Anticipating Stochastic Integration

escrito por

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Abstract

Abstract en español: En esta tesis, se estudia la teoría de integración estocástica de Itô, así como, su aplicación en la modelización financiera a partir de las ecuaciones diferenciales estocásticas. A continuación, se presentan dos nuevas teorías de integración, la integral estocástica de Ayed-Kuo y la de Russo-Vallois, que generalizan la de Itô en el sentido de que introducen el cálculo estocástico anticipante. Se analizan algunas de sus propiedades más importantes, así como sus respectivas extensiones de la formula de Itô. Finalmente, se transponen ambas integrales a la teoría de ecuaciones diferenciales estocásticas y se introduce el problema de inversión con información privilegiada, cuyas hipotésis están relacionadas con la condición anticipante. Para este último punto, se proponen dos nuevos teoremas que se han demostrado en este trabajo.

Abstract in English: In this thesis, the Itô theory of stochastic integration is studied, as well as its application in financial modeling based on stochastic differential equations. Then, two new integration theories are presented, the Ayed-Kuo and the Russo-Vallois stochastic integrals, which generalize the Itô one in the sense that they deal with anticipating stochastic calculus. Some of their most remarkable properties are discussed, as well as their respectively extensions of the Itô formula. Finally, both integrals are transposed to the stochastic differential equations theory and the insider trading problem is introduced, whose hypothesis are related to the anticipating condition. For this final point, two new theorems, which have been proved in this work, are proposed.

Keywords: Brownian motion; Itô Stochastic Integration; Stochastic Differential Equations; Anticipating Stochastic Calculus; Ayed-Kuo Integral; Russo-Vallois Integral; Insider Trading

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To my family.

Thank you for your unconditional and strong support in each stage of my academic, professional and personal life. Thank you for your education based on the effort and respect principles. For your dedication and determination. For your love and your care.

"Tomorrow belongs to those who can hear it coming."

(David Bowie)

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"Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two facilities, which we may call intuition and ingenuity."

(Alan Mathison Turing)

Contents

Introduction

In the theory of stochastic integration, the mathematician K. Itô in 1944 introduced the Itô stochastic integral in order to obtain a method to construct diffusion processes as solutions to stochastic differential equations. The main problem of the integration with respect to Brownian motion, is the fact that the Riemann-Stieltjes integration fails. This need inspired Itô to construct a theory of stochastic integration.

The Itô stochastic integral has a rage of applications. The most famous are those related to financial modeling, as the Black-Scholes-Merton model. A continuos-time model, which aim is to describe the behaviour of stock price. Specifically, it faces the problem of pricing European options. However, this integration theory requires that the stochastic process must be adapted.

In this dissertation, our aim is to study an extension of the Itô theory considering the fact of having information from the future, it means, that the integrand is anticipating. This idea is mainly motivated by the problem of insider trading, in which a trader is considered to have privileged information about future prices of assets. Hence, an extension of the Itô theory is required.

This thesis introduce two anticipating stochastic integration theories. On the one hand, the Ayed-Kuo stochastic integral, which was proposed by W. Ayed and H.-H. Kuo in 2008. On the other hand, the Russo-Vallois stochastic integral, which was first introduced by F. Russo and P. Vallois in 1993. The first one keeps most of the properties of the Itô stochastic integral, while the second one does not have the analytical structure of the Ayed-Kuo one. However, as we will discuss in this thesis, if we consider anticipating condition in the financial model, the Ayed-Kuo solution seems to be counterintuitive, while the Russo-Vallois solution works in the financial context.

This dissertation is organized in six chapters. In the first one, we give an introduction to Brownian motion. We provide a definition of it and study some of its most important properties, specifically those related to stochastic integration, as the martingale property. In the second chapter, we construct the Itô stochastic integral and discuss some of its most remarkable properties. Then, we present the Itô formula and an important result, the Girsanov theorem. The third chapter is focused on the stochastic differential equations theory. We introduce the notion of solution of an stochastic differential equation and we prove the existence and uniqueness of this solution. We do also describe the well-known Black-Scholes-Merton model, as an example of financial modeling based on this theory.

The next three chapters are related to the anticipating stochastic calculus. The aim is to give a theory to the stochastic integrals whose integrand is anticipating, and consequently non-adapted. In the fourth chapter, we construct the Ayed-Kuo stochastic integral and we discuss some of its most remarkable results. In the same form, the fifth chapter examines the Russo-Vallois stochastic integral.

This thesis ends with the description of the insider trading problem, which is one of the motivations of the anticipating stochastic calculus. We present the problem and discuss the solutions for the Ayed-Kuo and the Russo-Vallois stochastic integrals. For this final point, we propose two new theorems that we have proved in this work, which deal with the optimal investment strategy for insider trading.

The appendixes include a summary on Normal random variables, conditional expectation and the Borell-Cantelli lemma and the Chebyshev inequality.

Chapter 1

Brownian Motion

Brownian motion was first defined by the botanist Robert Brown in 1828 while he was studying the irregular random movement of a particle of pollen. The motion was later explained by Albert Einstein in 1905 to refer to the random collisions with the molecules of the liquid. In 1900, Louis Bachelier was the first mathematician to used Brownian motion as a model for movement of stock prices in his PhD thesis *The Theory of Speculation*. Nevertheless, it was not until 1931 when the establishment of Brownian motion as a stochastic process was done by Norbert Wiener.

Brownian motion $B(t)$ is fundamental in the theory of stochastic integration because of its definition and properties. It has a range of applications, those including financial modeling, as it is an important stochastic process. Hence, the aim of this chapter is to discuss the main notions of Brownin motion, which will be necessary in the stochastic integration theory.

This chapter is organized as follows. First, we provide elementary notions of stochastic processes and martingales. Next, we give a definition of Brownian motion and discuss some of its most remarkable properties, as the martingale one and those related to sample paths. Finally, we examine the quadratic variation and the Markov property for this stochastic process.

1.1 Preliminaries

This section is, in turn, divided into two parts. First, we provide fundamental notions of the stochastic processes in order to, in the following section, be able to give a definition of Brownian motion. Then, we introduce the concept of martingale and its importance as a property of the Brownian motion.

1.1.1 Stochastic Processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a *probability space*, such that

- Ω is a set of outcomes;
- $\mathcal{F}$ is a set of events;
- $\mathbb{P} : \mathcal{F} \to [0,1]$ is a function that assigns probabilities to events.

Definition 1.1. Consider a set Ω. Then, the *σ-algebra* $\mathcal{F}$ on Ω is defined as a family $\mathcal{F}$ of subsets of Ω satisfying the following properties

(i) $\emptyset \in \mathcal{F}$;

(ii) $\mathcal{A} \in \mathcal{F} \to \mathcal{A}^c \in \mathcal{F}$;

(iii) $\mathcal{A}_1, \mathcal{A}_2, \ldots \in \mathcal{F} \to \mathcal{A} = \bigcup_{i=1}^{\infty} \mathcal{A}_i \in \mathcal{F}$.

Definition 1.2. Let $X : \Omega \to \mathbb{R}$. The random variable X is *$\mathcal{F}$-measurable*, or equivalently, *measurable with respect to $\mathcal{F}$*, if

$$X^{-1}(U) = \{\omega \in \Omega,\ X(\omega) \in U\} \in \mathcal{F},$$

for all *Borel* sets such that $U \in \mathcal{B}(\mathbb{R})$.

Proposition 1.1 (Doob-Dynkin lemma). *Let $X, Y : \Omega \to \mathbb{R}$ be two random variables. Then, Y is $\mathcal{H}_x$-measurable, where $\mathcal{H}_x$ is the smallest σ-algebra generated by X, if and only if there exists a Borel measurable function $g : \mathbb{R} \to \mathbb{R}$, such that*

$$Y = g(x).$$

Definition 1.3 (Stochastic process). A *stochastic process* is a parametrized collection of random variables

$$\{X_t, t \in T\} = \{X_t(\omega), t \in T, \omega \in \Omega\},$$

defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assuming values in $\mathbb{R}$.

Remark 1.1. For each $t \in T$ fixed, we have a random variable

$$\omega \to X_t(\omega), \qquad \omega \in \Omega.$$

Hence, we called *realization*, *trajectory* or *sample path* of the process X_t to the function

$$t \to X_t(\omega), \qquad t \in T,$$

for each $\omega \in \Omega$ fixed.

Definition 1.4. A *filtration* on the measurable space $(\Omega, \mathcal{F})$ is an increasing family $\{\mathcal{F}_t, t \geq 0\}$ of sub-σ-algebras of $\mathcal{F}$.

Definition 1.5. A stochastic process X_t is *adapted* to the filtration $\{\mathcal{F}_t, t \geq 0\}$, if the random variable X_t is $\mathcal{F}_t$-measurable for each t.

1.1.2 Martingales

Definition 1.6 (Martingale)**.** A stochastic process $\{X_t, t \geq 0\}$ is a *martingale* with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$, if and only if

(i) $\mathbb{E}|X_t| < \infty$, for all $t \geq 0$;

(ii) X_t is $\mathcal{F}_t$-measurable, for any $t \geq 0$;

(iii) $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ almost surely, for all $0 \leq s \leq t$.

Remark 1.2. Note that, the expectation of a martingale is constant. Then, by the properties of the conditional expectation (see Appendix B) and the martingale definition, we have, for all $t \geq 0$

$$\mathbb{E}(X_t) = \mathbb{E}(\mathbb{E}(X_t|\mathcal{F}_0)) = \mathbb{E}(X_0).$$

This result is called the *fair game* property. It can be used as an argument to prove that a stochastic process is not a martingale. However, we cannot use it in the opposite sense, since a not martingale process can have a constant expectation.

Definition 1.7. Let $\{X_t, t \geq 0\}$ be a stochastic process and $\{\mathcal{F}_t, t \geq 0\}$ a filtration. If the stochastic process satisfies the conditions (i) and (ii) of Definition 1.6, we have

(i) The stochastic process $\{X_t, t \geq 0\}$ is a *submartingale* with respect to the filtration if and only if $\mathbb{E}(X_t|\mathcal{F}_s) \geq X_s$ almost surely, for all $0 \leq s \leq t$.

(ii) The stochastic process $\{X_t, t \geq 0\}$ is a *supermartingale* with respect to the filtration if and only if $\mathbb{E}(X_t|\mathcal{F}_s) \leq X_s$ almost surely, for all $0 \leq s \leq t$.

Next, let us establish a relationship between martingales and submartingales in the following proposition. This result can be proved by the martingale definition and the conditional *Jensen inequality* (see Appendix B).

Proposition 1.2. *Let $\{X_t, 0 \leq t \leq T\}$ be a martingale and $\varphi : \mathbb{R} \to \mathbb{R}$ a convex function. Let us assume that, for each t, $\mathbb{E}|\varphi(X_t)|$ is finite. Hence, we have*

$$\{\varphi(X_t), 0 \leq t \leq T\},$$

is a submartingale.

For continuous martingales we have the following important result, the *Doob Martingale inequality*, which is due to J. L. Doob. The proof can be found in [35].

Theorem 1.1 (Doob martingale inequality)**.** *If M_t is a martingale, such that $t \to M_t(\omega)$ is continuous almost surely. Then, we have*

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} |M_t| \geq \lambda\right) \leq \frac{1}{\lambda^p}\mathbb{E}\left(|M_t|^p\right),$$

for all $p \geq 1$, $T \geq 0$ and $\lambda > 0$.

1.2 Definition of Brownian Motion

In this section, we give a definition of Brownian motion. Next, we discuss that it can also be defined as a Gaussian process. Then, we study some of its most remarkable properties and we prove that it is a martingale.

Definition 1.8 (Brownian motion)**.** A stochastic process $\{B(t,\omega), t \geq 0\}$ is a standard one-dimensional *Brownian motion* or *Wiener process* if it satisfies the following properties

(i) (Independence of increments) For $0 \leq t_1 < t_2 ... < t_n$, the random variables

$$B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}},$$

are independent.

