vector Optimization. Set-Valued and Variational Analysis, Springer-Verlag, Berlin, 2005.

[21] C. Coello, G.B. Lamont, D.A. Van Veldhuizen, Evolutionary algorithms for solving multi-objective problems, Springer, New York, 2007.

[22] C. Combari, S. Marcellin, L. Thibault, On the graph convergence of $\varepsilon$-Fenchel subdifferentials of convex functions, J. Nonlinear Convex Anal. 4 (2003) no. 2 309–324.

[23] H. W. Corley, Duality theory for maximizations with respect to cones, J. Math. Anal. Appl. 84 (1981) 560–568.

[24] R. Correa, A. Hantoute, A. Jourani, Characterizations of convex approximate subdifferential calculus in Banach spaces, Trans. Amer. Math. Soc. (2014).

[25] A. Daniilidis, Arrow-Barankin-Blackwell theorems and related results in cone duality: A survey, Lecture Notes in Econom. and Math. Systems 481 (2000) 119–131.

[26] V. F. Dem'yanov, L. V. Vasil'ev, Nondifferentiable Optimization, Optimization Software, New York, 1985.

[27] A. L. Dontchev, T. Zolezzi, Well-Posed Optimization Problems, Lecture Notes in Math. 1543, Springer-Verlag, Berlin, 1993.

[28] J. Dutta, Necessary optimality conditions and saddle points for approximate optimization in Banach spaces, TOP 13 (2005) 127–143.

[29] J. Dutta, V. Vetrivel, On approximate minima in vector optimizacion, Numer. Funct. Anal. Optim. 22 (2001) 845–859.

[30] I. Ekeland, R. Témam, Convex Analysis and Variational Problems, North-Holland, Amsterdam, 1976.

[31] A. Engau, M. Wiecek, Generating $\varepsilon$-efficient solutions in multiobjective programming, European J. Oper. Res. 177 (2007) 1566–1579.

[32] S.P. Fitzpatrick, S. Simons, The conjugates, compositions and marginals of convex functions, J. Convex Anal. 8 (2001) no. 2 423–446.

[33] Y. Gao, X. Yang, K. L. Teo, Optimality conditions for approximate solutions of vector optimization problems, J. Ind. Manag. Optim. 7 (2011) 483–496.

[34] A. M. Geoffrion, Proper eficiency and the theory of vector maximization, J. Math. Anal. Appl. 22 (1968) 618–630.

[35] G. Giorgi, A. Guerraggio, J. Thierfelder, Mathematics of Optimization: Smooth and Nonsmooth Case, Elsevier, Amsterdam, 2004.

[36] G. Giorgi, B. Jiménez, V. Novo, Strong Kuhn-Tucker conditions and constraint qualifications in locally Lipschitz multiobjective optimization problems, Top 17 (2009) 288–304.

[37] M. Golestani, S. Nobakhtian, Nonsmooth multiobjective programming: strong Kuhn-Tucker conditions, Positivity 17 (2013) 711–732.

[38] A. Göpfert, H. Riahi, C. Tammer, C. Zălinescu, Variational Methods in Partially Ordered Spaces. Springer-Verlag, New York (2003).

[39] A. Göpfert, C. Tammer, C. Zălinescu, A new ABB theorem in normed vector spaces, Optimization 53 (2004) 369–376.

[40] M. G. Govil, A. Mehra, $\varepsilon$-optimality for multiobjective programming on a Banach space, European J. Oper. Res. 157 (2004) 106–112.

[41] D. Gupta, A. Mehra, Two types of approximate saddle points, Numer. Funct. Anal. Optim. 29 (2008) 532–550.

[42] C. Gutiérrez, L. Huerga, B. Jiménez, V. Novo, Proper approximate solutions and $\varepsilon$-subdifferentials in vector optimization: Basic properties and limit behaviour, Nonlinear Anal. 79 (2013) 52–67.

[43] C. Gutiérrez, L. Huerga, B. Jiménez, V. Novo, Proper approximate solutions and $\varepsilon$-subdifferentials in vector optimization: Moreau-Rockafellar type theorems, J. Convex Anal. 21 no. 3 (2014) online.

[44] C. Gutiérrez, L. Huerga, V. Novo, Scalarization and saddle points of approximate proper solutions in nearly subconvexlike vector optimization problems, J. Math. Anal. Appl. 389 (2012) 1046–1058.