(ii) (Normal increments) $B(t) - B(s)$ has a Normal distribution with mean 0 and variance $t - s$.

(iii) (Continuity of paths) $\{B(t,\omega), t \geq 0\}$ are continuous functions of t.

Note that, the properties (i) and (ii) determine all the finite dimensional distributions. Moreover, in Theorem 1.2 we prove that all them are Gaussian. On the other hand, the property (iii) deals with the continuity property of Brownian motion sample paths. We prove this property in Section 1.3 by the Kolmogorov continuity theorem.

Theorem 1.2 (Gaussian process)**.** *A stochastic process X_t is a Brownian motion if and only if it is a Gaussian process with zero mean function and covariance function $\Sigma(s,t) = \min(t,s)$.*

Proof. First, we prove the implication ($\Rightarrow$). The mean of the Brownian motion is zero, such that

$$\Sigma(s,t) = Cov(B(t), B(s)) = \mathbb{E}(B(t)B(s)).$$

Let us assume that $t < s$, such that $B(s) = B(t) + B(s) - B(t)$. Hence, we get

$$\mathbb{E}(B(t)B(s)) = \mathbb{E}(B(t)(B(t) + B(s) - B(t))) = \mathbb{E}\left(B(t)^2\right) + \mathbb{E}(B(t)(B(s) - B(t))) = \mathbb{E}\left(B(t)^2\right) = t,$$

by independent increments property. Next, let us assume that $t > s$, such that $\mathbb{E}(B(t)B(s)) = s$. Therefore, we have

$$\mathbb{E}(B(t)B(s)) = \min(t,s).$$

Next, we prove the implication ($\Leftarrow$). Let us assume that t is arbitrary and $s \geq 0$. We have that X_t is a Gaussian process, then the joint distribution of X_t and X_{t+s} is a bivariate Normal with zero mean. Moreover, $(X_t, X_{t+s} - X_t)$ is also a bivariate Normal. Furthermore, we have that $\Sigma(X_t, X_{t+s}) = \min(t,s)$, such that

$$\Sigma(X_t, X_{t+s} - X_t) = \Sigma(X_t, X_{t+s}) - \Sigma(X_t, X_t) = t - t = 0.$$

Hence, X_t and $X_{t+s} - X_t$ are not correlated. In addition, we can state that X_t and $X_{t+s} - X_t$ are independent. Then, the increment $X_{t+s} - X_t$ is independent of X_t and it has Normal distribution with zero mean and variance s. Indeed, it is a Brownian motion. □

Proposition 1.3 (Translation invariance). *Consider a fixed $t_0 \geq 0$. Then, the stochastic process $\tilde{B}(t) = B(t+t_0) - B(t_0)$ is a Brownian motion.*

Proof. Is easy to check that the stochastic process $\tilde{B}(t) = B(t+t_0) - B(t_0)$ has continuous sample paths. Hence, $\tilde{B}(t)$ satisfies the property (iii) of Definition 1.8. On the other hand, let us consider, for any $t > s$

$$\tilde{B}(t) - \tilde{B}(s) = B(t+t_0) - B(s+t_0). \tag{1.1}$$

Hence, $\tilde{B}(t) - \tilde{B}(s)$ has Normal distribution with zero mean and variance $(t+t_0)-(s+t_0) = t-s$. Hence, $\tilde{B}(t)$ satisfies property (ii). Finally, let us assume that $t_0 > 0$, such that, we have, for any $0 \leq t_1 < \ldots < t_n$

$$0 < t_0 \leq t_1 + t_0 < \ldots < t_n + t_0,$$

such that

$$B(t_k + t_0) - B(t_{k-1} + t_0), \qquad k = 1, 2, \ldots, n,$$

are independent random variables. Hence, by Equation (1.1), the increments $\tilde{B}(t_k) - \tilde{B}(t_{k-1})$, for $k = 1, 2, \ldots, n$, are independent random variables. Hence, $\tilde{B}(t)$ satisfies property (i). Consequently, $\tilde{B}(t)$ is a Brownian motion. □

Proposition 1.4 (Scaling invariance). *Let $\lambda > 0$ be any real number. Then, the stochastic process $\tilde{B}(t) = B(\lambda t)/\sqrt{\lambda}$ is a Brownian motion.*

Proof. Is easy to check that $\tilde{B}(t)$ satisfies property (i) and property (iii) of Definition 1.8. Next, let us consider, for any $t > s$

$$\tilde{B}(t) - \tilde{B}(s) = \frac{1}{\sqrt{\lambda}}(B(\lambda t) - B(\lambda s)).$$

Then, $\tilde{B}(t) - \tilde{B}(s)$ has Normal distribution with zero mean and variance $\frac{1}{\lambda}(\lambda t - \lambda s) = t - s$. Hence, $\tilde{B}(t)$ satisfies property (ii). Consequently, $\tilde{B}(t)$ is a Brownian motion. □

Theorem 1.3. *The Brownian motion $B(t)$ is a martingale.*

Proof. Let $0 \leq s \leq t$. Then, we have

$$\begin{aligned}
\mathbb{E}(B(t)|\mathcal{F}_s) &= \mathbb{E}((B(t) - B(s)) + B(s)|\mathcal{F}_s) \\
&= \mathbb{E}(B(t) - B(s)|\mathcal{F}_s) + \mathbb{E}(B(s)|\mathcal{F}_s) \\
&= \mathbb{E}(B(t) - B(s)|\mathcal{F}_s) + B(s) \\
&= B(s),
\end{aligned}$$

by independent increments property and Normal increments property. Hence, $B(t)$ satisfies the martingale property. □

In the following result, we give two examples of martingales that are associated with Brownian motion.

Theorem 1.4. *Let $B(t)$ be a Brownian motion. Then, we have*

(i) $B(t)^2 - t$ *is a martingale;*

(ii) For any σ, $e^{\sigma B(t) - \frac{1}{2}\sigma^2 t}$ *is a martingale.*

Proof. Let us consider the function f and the filtration $\mathcal{F}_t$, such that

$$\mathbb{E}(f(B(t+s)-B(t))|\mathcal{F}_t) = \mathbb{E}(f(B(t+s)-B(t))),$$

since $B(t+s)-B(t)$ and $\mathcal{F}_t$ are independent. Indeed, the latter expectation is just $\mathbb{E}(f(X))$, where X is a random variable with zero mean and variance s.

(i) Note that, $\mathbb{E}\left(B(t)^2\right) = t < \infty$. Then, $B(t)^2$ is integrable. Moreover, we have

$$\begin{aligned} B(t+s)^2 &= (B(t) + B(t+s) - B(t))^2 \\ &= B(t)^2 + 2B(t)(B(t+s)-B(t)) + (B(t+s)-B(t))^2. \end{aligned}$$

Then, we use that $B(t+s)-B(t)$ is independent of $\mathcal{F}_t$ and it has zero mean. In addition, we take $g(x) = x^2$, such that

$$\begin{aligned} \mathbb{E}\left(B(t+s)^2 \middle| \mathcal{F}_t\right) &= B(t)^2 + 2\mathbb{E}\left(B(t)(B(t+s)-B(t)) \middle| \mathcal{F}_t\right) + \mathbb{E}\left((B(t+s)-B(t))^2 \middle| \mathcal{F}_t\right) \\ &= B(t)^2 + s. \end{aligned}$$

Hence, the stochastic process $B(t)^2 - t$ satisfies the martingale property.

(ii) Let us consider the moment generating function of $B(t)$. Then, since $B(t) \sim \mathcal{N}(0,t)$, we have $\mathbb{E}\left(e^{\sigma B(t)}\right) = e^{\frac{1}{2}\sigma^2 t} < \infty$. Then, $e^{\sigma B(t) - \frac{1}{2}\sigma^2 t}$ is integrable. Hence, we get

$$\mathbb{E}\left(e^{\sigma B(t) - \frac{1}{2}\sigma^2 t}\right) = 1.$$

Then, we take $g(x) = e^{\sigma x}$ and use the fact that $B(t)$ is $\mathcal{F}_t$-measurable, such that, we have

$$\begin{aligned} \mathbb{E}\left(e^{\sigma B(t+s)} \middle| \mathcal{F}_t\right) &= \mathbb{E}\left(e^{\sigma B(t) + \sigma(B(t+s)-B(t))} \middle| \mathcal{F}_t\right) \\ &= e^{\sigma B(t)} \mathbb{E}\left(e^{\sigma(B(t+s)-B(t))} \middle| \mathcal{F}_t\right). \end{aligned}$$

Finally, since the increment $B(t+s)-B(t)$ is independent of $\mathcal{F}_t$, we get

$$\begin{aligned} \mathbb{E}\left(e^{\sigma B(t+s)} \middle| \mathcal{F}_t\right) &= e^{\sigma B(t)} \mathbb{E}\left(e^{\sigma(B(t+s)-B(t))} \middle| \mathcal{F}_t\right) \\ &= e^{\sigma B(t)} \mathbb{E}\left(e^{\sigma(B(t+s)-B(t))}\right) \\ &= e^{\sigma B(t) + \frac{1}{2}\sigma^2 s}. \end{aligned}$$

Hence, the stochastic process $e^{\sigma B(t) - \frac{1}{2}\sigma^2 t}$ satisfies the martingale property.

□

Remark 1.3. The three stochastic processes from Theorem 1.3 and Theorem 1.4 are important martingales in the stochastic calculus theory. The first one $B(t)$ is the Brownian motion. The second one $B(t)^2 - t$ provides the Levy characterization of Brownian motion. Additionally, the martingale $e^{\sigma B(t)-\frac{1}{2}\sigma^2 t}$ is called the exponential Brownian motion or the exponential martingale, and it is used in the establishment of the distribution properties of Brownian motion.

1.3 Properties of Brownian Motion Paths

In this section, our aim is to prove the continuity of Brownian motion sample paths, which is stated in property (iii) from Definition 1.8, and the nowhere differentiability of these trajectories.

1.3.1 Hölder Continuity

Next, we prove that Brownian motion $B(t)$ always has a version of it with uniformly Hölder continuous sample paths for each exponent $\gamma < \frac{1}{2}$, but these paths are nowhere Hölder continuous with any exponent $\gamma \geq \frac{1}{2}$.

Let us first introduce some elementary notions.

Definition 1.9. Let $\{X_t\}$ and $\{Y_t\}$ be two stochastic processes on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A process $\{X_t\}$ is called a *version* of $\{Y_t\}$, if, for all t

$$\mathbb{P}(\{X_t(\omega) = Y_t(\omega), \omega \in \Omega\}) = 1.$$

Definition 1.10. Let $0 \leq \gamma \leq 1$ and let $f : [0,T] \to \mathbb{R}$. The function f is called *uniformly Hölder continuous* with exponent $\gamma > 0$, if there exists a constant K, such that, for all s, $t \in [0,T]$

$$|f(t) - f(s)| \leq K|t-s|^{\gamma}.$$

In addition, the function f is called *Hölder continuous* with exponent $\gamma > 0$ at the point s, if there exists a constant K, such that, for all $t \in [0,T]$

$$|f(t) - f(s)| \leq K|t-s|^{\gamma}.$$

Next, we present the *Kolmogorov continuity theorem*, which allows us to prove Hölder continuity of Brownian motion sample paths. The proof can be found in [22].

Theorem 1.5 (Kolmogorov continuity theorem). *Let $\{X_t, 0 \leq t \leq T\}$ be a stochastic process. If there exist three constants $K, \alpha, \beta > 0$, such that, for all $s, t \in [0,T]$*

$$\mathbb{E}\left(|X_t - X_s|^{\beta}\right) \leq K|t-s|^{1+\alpha}.$$

Then, there exists a version $\{\tilde{X}_t, t \geq 0\}$ of the process with almost surely Hölder continuous sample paths with exponent $\gamma \in \left(0, \frac{\alpha}{\beta}\right)$.

Theorem 1.6 (Hölder continuity)**.** *Let $\{B(t), t \geq 0\}$ be a Brownian motion. There exists a version $\{\bar{B}(t), t \geq 0\}$ of the process, such that, for almost all $\omega \in \Omega$ and all $T > 0$, the sample paths $t \to B(t,\omega)$ are Hölder continuous on $[0,T]$ with exponent $0 < \gamma < \frac{1}{2}$.*

Proof. In order to prove this result, we use the Kolmogorov continuity theorem. Our aim is to check that any Brownian motion satisfies the hypothesis of this theorem. Let us recall that $B(t) - B(s) \sim \mathcal{N}(0, t-s)$. Hence, by Normal distribution properties (see Appendix A), we have

$$\begin{aligned}\mathbb{E}\left(|B(t)-B(s)|^{2n}\right) &= \frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty} x^{2n} e^{-\frac{x^2}{2(t-s)}}dx \\ &= \frac{1}{\sqrt{2\pi(t-s)}}\int_{-\infty}^{\infty} y^{2n}(t-s)e^{-\frac{y^2}{2}}\sqrt{t-s}dy \\ &= K(t-s)^n.\end{aligned}$$

Then, by Theorem 1.5, we know that there exists a version of the process with Hölder continuous sample paths, whose exponent γ satisfies

$$0 < \gamma < \frac{n-1}{2n} = \frac{1}{2} - \frac{1}{2n}.$$

Therefore, taking $n \to \infty$, we have

$$0 < \gamma < \frac{1}{2}.$$

□

1.3.2 Nowhere Differentiability

Next, we prove that Brownian motion sample paths are nowhere differentiable. In order to do so, we first state that, with probability one, all sample paths of the process are nowhere Hölder continuous with exponent greater than or equal to $\frac{1}{2}$.

Theorem 1.7. *Let $1/2 \leq \gamma \leq 1$. Almost all sample paths of Brownian motion $B(t)$ are nowhere Hölder continuous with exponent γ.*

We do not prove this result as it is highly extensive. The proof can be found in [13]. Moreover, by Theorem 1.7, we can establish the following result, whose proof leads immediately.

Theorem 1.8 (Nowhere differentiability)**.** *Any Brownian motion has almost surely nowhere differentiable sample paths.*

Proof. By reducing to absurd, let us consider that the sample paths of Brownian motion are differentiable at some point s. Then, they are Hölder continuous with exponent $\gamma = 1$ at s. However, Theorem 1.7 states that this is almost surely not so. Hence, for almost all ω, $t \to B_t(t,\omega)$ is nowhere differentiable. □

Remark 1.4. As a consequence of the nowhere differentiability of Brownian motion sample paths, with probability one, these paths are not of bounded variation, it means, the trajectories are not monotone.

1.4 Quadratic Variation of Brownian Motion

In this section, we derive the quadratic variation of Brownian motion. This result has a great importance in the following chapter, where we introduce the stochastic integral. As a consequence of Remark 1.4 and the quadratic variation of brownian motiom, we have that it is not possible to integrate with respect to Brownian motion in a Riemann-Stieltjes sense.

Definition 1.11. The *quadratic variation* of Brownian motion $[B,B](t)$ is defined by

$$[B,B](t) = \lim_{||\Delta_n|| \to 0} \sum_{i=1}^{n} \left| B\left(t_i^n\right) - B\left(t_{i-1}^n\right) \right|^2, \tag{1.2}$$

where the limit is taken over the partition Δ of the interval $[0,t]$ and $||\Delta_n|| = \max_{1 \le i \le n} \left(t_i^n - t_{i-1}^n\right)$ as $n \to \infty$.

Theorem 1.9 (Quadratic variation)**.** *The quadratic variation of a Brownian motion over the interval* $[s,t]$ *is* $t-s$.

Proof. In order to prove the result above, we give the proof for a sequence of partitions, such that $\sum_n ||\Delta_n|| < \infty$. Let us consider

$$T_n = \sum_i \left| B\left(t_i^n\right) - B\left(t_{i-1}^n\right) \right|^2.$$

Then, we have

$$\begin{aligned} \mathbb{E}(T_n) &= \mathbb{E}\left(\sum_i \left| B\left(t_i^n\right) - B\left(t_{i-1}^n\right) \right|^2 \right) \\ &= \sum_{i=1}^{n} \left(t_i^n - t_{i-1}^n\right) \\ &= t - s. \end{aligned}$$

The variance of T_n, using the fourth moment of the Normal distribution with zero mean and standard deviation σ (see Appendix A), is

$$\begin{aligned} V(T_n) &= V\left(\sum_i \left| B\left(t_i^n\right) - B\left(t_{i-1}^n\right) \right|^2 \right) \\ &= \sum_i V\left(B\left(t_i^n\right) - B\left(t_{i-1}^n\right)\right)^2 \\ &= \sum_i 3\left(t_i^n - t_{i-1}^n\right)^2 \\ &\le 3 \max_{1 \le i \le n} \left(t_i^n - t_{i-1}^n\right)(t-s) \\ &= 3(t-s)||\Delta_n||. \end{aligned}$$

Hence, we get

$$\sum_{n=1}^{\infty} V(T_n) < \infty.$$

Therefore, by the *Monotone convergence theorem* (see Appendix B), we have

$$\mathbb{E}\left(\sum_{n=1}^{\infty}(T_n-\mathbb{E}(T_n))^2\right)<\infty.$$

Indeed, the series converges almost surely, which implies that its terms converge to zero and $T_n-\mathbb{E}(T_n)\to 0$ almost surely. Consequently, $T_n\to(t-s)$ almost surely. □

Remark 1.5. As we will discuss in Chapter 2, the stochastic integration, it means, integrating with respect to Brownian motion is such a complex task. The *Riemann-Stieltjes integration* fails. The sample paths of Brownian motion are nowhere differentiable, then they are not of bounded variation, as Theorem 1.9 states. This is one of motivations to construct a theory of stochastic integration.

1.5 Markov Property of Brownian Motion

In this final section, our aim is to prove that Brownian motion satisfies the Markov property, which refers to the loss of memory of the stochastic processes.

First, let us give a definition for Markov processes.

Definition 1.12 (Markov process)**.** For any t and $s>0$, the process X is a *Markov process* if

$$P(X_{t+s}\le y|\mathcal{F}_t)=P(X_{t+s}\le y|X_t),$$

almost surely.

Theorem 1.10 (Markov property)**.** *Brownian motion satisfies the Markov property.*

Proof. In order to prove the result above, let us use the moment generating function, such that, we have

$$\begin{aligned}\mathbb{E}\left(e^{\sigma B(t+s)}\middle|\mathcal{F}_t\right)&=e^{\sigma B(t)}\mathbb{E}\left(e^{\sigma(B(t+s)-B(t))}\middle|\mathcal{F}_t\right)\\&=e^{\sigma B(t)}\mathbb{E}\left(e^{\sigma(B(t+s)-B(t))}\right).\end{aligned}$$

Note that $e^{\sigma(B(t+s)-B(t))}$ and $\mathcal{F}_t$ are independent. In addition, the increment $B(t+s)-B(t)$ has a Normal distribution with zero mean and variance s. Then, we get

$$\begin{aligned}\mathbb{E}\left(e^{\sigma B(t+s)}\middle|\mathcal{F}_t\right)&=e^{\sigma B(t)}\mathbb{E}\left(e^{\sigma(B(t+s)-B(t))}\right)\\&=e^{\sigma B(t)}e^{\sigma^2 s/2}\\&=e^{\sigma B(t)}\mathbb{E}\left(e^{\sigma(B(t+s)-B(t))}\middle|B(t)\right)\\&=\mathbb{E}\left(e^{\sigma B(t+s)}\middle|B(t)\right).\end{aligned}$$

□

The *Strong Markov property* follows the same argument as the Markov Property, except that the fixed time t is replaced by a stopping time τ. We do not prove this result as it is highly extensive. The proof can be found in many of the given references as in [11], as well as some applications of these two properties.

CHAPTER 2

THEORY OF STOCHASTIC INTEGRATION

The aim of this chapter is to study the Itô stochastic integral, which was defined by K. Itô in [14] in 1944. This setting was the very first stochastic integral proposed and it plays a fundamental role in the theory of stochastic processes as we will see among this chapter and the following ones.

The theory of stochastic integration of Itô was originally motivated by the need of constructing diffusion processes as solutions to stochastic differential equations. As we have shown in Chapter 1, the main problem of the integration with respect to Brownian motion, is the fact that the Riemann-Stieltjes integration fails. This need inspired Itô to construct a theory of stochastic integration.

This chapter is organized as follows. First, we construct the Itô stochastic integral. We show that it can be also understood in terms of Riemann sums, by evaluating the integrand at the left endpoints of the intervals of the partition, and we calculate some examples. Then, we study some of its most remarkable properties as the zero mean property, the martingale property and the Itô isometry. Next, we present the Itô formula and calculate some examples in order to show how it works. Finally, we introduce the Girsanov theorem and explain its importance in the stochastic processes theory.

2.1 Definition of the Itô Stochastic Integral

Let $B(t)$ be a Brownian motion $\{B(t), t \geq 0\}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\{\mathcal{F}_t, t \geq 0\}$ be the associated filtration, i.e., $\mathcal{F}_t = \sigma\{B(s), t \geq s \geq 0\}$, such that

(i) For each $t \geq 0$, $B(t)$ is $\mathcal{F}_t$-measurable;

(ii) For any $0 \leq s \leq t$, $B(t) - B(s)$ is independent of $\mathcal{F}_s$.

2.1.1 Construction of the Itô Stochastic Integral

In this subsection, our aim is to construct the Itô stochastic integral. We follow K. Itô original ideas from [14] in order to define the integral

$$\int_a^b f(t)dB(t),$$

for $f \in L^2_{ad}([a,b]\times\Omega)$.

Definition 2.1. Let $L^2_{ad}([a,b]\times\Omega)$ be the set of stochastic processes X, such that

(i) $(t,\omega) \to X(t,\omega)$ is $\mathcal{B}([a,b])\times\mathcal{F}$-measurable;

(ii) X is adapted to the filtration $\{\mathcal{F}_t, t\in[a,b]\}$;

(iii) $\int_a^b \mathbb{E}\left(X_t^2\right)dt < \infty$.

The argument is, in turn, divided into three parts. First, we construct the stochastic integral for step processes in $L^2_{ad}([a,b]\times\Omega)$. Next, we examine a result, which is key for the next step. Finally, we construct the stochastic integral for general stochastic processes in $L^2_{ad}([a,b]\times\Omega)$.

Step 1. ***f is a step process in*** $\boldsymbol{L^2_{ad}([a,b]\times\Omega)}$**.**

Let us assume that f is a *step stochastic process*

$$f(t,\omega) = \sum_{i=1}^{n} \xi_{i-1}(\omega)\mathbb{1}_{[t_{i-1},t_i)}(t),$$

such that ξ_{i-1} is $\mathcal{F}_{t_{i-1}}$-measurable and $\mathbb{E}(\xi_{i-1}^2)<\infty$. Then, we construct

$$I(f) = \sum_{i=1}^{n} \xi_{i-1}(B(t_i)-B(t_{i-1})). \tag{2.1}$$

Remark 2.1. Note that, for any a, $b\in\mathbb{R}$ and any step stochastic processes f and g, I is clearly linear. Then, we have

$$I(af+bg) = aI(f)+bI(g).$$

Hence, let us propose the following result.