[45] C. Gutiérrez, L. Huerga, V. Novo, C. Tammer, Duality related to approximate proper solutions of vector optimization problems (preprint).

[46] C. Gutiérrez, L. Huerga, V. Novo, L. Thibault, Chain rules for a proper $\varepsilon$-subdifferential of vector mappings (preprint).

[47] C. Gutiérrez, B. Jiménez, V. Novo, Multiplier rules and saddle-point theorems for Helbig's approximate solutions in convex Pareto problems, J. Global Optim. 32 (2005) 367–383.

[48] C. Gutiérrez, B. Jiménez, V. Novo, On approximate efficiency in multiobjective programming, Math. Methods Oper. Res. 64 (2006) 165–185.

[49] C. Gutiérrez, B. Jiménez, V. Novo, A unified approach and optimality conditions for approximate solutions of vector optimization problems, SIAM J. Optim. 17 (2006) 688–710.

[50] C. Gutiérrez, B. Jiménez, V. Novo, Optimality conditions via scalarization for a new $\varepsilon$-efficiency concept in vector optimization problems. European J. Oper. Res. 201 (2010) 11–22.

[51] C. Gutiérrez, B. Jiménez, V. Novo, Improvement sets and vector optimization. European J. Oper. Res. 223 (2012) 304–311.

[52] C. Gutiérrez, B. Jiménez, V. Novo, Equivalent $\varepsilon$-efficiency notions in vector optimization, Top 20 (2012) 437–455.

[53] S. Helbig, On a new concept for $\varepsilon$-efficiency, talk at "Optimization Days 1992", Montreal, 1992.

[54] M. I. Henig, Proper efficiency with respect to cones, J. Optim. Theory Appl. 36 (1982) 387–407.

[55] J.-B. Hiriart-Urruty, Lipschitz $r$-continuity of the approximate subdifferential of a convex function, Math. Scand. 47 (1980) 123–134.

[56] J.-B. Hiriart-Urruty, $\varepsilon$-subdifferential calculus, Res. Notes in Math., 57, Pitman (1982) 43–92.

[57] J.-B. Hiriart-Urruty, From convex optimization to nonconvex optimization. Necessary and sufficient conditions for global optimality. Nonsmooth optimization and related topics (Erice, 1988), 219–239, Ettore Majorana Internat. Sci. Ser. Phys. Sci. 43, Plenum, New York, 1989.

[58] J.-B. Hiriart-Urruty, C. Lemaréchal, Convex Analysis and Minimization Algorithms I, Springer-Verlag, Berlin, 1993.

[59] J.-B. Hiriart-Urruty, C. Lemaréchal, Convex Analysis and Minimization Algorithms II, Springer-Verlag, Berlin, 1993.

[60] J.-B. Hiriart-Urruty, M. Moussaoui, A. Seeger, M. Volle, Subdifferential calculus without qualification conditions, using approximate subdifferentials: a survey, Nonlinear Anal. 24 (1995) 1727–1754.

[61] J.-B. Hiriart-Urruty, R. R. Phelps, Subdifferential calculus using $\varepsilon$-subdifferentials, J. Funct. Anal. 118 (1993) 154–166.

[62] R. Horst, H. Tuy, Global Optimization. Springer, Berlin (1990).

[63] J. Jahn, Duality in vector optimization, Math. Program. 25 (1983) 343–353.

[64] J. Jahn, Vector Optimization. Theory, Applications and Extensions, Springer-Verlag, Berlin, 2011.

[65] J.-H. Jia, Z.-F. Li, $\varepsilon$-Conjugate maps and $\varepsilon$-conjugate duality in vector optimization with set-valued maps, Optimization 57 (2008) 621–633.

[66] H. Kuhn, A. Tucker, Nonlinear programming, Proceedings of the second Berkeley symposium on mathematical statistics and probability, Berkeley (1951) 481–492.

[67] A. J. Kurdila, M. Zabarankin, Convex Functional Analysis, Birkhäuser Verlag, Basel, 2005.

[68] A. G. Kusraev, S. S. Kutateladze, Local convex analysis, J. Math. Sci. 26 (1984) 2048–2087.

[69] A. G. Kusraev, S. S. Kutateladze, Subdifferentials: Theory and Applications, Kluwer Academic Publishers, Dordrecht, 1995.