Lemma 2.1. *Let $I(f)$ be defined as in Equation (2.1). Then, we have*

$$\mathbb{E}(I(f)) = 0 \qquad and \qquad \mathbb{E}\left(|I(f)|^2\right) = \int_a^b \mathbb{E}\left(|f(t)|^2\right)dt.$$

Proof. For each $1\le i\le n$ in Equation (2.1), we have

$$\begin{aligned}
\mathbb{E}(\xi_{i-1}(B(t_i)-B(t_{i-1}))) &= \mathbb{E}\left(\mathbb{E}\left(\xi_{i-1}(B(t_i)-B(t_{i-1}))\,\middle|\mathcal{F}_{t_{i-1}}\right)\right)\\
&= \mathbb{E}\left(\xi_{i-1}\mathbb{E}\left(B(t_i)-B(t_{i-1})\middle|\mathcal{F}_{t_{i-1}}\right)\right)\\
&= \mathbb{E}(\xi_{i-1}\mathbb{E}(B(t_i)-B(t_{i-1})))\\
&= 0.
\end{aligned}$$

Hence, $\mathbb{E}(I(f)) = 0$, it means, $I(f)$ satisfies the zero mean property. Furthermore, let

$$\delta B_i = (B(t_i) - B(t_{i-1})) \qquad and \qquad \delta B_j = \left(B(t_j) - B(t_{j-1})\right).$$

Then, we have

$$|I(f)|^2 = \sum_{i,j=1}^{n} \xi_{i-1}\xi_{j-1}\delta B_i \delta B_j.$$

Firstly, suppose $i < j$ such that, we have

$$\begin{aligned}\mathbb{E}\left(\xi_{i-1}\xi_{j-1}\delta B_i \delta B_j\right) &= \mathbb{E}\left(\mathbb{E}\left(\xi_{i-1}\xi_{j-1}\delta B_i \delta B_j | \mathcal{F}_{t_{j-1}}\right)\right) \\ &= \mathbb{E}\left(\xi_{i-1}\xi_{j-1}\delta B_i \mathbb{E}\left(\delta B_j | \mathcal{F}_{t_{j-1}}\right)\right) \\ &= 0,\end{aligned}$$

as $\mathbb{E}\left(\delta B_j | \mathcal{F}_{t_{j-1}}\right) = \mathbb{E}\left(\delta B_j\right) = 0$. Next, suppose $i = j$ such that, we have

$$\begin{aligned}\mathbb{E}\left(\xi_{i-1}^2 \delta B_i^2\right) &= \mathbb{E}\left(\mathbb{E}\left(\xi_{i-1}^2 \delta B_i^2 | \mathcal{F}_{t_{i-1}}\right)\right) \\ &= \mathbb{E}\left(\xi_{i-1}^2 \mathbb{E}\left(\delta B_i^2\right)\right) \\ &= \mathbb{E}\left(\xi_{i-1}^2 (t_i - t_{i-1})\right) \\ &= (t_i - t_{i-1})\mathbb{E}\left(\xi_{i-1}^2\right).\end{aligned}$$

Hence, the isometry from Lemma 2.1 holds. □

Step 2. ***An approximation lemma.***

Next, we propose the following lemma, which is key in the construction of the stochastic integral from Equation (2.1) for general stochastic processes in $L^2_{ad}([a,b] \times \Omega)$.

Lemma 2.2. *Let $f \in L^2_{ad}([a,b] \times \Omega)$. There exists a sequence $\{f_n(t), n \geq 1\}$ of the step stochastic processes in $L^2_{ad}([a,b] \times \Omega)$ satisfying*

$$\lim_{n \to \infty} \int_a^b \mathbb{E}\left(|f(t) - f_n(t)|^2\right) dt = 0. \tag{2.2}$$

We do not prove it as it is highly extensive. The proof can be found in [22].

Step 3. ***Stochastic integral $\int_a^b f(t)dB(t)$ for $f \in L^2_{ad}([a,b] \times \Omega)$.***

Finally, by the results obtained in **Step 1** and **Step 2**, we construct, for $f \in L^2_{ad}([a,b] \times \Omega)$, the stochastic integral

$$\int_a^b f(t)dB(t).$$

Let $\{f_n(t,\omega), n \geq 1\}$ be the sequence of adapted step stochastic processes, such that expression from Equation (2.2) holds. Then, $I(f_n)$ is defined by **Step 1**. Hence, by Lemma 2.1, we get

$$\mathbb{E}\left(\left|I(f_n) - I(f_m)\right|^2\right) = \int_a^b \mathbb{E}\left(\left|f_n(t) - f_m(t)\right|^2\right) dt \to 0,$$

as $n, m \to \infty$. Hence, the sequence $\{I(f_n)\}$ is a Cauchy sequence in $L^2(\Omega)$.

Definition 2.2 (Itô stochastic integral)**.** The limit $I(f)$ defined by

$$I(f) = \lim_{n\to\infty} I(f_n), \quad \text{in } L^2(\Omega), \tag{2.3}$$

is called the *Itô stochastic integral*, and it is denoted by

$$I(f) = \int_a^b f(t)dB(t). \tag{2.4}$$

Remark 2.2. Note that, the Itô integral $I(f)$ is defined for $f \in L^2_{ad}([a,b]\times\Omega)$. For any $a,\ b \in \mathbb{R}$ and any $f,\ g \in L^2_{ad}([a,b]\times\Omega)$, I is clearly linear. Then, we have

$$I(af + bg) = aI(f) + bI(g).$$

2.1.2 The Itô Stochastic Integral as Riemann Sums

As we have mentioned at the beginning of this chapter, the Itô integration can be understood in terms of *Riemann sums* by evaluating the integrand at the left endpoints of the intervals of the partition. The aim of this subsection is to prove this statement in the following result.

Theorem 2.1. *Let $f \in L^2_{ad}([a,b]\times\Omega)$ and let $\mathbb{E}(f(t)f(s))$ be a continuous function of t and s. Then, we have*

$$\int_a^b f(t)dB(t) = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} f(t_{i-1})(B(t_i) - B(t_{i-1})), \quad \textit{in } L^2(\Omega), \tag{2.5}$$

where $\Delta = \{a = t_0 < t_1 < \dots < t_{n-1} < t_n = b\}$ is a partition of $[a,b]$ and $\|\Delta_n\| = \max_{1\le i\le n}(t_i - t_{i-1})$.

Proof. Let $f \in L^2_{ad}([a,b]\times\Omega)$ and let $\Delta_n = \{t_0, t_1, \dots, t_n\}$ be a partition of the interval $[a,b]$. The Riemann sum of f with respect to $B(t)$ is given by

$$\sum_{i=1}^{n} f(t_{i-1})(B(t_i) - B(t_{i-1})). \tag{2.6}$$

Our aim is to prove that this sequence of Riemann sums converges to the Itô stochastic integral from Equation (2.4). Let $\mathbb{E}(f(t)f(s))$ be a continuous function of t and s. Consider the stochastic process f_n of the form

$$f_n(t,\omega) = f(t_{i-1},\omega), \quad t_{i-1} < t \le t_i.$$

Then, we have

$$\lim_{n\to\infty} \int_a^b \mathbb{E}\left(|f(t) - f_n(t)|^2\right) dt = 0.$$

Hence, by Equation (2.3), we get

$$\int_a^b f(t)dB(t) = \lim_{n\to\infty} I(f_n), \quad \text{in } L^2(\Omega).$$

In addition, by Equation (2.1), we have

$$\begin{aligned} I(f_n) &= \sum_{i=1}^{n} f_n(t_{i-1})\big(B(t_i) - B_{t_{i-1}}\big) \\ &= \sum_{i=1}^{n} f(t_{i-1})\big(B(t_i) - B_{t_{i-1}}\big), \end{aligned}$$

which is exactly the Riemann sum in Equation (2.6). Consequently, the statement from Theorem 2.1 is proved. □

Next, we calculate some stochastic processes in order to show that the Itô stochastic integral allows us to compute some stochastic integrals. In Section 2.3, we will check that the results obtained by the definition coincide with the ones calculated by the Itô formula.

Example 2.1. Consider the stochastic process

$$\int_0^t B(t)dB(t).$$

By definition, we have

$$\int_0^t B(t)dB(t) = \lim_{||\Delta_n||\to 0} \sum_{i=1}^{n} B(t_{i-1})(B(t_i) - B(t_{i-1})).$$

We need to take into account that the quadratic variation of Brownian motion on the interval $[0,t]$ is equal to t (see Theorem 1.7). In addition, we have to consider

$$a(b-a) = \frac{1}{2}(b^2 - a^2 - (b-a)^2),$$

and

$$\sum_{i=1}^{n} \big(B(t_i)^2 - B(t_{i-1})^2\big) = B(t_n)^2 - B(t_0)^2.$$

Hence, we get

$$\begin{aligned} \boxed{\int_0^t B(t)dB(t)} &= \lim_{||\Delta_n||\to 0} \sum_{i=1}^{n} B(t_{i-1})(B(t_i) - B(t_{i-1})) \\ &= \frac{1}{2} \lim_{||\Delta_n||\to 0} \sum_{i=1}^{n} \big(B(t_i)^2 - B(t_{i-1})^2 - (B(t_{i-1}) - B(t_i))^2\big) \\ &= \boxed{\frac{1}{2}\big(B(t)^2 - t\big).} \end{aligned}$$

Example 2.2. Consider the stochastic process

$$\int_0^t B(t)^2 dB(t).$$

By definition, we have

$$\int_0^t B(t)^2 dB(t) = \lim_{||\Delta_n||\to 0} \sum_{i=1}^{n} B(t_{i-1})^2 (B(t_i) - B(t_{i-1})).$$

We have to consider

$$a^2(b-a) = \frac{1}{3}(b^3-a^3) - a(b-a)^2 - \frac{1}{3}(b-a)^3.$$

Hence, we get

$$\begin{aligned}
\boxed{\int_0^t B(t)^2 dB(t)} &= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^n B(t_{i-1})^2 (B(t_i)-B(t_{i-1})) \\
&= \frac{1}{3}\sum_{i=1}^n \left(B(t_i)^3 - B(t_{i-1})^3\right) - \sum_{i=1}^n B(t_{i-1})(B(t_i)-B(t_{i-1}))^2 - \frac{1}{3}\sum_{i=1}^n (B(t_i)-B(t_{i-1}))^3 \\
&= \frac{1}{3}B(t)^3 - \sum_{i=1}^n B(t_{i-1})(t_i - t_{i-1}) - \sum_{i=1}^n B(t_{i-1})\left((B(t_i)-B(t_{i-1}))^2 - (t_i - t_{i-1})\right) \\
&\quad - \frac{1}{3}\sum_{i=1}^n (B(t_i)-B(t_{i-1}))^3 \\
&= \boxed{\frac{1}{3}B(t)^3 - \int_0^t B(t)dt.}
\end{aligned}$$

Example 2.3. Consider the stochastic process

$$\int_0^t t dB(t).$$

By definition, we have

$$\int_0^t t dB(t) = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^n t_{i-1}(B(t_i)-B(t_{i-1})).$$

We have to consider

$$c(b-a) = db - ca - b(d-c).$$

Hence, we get

$$\begin{aligned}
\boxed{\int_0^t t dB(t)} &= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^n t_{i-1}(B(t_i)-B(t_{i-1})) \\
&= \sum_{i=1}^n (t_i B(t_i) - t_{i-1}B(t_{i-1})) - \sum_{i=1}^n B(t_i)(t_i - t_{i-1}) \\
&= tB(t) - \sum_{i=1}^n B(t_i)(t_i - t_{i-1}) \\
&= \boxed{tB(t) - \int_0^t B(t)dt.}
\end{aligned}$$

Example 2.4. Consider the stochastic process

$$\int_0^t B(T)dB(t), \qquad 0 \le t \le T.$$

Note that, $B(T)$ is not adapted to the filtration. Hence, the Itô integral does not exist. This example shows that, the assumption of the integrand to be adapted to the filtration is fundamental in the existence of the Itô integral. In the following chapters, we will find a solution to *anticipating* stochastic integrals, it means non-adapted.

2.2 Properties of the Itô Stochastic Integral

In this section, our aim is to prove that the defined Itô stochastic integral satisfies three properties that are considered fundamental in stochastic analysis. These properties are the *Zero Mean property*, the *Martingale property* and the *Itô isometry*. Some of these have already been proved for the step stochastic processes in the previous section, but now we consider the Itô stochastic integral for general stochastic processes in $L^2_{ad}([a,b]\times\Omega)$.

2.2.1 Zero Mean Property

In the next theorem we prove that the Itô stochastic integral satisfies the zero mean property.

Theorem 2.2 (Zero mean property). *Let $f \in L^2_{ad}([a,b]\times\Omega)$. The expectation of the Itô integral of $f(t)$ is zero, it means, the following equality holds*

$$\mathbb{E}\left(\int_a^b f(t)dB(t)\right) = 0.$$

Proof. By the definition of the Itô integral, we have

$$\begin{aligned}\mathbb{E}\left(\int_a^b f(t)dB(t)\right) &= \mathbb{E}\left(\lim_{||\Delta_n||\to 0}\sum_{i=1}^n f(t_{i-1})(B(t_i)-B(t_{i-1}))\right)\\ &= \lim_{||\Delta_n||\to 0}\sum_{i=1}^n \mathbb{E}(f(t_{i-1})(B(t_i)-B(t_{i-1}))).\end{aligned}$$

In order to prove that the expectation of the Itô integral is zero, we have to show that, for all $i \in \{1,2,...,n\}$ and all $n \in \mathbb{B}$, $\mathbb{E}(f(t_{i-1})(B(t_i)-B(t_{i-1}))) = 0$. Then, by the conditional expectation properties (see Appendix B), we have

$$\begin{aligned}\mathbb{E}(f(t_{i-1})(B(t_i)-B(t_{i-1}))) &= \mathbb{E}\left(\mathbb{E}\left(f(t_{i-1})(B(t_i)-B(t_{i-1}))\,\middle|\,\mathcal{F}_{t_{i-1}}\right)\right)\\ &= \mathbb{E}\left(f(t_{i-1})\mathbb{E}\left(B(t_i)-B(t_{i-1})\middle|\mathcal{F}_{t_{i-1}}\right)\right)\\ &= \mathbb{E}(f(t_{i-1})\mathbb{E}(B(t_i)-B(t_{i-1})))\\ &= \mathbb{E}(f(t_{i-1}))\mathbb{E}(B(t_i)-B(t_{i-1}))\\ &= 0.\end{aligned}$$

Hence, the Itô stochastic integral satisfies the zero mean property. □

2.2.2 Martingale Property

Next, we present a theorem that states that the Itô stochastic integral is a martingale. Furthermore, this allows us to prove the continuity of the stochastic process, it means, that almost all sample paths of the stochastic integral are continuous functions on the interval.

Theorem 2.3 (Martingale property)**.** *Let $f \in L^2_{ad}([a,b]\times\Omega)$. The stochastic process*

$$X_t = \int_a^t f(s)dB(s), \quad a \le t \le b, \tag{2.7}$$

is a martingale with respect to the filtration $\{\mathcal{F}_t, a \le t \le b\}$.

Proof. Let us consider the case that f is a step stochastic process. Our aim is to show that, for any $a \le s \le t \le b$

$$\mathbb{E}(X_t|\mathcal{F}_s) = X_s,$$

almost surely. However, we have

$$X_t = X_s + \int_s^t f(u)dB(u).$$

Hence, our aim is to prove

$$\mathbb{E}\left(\int_s^t f(u)dB(u)\middle|\mathcal{F}_s\right) = 0, \tag{2.8}$$

almost surely. Let us assume that f is given by

$$f(u,\omega) = \sum_{i=1}^n \xi_{i-1}(\omega)\mathbb{1}_{[t_{i-1},t_i)}(u),$$

where $s = t_0 < t_1 < \ldots < t_n = t$, ξ_{i-1} is $\mathcal{F}_{t_{i-1}}$-measurable and $\xi_{i-1} \in L^2(\Omega)$. Then, we get

$$\int_s^t f(u)dB(u) = \sum_{i=1}^n \xi_{i-1}(B(t_i) - B(t_{i-1})).$$

Note that $\mathbb{E}\left(B(t_i) - B(t_{i-1})|\mathcal{F}_{t_{i-1}}\right) = 0$. Then, for any $i = 1,2,\ldots,n$, we have

$$\begin{aligned}\mathbb{E}(\xi_{i-1}(B(t_i)-B(t_{i-1}))|\mathcal{F}_s) &= \mathbb{E}\left(\mathbb{E}\left(\xi_{i-1}(B(t_i)-B(t_{i-1}))\middle|\mathcal{F}_{t_{i-1}}\right)\middle|\mathcal{F}_s\right)\\ &= \mathbb{E}\left(\xi_{i-1}\mathbb{E}\left(B(t_i)-B(t_{i-1})\middle|\mathcal{F}_{t_{i-1}}\right)\middle|\mathcal{F}_s\right)\\ &= 0.\end{aligned}$$

Therefore, Equation (2.8) holds. Next, let us consider the general case. Let $f \in L^2_{ad}([a,b]\times\Omega)$ and let $\{f_n\}$ be a sequence of step stochastic processes in $L^2_{ad}([a,b]\times\Omega)$. Hence, we have

$$\lim_{n\to\infty}\int_a^b \mathbb{E}\left(\left|f(u)-f_n(u)\right|^2\right)du = 0.$$

Consider the stochastic process

$$X_t^{(n)} = \int_a^t f_n(u)dB(u).$$

We have already proved that $X_t^{(n)}$ is a martingale. Then, for $s < t$, we have

$$X_t - X_s = \left(X_t - X_t^{(n)}\right) + \left(X_t^{(n)} - X_s^{(n)}\right) + \left(X_s^{(n)} - X_s\right).$$

Next, let us take the conditional expectation, in order to get

$$\mathbb{E}(X_t - X_s|\mathcal{F}_s) = \mathbb{E}\left(X_t - X_t^{(n)}\Big|\mathcal{F}_s\right) + \mathbb{E}\left(X_s^{(n)} - X_s\Big|\mathcal{F}_s\right), \tag{2.9}$$

such that

$$\begin{aligned}\mathbb{E}\left(\left|\mathbb{E}\left(X_t - X_t^{(n)}\Big|\mathcal{F}_s\right)\right|^2\right) &\leq \mathbb{E}\left(\mathbb{E}\left(|X_t - X_t^{(n)}|^2\Big|\mathcal{F}_s\right)\right)\\ &= \mathbb{E}\left(|X_t - X_t^{(n)}|^2\right).\end{aligned}$$

Note that $\mathbb{E}\left(|I(f)|^2\right) = \int_a^b \mathbb{E}\left(|f(t)|^2\right) dt$. Then, we get

$$\begin{aligned}\mathbb{E}\left(\left|\mathbb{E}\left(X_t - X_t^{(n)}\Big|\mathcal{F}_s\right)\right|^2\right) &\leq \int_a^t \mathbb{E}\left(\left|f(u) - f_n(u)\right|^2\right) du\\ &\leq \int_a^b \mathbb{E}\left(\left|f(u) - f_n(u)\right|^2\right) du \to 0,\end{aligned}$$

as $n \to \infty$. Then, $\mathbb{E}\left(X_t - X_t^{(n)}|\mathcal{F}_s\right)$ and $\mathbb{E}\left(X_s - X_s^{(n)}|\mathcal{F}_s\right)$ converge almost surely to 0. Hence, by Equation (2.9), we have

$$\mathbb{E}(X_t - X_s|\mathcal{F}_s) = 0,$$

almost surely. Hence, X_t is a martingale, it means, the Itô stochastic integral satisfies the martingale property. □

The following result allows us to state that the stochastic process X_t defined in Equation (2.7) satisfies the continuity property, such that almost all sample paths of X_t are continuous functions on the interval. We do not prove this result as it is highly extensive. The proof can be found in [22].

Theorem 2.4 (Continuity property). *Let $f \in L^2_{ad}([a,b] \times \Omega)$. The stochastic process*

$$X_t = \int_a^b f(s)dB(s), \quad a \leq t \leq b,$$

is continuous, or equivalently, almost all of its sample paths are continuous functions on the interval $[a,b]$.

2.2.3 Itô Isometry

In the following theorem, we compute the second moment of the Itô integral in order to find its variance. This statement is also known as the Itô isometry, because of its importance in the stochastic analysis.

Theorem 2.5 (Itô isometry). *Let $f \in L^2_{ad}([a,b] \times \Omega)$. The following isometry holds*

$$\mathbb{E}\left(\left(\int_a^b f(t)dB(t)\right)^2\right) = \mathbb{E}\left(\int_a^b f^2(t)dt\right). \tag{2.10}$$

Proof. By definition, we have

$$\mathbb{E}\left(\left(\int_a^b f(t)dB(t)\right)^2\right) = \sum_{i,j=1} \mathbb{E}\left(\xi_{i-1}\xi_{j-1}(B(t_i)-B(t_{i-1}))\left(B(t_j)-B(t_{j-1})\right)\right).$$

Moreover, let

$$\delta B_i = (B(t_i)-B(t_{i-1})) \qquad and \qquad \delta B_j = \left(B(t_j)-B(t_{j-1})\right).$$

Firstly, suppose $i < j$ such that, we have

$$\begin{aligned}\mathbb{E}\left(\xi_{i-1}\xi_{j-1}\delta B_i \delta B_j\right) &= \mathbb{E}\left(\mathbb{E}\left(\xi_{i-1}\xi_{j-1}\delta B_i\delta B_j \middle| \mathcal{F}_{t_{j-1}}\right)\right)\\ &= \mathbb{E}\left(\xi_{i-1}\xi_{j-1}\delta B_i \mathbb{E}\left(\delta B_j \middle| \mathcal{F}_{t_{j-1}}\right)\right)\\ &= \mathbb{E}\left(\xi_{i-1}\xi_{j-1}\delta B_i \mathbb{E}\delta B_j\right)\\ &= 0,\end{aligned}$$

where $\xi_{i-1}\xi_{j-1}$ and δB_i are $\mathcal{F}_{t_{j-1}}$-measurable. Hence, suppose $i = j$ such that, we get

$$\begin{aligned}\mathbb{E}\left(\left(\int_a^b f(t)dB(t)\right)^2\right) &= \sum_{i=1}^n \mathbb{E}\left(\xi_{i-1}^2 \delta B_i^2\right)\\ &= \sum_{i=1}^n \mathbb{E}\left(\mathbb{E}\left(\xi_{i-1}^2\delta B_i^2 \middle| \mathcal{F}_{t_{i-1}}\right)\right)\\ &= \sum_{i=1}^n \mathbb{E}\left(\xi_{i-1}^2 \mathbb{E}\left(\delta B_i^2 \middle| \mathcal{F}_{t_{i-1}}\right)\right)\\ &= \sum_{i=1}^n \mathbb{E}\left(\xi_{i-1}^2 \mathbb{E}\left(\delta B_i^2\right)\right)\\ &= \sum_{i=1}^n \mathbb{E}\left(\xi_{i-1}^2\right)(t_i - t_{i-1})\\ &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \mathbb{E}\left(\xi_{i-1}^2\right) dt\\ &= \int_a^b \mathbb{E}\left(\sum_{i=1}^n \xi_{i-1}^2 \mathbb{1}_{[t_{i-1},t_i)}(t)\right) dt\\ &= \int_a^b \mathbb{E}\left(f^2(t)\right) dt.\end{aligned}$$

Hence, the Itô stochastic integral satisfies the isometry from Equation (2.10). □

2.3 The Itô formula

As we have check in the examples calculated in Section 2.1, the evaluation of the stochastic integrals may be a quite complex exercise. This fact also happens in the *Newton-Leibniz calculus*, where the *Fundamental Theorem of Calculus* give us a method to evaluate definite integrals, which simplifies the calculus notoriously.