[70] S. S. Kutateladze, Convex $\varepsilon$-programming, Soviet Math. Dokl. 20 (1979) 391–393.

[71] S. S. Kutateladze, $\varepsilon$-subdifferentials and $\varepsilon$-optimality, Siberian Math. J. 21 (1980) no. 3 (1981) 404–411.

[72] Z. F. Li, Benson proper efficiency in the vector optimization of set-valued maps, J. Optim. Theory Appl. 98 (1998) 623–649.

[73] Z. F. Li, S. Y. Wang, Lagrange multipliers and saddle points in multiobjective programming, J. Optim. Theory Appl. 83 (1994) 63–81.

[74] Z. F. Li, S. Y. Wang, $\varepsilon$-approximate solutions in multiobjective optimization, Optimization 44 (1998) 161–174.

[75] X.F. Li, J.Z. Zhang, Stronger Kuhn-Tucker type conditions in nonsmooth multiobjective optimization: locally Lipschitz case. J. Optim. Theory Appl. 127 (2005) 367–388.

[76] J.-C. Liu, $\varepsilon$-properly efficient solutions to nondifferentiable multiobjective programming problems, Appl. Math. Lett. 12 (1999) 109–113.

[77] P. Loridan, Necessary conditions for $\varepsilon$-optimality, Math. Programming Stud. 19 (1982) 140–152.

[78] P. Loridan, $\varepsilon$-solutions in vector minimization problems, J. Optim. Theory Appl. 43 (1984) 265–276.

[79] P. Loridan, J. Morgan, Penalty functions in $\varepsilon$-programming and $\varepsilon$-minimax problems, Math. Programming 26 (1983) 213–231.

[80] D. T. Luc, On duality theory in multiobjective programming, J. Optim. Theory Appl. 43 (1984) 557–582.

[81] D.T. Luc, Theory of Vector Optimization, Lecture Notes in Econom. and Maht. Systems 319, Springer-Verlag, Berlin, 1989.

[82] T. Maeda, Constraint qualification in multiobjective optimization problems: differentiable case. J. Optim. Theory Appl. 80 (1994) 483–500.

[83] M. El Maghri, Pareto-Fenchel $\varepsilon$-subdifferential sum rule and $\varepsilon$-efficiency, Optim. Lett. 6 (2012) 763–781.

[84] M. El Maghri, Pareto-Fenchel $\varepsilon$-subdifferential composition rule and $\varepsilon$-efficiency, Numer. Func. Anal. Optim. 35(1) (2014) 1–19.

[85] M. El Maghri, ($\epsilon$-)Efficiency in difference vector optimization, J. Glob. Optim., DOI 10.1007/s10898-014-0204-0.

[86] M. El Maghri, M. Laghdir, Pareto subdifferential calculus for convex vector mappings and applications to vector optimization, SIAM J. Optim. 19 (2008) 1970–1994.

[87] A. Mehra, Super efficiency in vector optimization with nearly convexlike set-valued maps, J. Math. Anal. Appl. 276 (2002) 815–832.

[88] K. M. Miettinen, Nonlinear Multiobjective Optimization, Kluwer Academic Publishers, Dordrecht, 1999.

[89] A. B. Németh, A nonconvex vector minimization problem, Nonlinear Anal. 10 (1986) no. 7 669–678.

[90] E. A. Nurminskiĭ, The continuity of $\varepsilon$-subgradient mappings, Kibernetika (Kiev) 5 (1977) 148–149.

[91] V. Pareto, Manuale di economia politica, Società Editrice Libraria, Milano, 1906.

[92] J. H. Qiu, On solidness of polar cones, J. Optim. Theory Appl. 109 (2001) 199–214.

[93] J. H. Qiu, Dual characterization and scalarization for Benson proper efficiency, SIAM J. Optim. 19 (2008) 144–162.

[94] J.-H. Qiu, Y. Hao, C. Wang, Separable conditions and scalarization of Benson proper efficiency, Acta Math. Sci. Ser. B Engl. 30 (2010) 1154–1166.

[95] C. Raffin, Contribution à l'Étude des Programmes Convexes Définis dans des Espaces Vectoriels Topologiques, Thèse, Université Pierre et Marie Curie, Paris, 1969.

[96] N. Rastegar, E. Khorram, A combined scalarizing method for multiobjective programming problems, European J. Oper. Res. 236 (2014) 229–237.