In the same form, the aim of this section is to give a method that simplifies computations for the stochastic integration. The *Itô formula* is such a tool. It is a fundamental method that generalizes the well-known chain rule in the classical calculus for the stochastic integrals. Let us recall that, for f and g differentiable functions, the *chain rule* establishes

$$\frac{\partial f}{\partial t}(g(t)) = f'(g(t))g'(t).$$

Remark 2.3. Note that, we can write the above equation as

$$f(g(b) - f(g(a)) = \int_a^b f'(g(t))g'(t)dt = \int_a^b f'(g(t))dg(t), \tag{2.11}$$

where the last integral is of the Riemann-Stieltjes form.

In order to introduce the Itô formula, first let us give a definition of Itô processes, which is restricted to those from the $L^2_{ad}([a,b]\times\Omega)$ class.

Definition 2.3 (Itô process)**.** The stochastic process X_t is an *Itô process* if

$$X_t = X_a + \int_a^t f(s)dB(s) + \int_a^t g(s)ds, \quad a \le t \le b. \tag{2.12}$$

In differential form,

$$dX_t = f(t)dB(t) + g(t)dt, \tag{2.13}$$

where X_a is an $\mathcal{F}_a$-measurable random variable and $f,\, g \in L^2_{ad}([a,b]\times\Omega)$.

The next theorem establishes the Itô formula for the Itô processes defined in Equation (2.12). The proof can be found in many of the references proposed, we suggest to read the one in [22].

Theorem 2.6 (Itô formula)**.** *Let X_t be an Itô process given by Equation (2.12) and let $\theta(t,x)$ be a continuous function with continuous partial derivatives $\frac{\partial\theta}{\partial t}$, $\frac{\partial\theta}{\partial x}$ and $\frac{\partial^2\theta}{\partial x^2}$. Then, we have*

$$\begin{aligned} \theta(t,X_t) = \theta(a,X_a) &+ \int_a^t \frac{\partial\theta}{\partial x}(s,X_s)f(s)dB(s) \\ &+ \int_a^t \left(\frac{\partial\theta}{\partial t}(s,X_s) + \frac{\partial\theta}{\partial x}g(s) + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(s,X_s)f(s)^2\right)ds. \end{aligned} \tag{2.14}$$

In differential form,

$$d\theta(t,X_t) = \frac{\partial\theta}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(t,X_t)(dX_t)^2 + \frac{\partial\theta}{\partial t}(t,X_t)dt. \tag{2.15}$$

In order to compute some examples with the Itô formula, it is often helpful to use the so-called *Itô table* that we present below.

	$dB(t)$	dt
$dB(t)$	dt	0
dt	0	0

Table 2.1: Itô table

Remark 2.4. Note that, in the Itô table the term $(dX_t)^2$ is equal to dt. Hence, by Equation (2.13), we have

$$\begin{aligned}(dX_t)^2 &= (f(t)dB(t)+g(t)dt)^2 \\ &= f^2(t)(dX_t)^2 + 2f(t)g(t)(dX_t)(dt) + g(t)(dt)^2 \\ &= f^2(t)dt.\end{aligned} \tag{2.16}$$

By combining Equation (2.14) and Equation (2.16), we get

$$\begin{aligned}d\theta(t,X_t) &= \frac{\partial\theta}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(t,X_t)(dX_t)^2 + \frac{\partial\theta}{\partial t}(t,X_t)dt \\ &= \frac{\partial\theta}{\partial x}(t,X_t)(f(t)dB(t)+g(t)dt) + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(t,X_t)\left(f^2(t)dt\right) + \frac{\partial\theta}{\partial t}(t,X_t)dt \\ &= \frac{\partial\theta}{\partial x}(t,X_t)f(t)dB(t) + \left(\frac{\partial\theta}{\partial x}(t,X_t)g(t) + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(t,X_t)f^2(t) + \frac{\partial\theta}{\partial t}(t,X_t)\right)dt.\end{aligned} \tag{2.17}$$

Taking into account the fact that $B(t)$ is nowhere differentiable (see Section 1.3), we have that Equation (2.17) is the same as Equation (2.14).

Next, we apply the Itô formula to the stochastic processes introduced in Section 2.1 and we check that the results obtained by the definition and this method coincide.

Example 2.5. Consider the stochastic process introduced in Example 2.1

$$X_t = \int_0^t B(t)dB(t).$$

According to Theorem 2.6, we consider the function $\theta(t,x)$, such that

$$\frac{\partial\theta}{\partial x}(t,x) = x.$$

Hence, we take $\theta(t,x) = x^2/2$, whose partial derivatives are

$$\frac{\partial\theta}{\partial t} = 0, \qquad \frac{\partial\theta}{\partial x} = x, \qquad \frac{\partial^2\theta}{\partial x^2} = 1.$$

Then, we get

$$\begin{aligned}d\left(B(t)^2/2\right) &= B(t)dB(t) + \frac{1}{2}(dB(t))^2 \\ &= B(t)dB(t) + \frac{1}{2}dt.\end{aligned}$$

Integrating on both sides of the equality from 0 to t, we have

$$\int_0^t d\left(B(t)^2/2\right) = \int_0^t B(t)dB(t) + \int_0^t \frac{1}{2}dt.$$

Hence, we get

$$\boxed{\int_0^t B(t)dB(t) = \frac{1}{2}(B(t) - t),}$$

which coincides with the result obtained in Example 2.1.

Example 2.6. Consider the stochastic process introduced in Example 2.2

$$X_t = \int_0^t B(t)^2 dB(t).$$

According to Theorem 2.6, we consider the function $\theta(t,x)$, such that

$$\frac{\partial\theta}{\partial x}(t,x) = x^2.$$

Hence, we take $\theta(t,x) = x^3/3$, whose partial derivatives are

$$\frac{\partial\theta}{\partial t} = 0, \qquad \frac{\partial\theta}{\partial x} = x^2, \qquad \frac{\partial^2\theta}{\partial x^2} = 2x.$$

Then, we get

$$\begin{aligned} d\left(B(t)^3/3\right) &= B(t)^2 dB(t) + \frac{1}{2}(2B(t))(dB(t))^2 \\ &= B(t)^2 dB(t) + B(t)dt. \end{aligned}$$

Integrating on both sides of the equality from 0 to t, we have

$$\int_0^t d\left(B(t)^3/3\right) = \int_0^t B(t)^2 dB(t) + \int_0^t B(t)dt.$$

Hence, we get

$$\boxed{\int_0^t B(t)^2 dB(t) = \frac{1}{3}B(t) - \int_0^t B(t)dt,}$$

which coincides with the result obtained in Example 2.2.

Example 2.7. Consider the stochastic process introduced in Example 2.3

$$X_t = \int_0^t t dB(t).$$

According to Theorem 2.6, we consider the function $\theta(t,x)$, such that

$$\frac{\partial\theta}{\partial x}(t,x) = t.$$

Hence, we take $\theta(t,x) = tx$, whose partial derivatives are

$$\frac{\partial\theta}{\partial t} = x, \qquad \frac{\partial\theta}{\partial x} = t, \qquad \frac{\partial^2\theta}{\partial x^2} = 0.$$

Then, we get

$$d(tB(t)) = B(t)dt + tdB(t).$$

Integrating on both sides of the equality from 0 to t, we have

$$\int_0^t d(tB(t)) = \int_0^t B(t)dt + \int_0^t tdB(t).$$

Hence, we get

$$\boxed{\int_0^t tdB(t) = tB(t) - \int_0^t B(t)dt,}$$

which coincides with the result obtained in Example 2.3.

2.4 The Girsanov Theorem

We end this chapter by discussing an important result called the *Girsanov theorem*, which plays a fundamental role in the theory of stochastic processes. This theorem states that if we change the drift coefficient of a given Itô process, then the law of the process will not change dramatically.

In order to state the Girsansov theorem, first let us give some elementary notions.

Definition 2.4. Let $\mathbb{P}$ and $\mathbb{Q}$ be two probabilities on $(\Omega, \mathcal{F})$. We state that this probabilities are equivalent if, for any $\mathcal{G} \subset \mathcal{F}$

$$\mathbb{P}(\mathcal{G}) > 0 \leftrightarrow \mathbb{Q}(\mathcal{G}) > 0, \quad \text{or equivalently,} \quad \mathbb{P}(\mathcal{G}) = 0 \leftrightarrow \mathbb{Q}(\mathcal{G}) = 0.$$

Remark 2.5. The *Radon-Nikodym theorem* states that this is equivalent to the existence of a random variable $M > 0$, which is $\mathcal{F}_t$-measurable and $\mathbb{E}_{\mathbb{P}}(M) = 1$, such that, for any random variable X, which is also $\mathcal{F}_t$-measurable, we have

$$\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{P}}(XM),$$

Then, M is defined as the density of $\mathbb{Q}$ respect to $\mathbb{P}$, or equivalently, as the *Radon-Nikodym derivative* of $\mathbb{Q}$ respect to $\mathbb{P}$

$$M = \frac{d\mathbb{Q}}{d\mathbb{P}}, \quad \text{such that} \quad \frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{1}{M}.$$

Then, let $B(t)$ be a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is adapted to the filtration $\mathcal{F}_t$ and fixed for any $\mu \in \mathbb{R}$. Consider the martingale

$$M_t = e^{-\mu B(t) - \mu^2 t/2},$$

where $M_t > 0$ and $\mathbb{E}(M_t) = \mathbb{E}(M_0) = 1$. Hence, M_T defines a new probability $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ given by

$$\mathbb{Q}(\mathcal{G}) = \mathbb{E}\left(\mathbb{1}_{\mathcal{G}} M_T\right), \quad \forall \mathcal{G} \in \mathcal{F}_T.$$

It leads immediately that M_t is the density of $\mathbb{Q}$ respect to $\mathbb{P}$ when both are restricted to the σ-algebra $\mathcal{F}_t$, as for any $\mathcal{G} \subset \mathcal{F}_t$, we have

$$\mathbb{Q}(\mathcal{G}) = \mathbb{E}\left(\mathbb{1}_{\mathcal{G}} M_T\right) = \mathbb{E}\left(\mathbb{E}\left(\mathbb{1}_{\mathcal{G}} M_T | \mathcal{F}_t\right)\right) = \mathbb{E}\left(\mathbb{1}_{\mathcal{G}} \mathbb{E}(M_t | \mathcal{F}_t)\right) = \mathbb{E}\left(\mathbb{1}_{\mathcal{G}} M_t\right). \tag{2.18}$$

The most simple version of the Girsanov theorem states that by changing the probability measure $\mathbb{P}$ to $\mathbb{Q}$, a Brownian motion with drift μ is converted into a Brownian motion without drift.

Theorem 2.7 (Girsanov theorem)**.** *The Brownian motion with drift μ on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$

$$\bar{B}(t) = B(t) + \mu t,$$

is a normalized Brownian motion in the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$.

This theorem can be prove by the Levy characterization. However, we do not prove this result as it is highly extensive. The proof can be found in [22].

An immediate generalization can be made by substituting the constant μ by a square integrable deterministic function $\mu(t)$ in $[0, T]$, such that

$$\bar{B}(t) = B(t) + \int_0^t \mu(s) ds.$$

Indeed, the final step would be to consider that $\{\mu_t\}$ is a stochastic process adapted to the filtration $\{\mathcal{F}_t\}$. We do not prove these results either as they are highly extensive. However, the proofs and details can be also found in [22].

As we said at the beginning, this result seems to be important in the stochastic process theory and in many of its applications. For example, in financial modeling, this result is used each time one needs to derive an asset or a rate dynamic under a new probability measure, as happens in the Black-Scholes-Merton model, in the Libor Market model or for the quanto adjustments.

CHAPTER 3

STOCHASTIC DIFFERENTIAL EQUATIONS

In Chapter 2 we study the Itô stochastic integration theory, whose original purpose was motivated by the need of a method to construct diffusion processes as solutions of stochastic differential equations. Hence, in this chapter we discuss the theory of stochastic differential equations, abbreviated as SDEs.

In stochastic integration theory, we understand by stochastic differential equation or SDE an expression like

$$X_t = X_a + \int_a^t \sigma(s, X_s) dB(s) + \int_a^t \mu(s, X_s) ds, \quad a \le t \le b,$$

where our aim is to find the solution X_t satisfying it. Moreover, this equation can be also interpreted in differential form as

$$dX_t = \sigma(t, X_t) dB(t) + \mu(t, X_t) dt.$$

This chapter is organized as follows. First, we give a definition for the notions of solution of a stochastic differential equation and calculate some examples. Next, we prove the existence and uniqueness of solutions of stochastic differential equations as in the classical theory of differential equations. Finally, we present a financial example, the Black-Scholes-Merton model, which is quite famous because of its importance in financial markets theory.

3.1 Definition of Stochastic Differential Equation

In this section, our aim is to give some notions related to the solutions of a stochastic differential equations. Then, we calculate several examples in order to show how it works.

Definition 3.1 (Stochastic differential equation)**.** Let $\sigma, \mu : [a,b] \times \mathbb{R} \to \mathbb{R}$ be two measurable functions. A *stochastic differential equation* is an expression of the form

$$\begin{cases} dX_t = \sigma(t, X_t)dB(t) + \mu(t, X_t)dt, \\ X_0 = X_a, \end{cases} \tag{3.1}$$

where X_a is a $\mathcal{F}_t$-measurable random variable. Moreover, it is interpreted as meaning the stochastic integral

$$X_t = X_a + \int_a^t \sigma(s, X_s)dB(s) + \int_a^t \mu(s, X_s)ds. \tag{3.2}$$

Next, we define the concept of strong solution. From now on, whenever we refer to solutions of stochastic differential equations, we mean in the strong sense.

Definition 3.2 (Strong solution)**.** A stochastic process $\{X_t, a \le t \le b\}$ is said to be a *strong solution* of SDE (3.1), if it is measurable, $\{\mathcal{F}_t\}$-adapted and satisfies the following conditions

(i) $\{\sigma(t, X_t), a \le t \le b\} \in L^2_{ad}([a,b] \times \Omega)$;

(ii) Almost all sample paths of the process $\{\mu(t, X_t), a \le t \le b\}$ belongs to $L^1[a,b]$;

(iii) For each $t \in [a,b]$, the SDE (3.1) holds almost surely.

Definition 3.3. The stochastic differential equation from SDE (3.1) has a *pathwise unique solution*, if given two strong solutions X_1 and X_2, they are indistinguishable, such that

$$\mathbb{P}(X_1(t) = X_2(t), \forall t \in [a,b]) = 1.$$

Next, let us calculate some well-known examples, as the *Langevin equation* and other classical SDEs in order to show how it works.

Example 3.1. Let us consider the *Langevin equation*

$$\begin{cases} dX_t = \alpha dB(t) - \beta X_t dt, \\ X_0 = x_0. \end{cases}$$

According to Theorem 2.6, it means, applying the Itô formula, we consider the function $\theta(t,x)$, such that

$$\frac{\partial \theta}{\partial x} = e^{\beta t}.$$

Hence, we take $\theta(t,x) = e^{\beta t}x$, whose partial derivatives are

$$\frac{\partial \theta}{\partial t} = \beta e^{\beta t}x, \qquad \frac{\partial \theta}{\partial x} = e^{\beta t}, \qquad \frac{\partial^2 \theta}{\partial x^2} = 0.$$

Then, we get

$$\begin{aligned} d\left(e^{\beta t}X_t\right) &= \beta e^{\beta t}X_t dt + e^{\beta t} dX_t \\ &= \beta e^{\beta t}X_t dt + e^{\beta t}\left(\alpha dB(t) - \beta X_t dt\right) \\ &= \cancel{\beta e^{\beta t}X_t dt} + \alpha e^{\beta t} dB(t) - \cancel{\beta e^{\beta t}X_t dt} \\ &= \alpha e^{\beta t} dB(t). \end{aligned}$$

Integrating on both sides of the equality from 0 to t, we have

$$e^{\beta t}X_t = x_0 + \alpha \int_0^t e^{\beta s} dB(s).$$

Hence, we get

$$\boxed{X_t = x_0 e^{\beta t} + \alpha \int_0^t e^{-\beta(t-s)} dB(s).}$$

Hence, the stochastic process X_t is measurable and $\{\mathcal{F}_t\}$-measurable. Therefore, we can conclude that the process X_t is a well-defined solution. In addition, the stochastic process X_t is called an *Ornstein-Uhlenbeck process*.

Example 3.2. Consider the stochastic differential equation

$$\begin{cases} dX_t = X_t^2 dB(t) + X_t^3 dt, \\ X_0 = 1. \end{cases}$$

According to Theorem 2.6, we consider the function $\theta(t,x)$, such that

$$\frac{\partial \theta}{\partial x} = -\frac{1}{x^2}.$$

Hence, we take $\theta(t,x) = 1/x$, whose partial derivatives are

$$\frac{\partial \theta}{\partial t} = 0, \qquad \frac{\partial \theta}{\partial x} = -\frac{1}{x^2}, \qquad \frac{\partial^2 \theta}{\partial x^2} = \frac{2}{x^3}.$$

Then, we get

$$\begin{aligned} d\left(\frac{1}{X_t}\right) &= -\frac{1}{X_t^2} dX_t + \frac{1}{2}\frac{2}{X_t^3}(X_t)^2 \\ &= -\frac{1}{X_t^2}\left(X_t^2 dB(t) + X_t^3 dt\right) + X_t dt \\ &= -dB(t) - \cancel{X_t dt} + \cancel{X_t dt} \\ &= -dB(t). \end{aligned}$$

Integrating on both sides of the equality from 0 to t, we have

$$\frac{1}{X_t} = 1 - B(t).$$

Hence, we get

$$\boxed{X_t = \frac{1}{1 - B(t)}.}$$

3.2 Existence and Uniqueness Theorem of the Solutions of the Stochastic Differential Equations

As in the classical theory of differential equations, we aim to have a unique strong solution of a stochastic differential equation. In order to prove the existence and uniqueness theorem, let us introduce some preliminaries which are required.
First, we need to impose conditions on the functions $\mu(t,x)$ and $\sigma(t,x)$ in order to ensure that the solution of the SDE from Equation (3.1) makes sense. We state the conditions in the next two definitions.

Definition 3.4 (Lipschitz condition)**.** A measurable function $g(t,x)$ on $[a,b]\times\mathbb{R}$ satisfies the *Lipschitz condition* in x if there exists a constant $K>0$, such that

$$|g(t,x)-g(t,y)|\le K|x-y|, \quad \forall a\le t\le b, x,y\in\mathbb{R}. \tag{3.3}$$

Definition 3.5 (Linear growth condition)**.** A measurable function $g(t,x)$ on $[a,b]\times\mathbb{R}$ satisfies the *linear growth condition* in x if there exists a constant $K>0$, such that

$$|g(t,x)|\le K(1+|x|), \quad \forall a\le t\le b, x\in\mathbb{R}. \tag{3.4}$$

Next, let us introduced two lemmas that are fundamental to prove the existence and uniqueness of SDEs solutions theorem. We do not prove them as they are highly extensive. Both proofs can be found in [22].

Lemma 3.1 (Bellman-Grownwall inequality)**.** *Let $\phi\in L^1[a,b]$ be, such that*

$$\phi(t)\le f(t)+\beta\int_a^t \Phi(s)ds, \tag{3.5}$$

where $f\in L^1[a,b]$ and $\beta>0$ is a constant. Then, we have

$$\phi(t)\le f(t)+\beta\int_a^t f(s)e^{\beta(t-s)}ds. \tag{3.6}$$

Lemma 3.2. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $L^1([a,b])$, such that*

$$f_{n+1}(t)\le\phi(t)+\beta\int_a^t f_n(s)ds, \quad \forall t\in[a,b], \tag{3.7}$$

where $\phi\in L^1([a,b])$ is non-negative and $\beta\ge 0$ is a constant. Then, $\forall n\ge 2$, the following expression holds

$$f_{n+1}(t)\le\phi(t)+\beta\int_a^t \phi(s)e^{\beta(t-s)}ds+\beta^n\int_a^t \frac{(t-s)^{n-1}}{(n-1)!}f_1(s)ds. \tag{3.8}$$

Now, we are ready to state and prove the theorem of existence and uniquenesss of solutions of the stochastic differential equations solutions.

Theorem 3.1 (Existence and Uniqueness of SDEs solutions)**.** *Let $\sigma, \mu : [a,b] \times \mathbb{R} \to \mathbb{R}$ be two measurable Lipschitz functions of linear growth in x. Suppose ξ is an $\mathcal{F}_a$-measurable random variable with $\mathbb{E}(\xi^2) < \infty$. Then, the stochastic differential equation*

$$\begin{cases} dX_t = \sigma(t, X_t) dB(t) + \mu(t, X_t) dt, \\ X_a = \xi, \end{cases} \tag{3.9}$$

has a unique continuous solution on X_t.

Proof. First, we prove the uniqueness of solutions of SDEs. Let X_t and $Y_t, \in [a,b]$, be two strong solutions of Equation (3.9). We denote by $Z_t = X_t - Y_t$ to a continuous stochastic process. Then, our aim is to prove

$$\mathbb{P}(Z_t = 0, \forall t \in [a,b]) = 1.$$

By definition, we have

$$Z_t = \int_a^t (\sigma(s, X_s) - \sigma(s, Y_s)) dB(s) + \int_a^b \big(\mu(s, X_s) - \mu(s, Y_s)\big) ds.$$

Using that $(a+b)^2 \leq 2(a^2 + b^2)$ and taking on both sides squares, we get

$$Z_t^2 \leq 2\left(\int_a^t (\sigma(s, X_s) - \sigma(s, Y_s)) dB(s)\right)^2 + 2\left(\int_a^b \big(\mu(s, X_s) - \mu(s, Y_s)\big) ds\right)^2.$$

Then, taking expectations on both sides, we have

$$\mathbb{E}\left(Z_t^2\right) \leq 2\mathbb{E}\left(\left(\int_a^t (\sigma(s, X_s) - \sigma(s, Y_s)) dB(s)\right)^2\right) + 2\mathbb{E}\left(\left(\int_a^b \big(\mu(s, X_s) - \mu(s, Y_s)\big) ds\right)^2\right). \tag{3.10}$$

Next, applying the Itô isometry, the Lipschitz condition and by the definition of strong solution to $\sigma(s, X_s) \in L^2_{ad}$, we get

$$\begin{aligned} \mathbb{E}\left(\left(\int_a^t (\sigma(s, X_s) - \sigma(s, Y_s)) dB(s)\right)^2\right) &= \int_a^t \mathbb{E}\left(|\sigma(s, X_s) - \sigma(s, Y_s)|^2\right) ds \\ &= K^2 \int_a^t \mathbb{E}\left(|X_s - Y_s|^2\right) ds \\ &= K^2 \int_a^b \mathbb{E}\left(Z_s^2\right) ds. \end{aligned} \tag{3.11}$$