[97] W. Rong, Proper $\varepsilon$-efficiency in vector optimization problems with cone-subconvexlikeness, Acta Sci. Natur. Univ. NeiMongol 28 (1997) 609–613.

[98] W. D. Rong, Y. N. Wu, $\varepsilon$-Weak minimal solutions of vector optimization problems with set-valued maps, J. Optim. Theory Appl. 106 (2000) 569–579.

[99] P. H. Sach, Nearly subconvexlike set-valued maps and vector optimization problems, J. Optim. Theory Appl. 119 (2003) 335–356.

[100] P. H. Sach, L. A. Tuan, N. B. Minh, Approximate duality for vector quasi-equilibrium problems and applications, Nonlinear Anal. 72 (2010) 3994–4004.

[101] Y. Sawaragi, H. Nakayama, T. Tanino, Theory of Multiobjective Optimization, Academic Press, Orlando, 1985.

[102] C. Scovel, D. Hush, I. Steinwart, Approximate duality, J. Optim. Theory Appl. 135 (2007) 429–443.

[103] J. J. Strodiot, V. H. Nguyen, N. Heukemes, $\varepsilon$-optimal solutions in nondifferentiable convex programming and some related questions, Math. Programming 25 (1983) 307–328.

[104] A. Taa, $\varepsilon$-Subdifferentials of set-valued maps and $\varepsilon$-weak Pareto optimality for multiobjective optimization. Math. Methods Oper. Res. 62 (2005) 187–209.

[105] C. Tammer, Approximate saddle point theorems. In Methods of Multicriteria Decision Theory, Verlag, Egelsbach (1997) 223–235.

[106] C. Tammer, C. Zălinescu, Minimal-point theorems in product spaces and applications, in http://www2.mathematik.uni-halle.de/institut/reports/ (2010).

[107] T. Tanaka, A new approach to approximation of solutions in vector optimization problems, Proceedings of APORS, 1994 (M. Fushimi and K. Tone, eds.), World Scientific Publishing, Singapore, 1995, 497–504.

[108] T. Tanino, Y. Sawaragi, Duality theory in multiobjective programming, J. Optim. Theory Appl. 27 (1979) 509–529.

[109] L. Thibault, Sequential convex subdifferential calculus and sequential Lagrange multipliers, SIAM J. Control Optim. 35 (1997) 1434–1444.

[110] L. A. Tuan, $\varepsilon$-optimality conditions for vector optimization problems with set-valued maps, Numer. Funct. Anal. Optim. 31 (2010) 78–95.

[111] H. Tuy, Convex Analysis and Global Optimization. Kluwer, Dordrecht (1998).

[112] M. Valadier, Sous-différentiabilité de fonctions convexes à valeurs dans un espace vectoriel ordonné. Math. Scand. 30 (1972) 65–74.

[113] I. Vályi, Approximate solutions of vector optimization problems, Systems Analysis and Simulation, Akademie-Verlag, Berlin, 1985, 246–250.

[114] I. Vályi, Approximate saddle-point theorems in vector optimization, J. Optim. Theory Appl. 55 (1987) 435–448.

[115] D. J. White, Epsilon efficiency, J. Optim. Theory Appl. 49 (1986) no. 2 319–337.

[116] X. M. Yang, D. Li, S. Y. Wang, Near-subconvexlikeness in vector optimization with set-valued functions, J. Optim. Theory Appl. 110 (2001) 413–427.

[117] K. Yokoyama, $\varepsilon$-optimality criteria for convex programming problems via exact penalty functions, Math. Program. 56 (1992) 233–243.

[118] P. L. Yu, Cone convexity, cone extreme points and nondominated solutions in decision problems with multiple objectives, J. Optim. Theory Appl. 14 (1974) no. 3 319–377.

[119] A. Zaffaroni, Superlinear separation for radiant and coradiant sets, Optimization 56 (2007) 267–285.

[120] C. Zălinescu, Convex Analysis in General Vector Spaces, World Scientific, Singapore, 2002.

[121] X. Y. Zheng, Proper efficiency in locally convex topological vector spaces, J. Optim. Theory Appl. 94 (1997) no. 2 469–486.

[122] J. Zowe, Subdifferentiability of convex functions with values in an ordered vector space, Math. Scand. 34 (1974) 69–83.

# Índice de notaciones