Next, we use the Hölder inequality, the Lipschtiz condition and the Fubini theorem over the Lebesgue integral in Equation (3.10), such that

$$\begin{aligned} \mathbb{E}\left(\left(\int_a^t \big(\mu(s, X_s) - \mu(s, Y_s)\big) ds\right)^2\right) &\leq t\mathbb{E}\left(\int_a^t |\mu(t, X_s) - \mu(s, Y_s)|^2 ds\right) \\ &\leq bK^2 \int_a^t \mathbb{E}\left(Z_s^2\right) ds. \end{aligned} \tag{3.12}$$

Substituting Equation (3.11) and Equation (3.12) into Equation (3.10), we get

$$\mathbb{E}\left(Z_t^2\right) \leq 2K^2(1+b)\int_a^t \mathbb{E}\left(Z_s^2\right) ds. \tag{3.13}$$

Since $Z_s \in L^2_{ad}$, we can apply Lemma 3.1, such that

$$\mathbb{E}\left(Z_t^2\right) \leq 0.$$

Indeed, $\mathbb{E}\left(Z_t^2\right) = 0, \forall t \in [a,b]$. Then, this fact implies that, for each $t \in [a,b]$, $Z_t(\omega) = 0$, for almost all $\omega \in \Omega$. Next, we denote $\mathbb{Q}_b = \mathbb{Q} \cap [a,b] = \{q_n\}_n$. Hence, we have, for each $q_n \in \mathbb{Q}_b$, that there exists $\Omega_n \subset \Omega$, such that $\mathbb{P}(\Omega_n) = 1$ and, for all $\omega \in \Omega_n$, $Z_{q_n}(\omega) = 0$.
Next, let us consider $\Omega' = \bigcap_{n=1}^{\infty} \Omega_n$, which has probability one and $\forall \omega \in \Omega'$ and $\forall n \in \mathbb{N}$, $Z_{q_n}(\omega) = 0$. Hence, taking into account that $t \to Z(t,\omega)$ is continuous almost surely, we have that there exists $\Omega'' \subset \Omega$ such that, $\mathbb{P}\left(\Omega''\right) = 1$ and $\forall \omega \in \Omega_0$, $Z(\cdot,\omega)$ is a continuous function, which vanishes on $\mathbb{Q}_b$. In addition, since $\mathbb{Q}_b$ is dense in $[a,b]$, $Z(\cdot,\omega)$ vanishes on $[a,b]$ for almost all $\omega \in \Omega_0$. Hence, we have

$$1 = \mathbb{P}(\Omega_0) \leq \mathbb{P}\{\omega \in \Omega : Z_t(\omega) = 0, \forall t \in [a,b]\}) \leq 1.$$

Indeed, the uniqueness of the solution is proved. Next, we aim to prove the existence of the solution of the SDE (3.9).

Let $\{X_t^{(n)}\}_{n=1}^{\infty}$ a sequence of stochastic processes given by

$$X_t^{(n+1)} = \xi + \int_a^t \sigma\left(s, X_s^{(n)}\right) dB(s) + \int_a^t \mu\left(s, X_s^{(n)}\right) ds. \tag{3.14}$$

For $n = 1$ and $\xi = X_t^{(1)}$, we have

Step 1. For all integers $n \leq 1$, $\{X_t^{(n)}, t \in [a,b]\} \in L^2_{ad}$ and has continuous sample paths almost surely.

By induction hypothesis. For $n = 1$, we have that ξ is $\mathcal{F}_a$-measurable, it means, $\{\mathcal{F}_t\}$-adapted, and

$$\int_a^b \mathbb{E}\left(\xi^2\right) dt = b\mathbb{E}\left(\xi^2\right) < \infty.$$

Let us assume that $X_t^{(n)} \in L^2_{ad}$ and it has continuous sample paths. Then, we have

$$\begin{aligned} \mathbb{E}\left(\int_a^b \sigma\left(t, X_t^{(n)}\right)^2 dt\right) &\leq \mathbb{E}\left(\int_a^b 2K\left(1 + \left(X_t^{(n)}\right)^2\right) dt\right) \\ &= 2Kb + 2K\int_a^b \mathbb{E}\left(\left(X_t^{(n)}\right)^2\right) dt < \infty. \end{aligned} \tag{3.15}$$

Indeed, $\sigma\left(t, X_t^{(n)}\right)$ is $\mathcal{F}_t$-measurable, so $\sigma\left(t, X_t^{(n)}\right) \in L^2_{ad}$. Hence, the Itô integral in Equation (3.14) makes sense, it is $\mathcal{F}_t$-measurable and has continuous sample paths.

For the Lebesgue integral in Equation (3.14), where $X_s^{(n)}$ has continuous sample paths almost surely, we have

$$\int_a^t \left|\mu\left(s, X_s^{(n)}\right)\right| ds \leq \sqrt{2Kb}\left(\int_a^t \left(1+\left(X_s^{(n)}\right)^2\right) ds\right)^{\frac{1}{2}} < \infty, \tag{3.16}$$

almost surely. Hence, it is continuous almost surely and is $\{\mathcal{F}_t\}$-adapted. Indeed, the stochastic process $X_t^{(n+1)}$ is $\{\mathcal{F}_t\}$-adapted with continuous sample paths almost surely. Moreover, since $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, we get

$$|X_t^{(n+1)}|^2 \leq 3\left(\xi^2 + \left(\int_a^t \sigma\left(s, X_s^{(n)}\right) dB(s)\right)^2 + \left(\int_a^t \mu\left(s, X_s^{(n)}\right) ds\right)^2\right).$$

By the linear growth condition (see Definition 3.5) and taking into account that $X_t^{(n)} \in L^2_{ad}$, we have

$$\int_a^b \mathbb{E}\left(|X_t^{(n+1)}|^2\right) dt < \infty.$$

Hence, the stochastic process $X_t^{(n+1)}$ belongs to $L^2_{ad}([a,b]\times\Omega)$.

Step 2. The sequence $\{X_t^{(n)}, t \in [a,b]\}_{n=1}^{\infty}$ converges uniformly on t almost surely.

Let

$$Y_t^{(n+1)} = \int_a^t \sigma\left(s, X_s^{(n)}\right) dB(s) \qquad and \qquad Z_t^{(n+1)} = \int_a^t \mu\left(s, X_s^{(n)}\right) ds,$$

such that

$$X_t^{(n+1)} = \xi + Y_t^{(n+1)} + Z_t^{(n+1)}.$$

Next, applying the Itô isometry and the Lipschitz condition, we get

$$\begin{aligned}
\mathbb{E}\left(|Y_t^{(n+1)} - Y_t^{(n)}|^2\right) &= \mathbb{E}\left(\left(\int_a^t \left(\sigma\left(s, X_s^{(n)}\right) - \sigma\left(s, X_s^{(n-1)}\right)\right) dB(s)\right)^2\right) \\
&= \int_a^t \mathbb{E}\left|\sigma\left(s, X_s^{(n)}\right) - \sigma\left(s, X_s^{(n-1)}\right)\right|^2 ds \\
&\leq K^2 \int_a^t \mathbb{E}|X_s^{(n)} - X_s^{(n-1)}|^2 ds.
\end{aligned} \tag{3.17}$$

By the Hölder continuity inequality and the Lipschitz condition, we have

$$\begin{aligned}
|Z_t^{(n+1)} - Z_t^{(n)}|^2 &= \left(\int_a^t \left(\mu\left(s, X_s^{(n)}\right) - \mu\left(s, X_s^{(n-1)}\right)\right) ds\right)^2 \\
&\leq bK^2 \int_a^t |X_s^{(n)} - X_s^{(n-1)}|^2 ds.
\end{aligned} \tag{3.18}$$

By Equation (3.17) and Equation (3.18), we get

$$\begin{aligned}
\mathbb{E}\left(|X_t^{(n+1)} - X_t^{(n)}|^2\right) &= \mathbb{E}\left(|Y_t^{(n+1)} - Y_t^{(n)} + Z_t^{(n+1)} - Z_t^{(n)}|^2\right) \\
&\leq 2\mathbb{E}\left(|Y_t^{(n+1)} - Y_t^{(n)}|^2\right) + 2\mathbb{E}\left(|Z_t^{(n+1)} - Z_t^{(n)}|^2\right) \\
&\leq 2K^2(1+b)\int_a^t \mathbb{E}|X_s^{(n)} - X_s^{(n-1)}|^2 ds.
\end{aligned} \tag{3.19}$$

Next, we apply Lemma 3.2 to get

$$\mathbb{E}\left(|X_t^{(n+1)} - X_t^{(n)}|^2\right) \leq \left(2K^2(1+b)\right)^{n-1} \int_a^t \frac{(t-s)}{(n-2)!} \mathbb{E}|X_s^{(2)} - X_s^{(1)}|^2 ds. \tag{3.20}$$

Hence, we have

$$\begin{aligned}
\mathbb{E}|X_s^{(2)} - X_s^{(1)}|^2 &= \mathbb{E}\left(\left(\int_a^s \sigma(u,\xi)\,dB(u) + \int_a^s \mu(u,\xi)\,du\right)^2\right) \\
&\leq 2\int_a^s \mathbb{E}\left(\sigma(u,\xi)^2\right)du + 2b\int_a^s \mathbb{E}\left(\mu(u,\xi)^2\right)du \\
&\leq 2K^2\int_a^s \left(1+\mathbb{E}\left(\xi^2\right)\right)du + 2b\int_a^s \left(1+\mathbb{E}\left(\xi^2\right)\right)du \\
&\leq 2K^2(1+b)\left(1+\mathbb{E}\left(\xi^2\right)\right)s.
\end{aligned} \tag{3.21}$$

Substituting Equation (3.21) into Equation (3.20), we get

$$\begin{aligned}
\mathbb{E}\left(|X_t^{(n+1)} - X_t^{(n)}|^2\right) &\leq \left(2K^2(1+b)\right)^n \left(1+\mathbb{E}\left(\xi^2\right)\right)\int_a^t \frac{(t-s)}{(n-2)!} s\,ds \\
&\leq \left(1+\mathbb{E}\left(\xi^2\right)\right)\left(2K^2(1+b)\right)^n \left(-\frac{s(t-s)^{n-1}}{(n-1)!}\Bigg|_a^t + \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} ds\right) \\
&\leq \left(1+\mathbb{E}\left(\xi^2\right)\right)\left(2K^2(1+b)\right)^n \frac{t^n}{n!}.
\end{aligned} \tag{3.22}$$

On the other hand, since

$$|X_t^{(n+1)} - X_t^{(n)}| \leq |Y_t^{(n+1)} - Y_t^{(n)}| + |Z_t^{(n+1)} - Z_t^{(n)}|,$$

it holds

$$\sup_{a\leq t\leq b} |X_t^{(n+1)} - X_t^{(n)}| \leq \sup_{a\leq t\leq b} |Y_t^{(n+1)} - Y_t^{(n)}| + \sup_{a\leq t\leq b} |Z_t^{(n+1)} - Z_t^{(n)}|.$$

Then, we have

$$\left\{\sup_{a\leq t\leq b} |X_t^{(n+1)} - X_t^{(n)}| > \frac{1}{n^2}\right\} \subset \left\{\sup_{a\leq t\leq b} |Y_t^{(n+1)} - Y_t^{(n)}| > \frac{1}{2n^2}\right\} \cup \left\{\sup_{a\leq t\leq b} |Z_t^{(n+1)} - Z_t^{(n)}| > \frac{1}{2n^2}\right\}.$$

Next, we take probabilities on both sides to get

$$\begin{aligned}
\mathbb{P}\left\{\sup_{a\leq t\leq b} |X_t^{(n+1)} - X_t^{(n)}| > \frac{1}{n^2}\right\} &\leq \mathbb{P}\left\{\sup_{a\leq t\leq b} |Y_t^{(n+1)} - Y_t^{(n)}| > \frac{1}{2n^2}\right\} \\
&\quad + \mathbb{P}\left\{\sup_{a\leq t\leq b} |Z_t^{(n+1)} - Z_t^{(n)}| > \frac{1}{2n^2}\right\}.
\end{aligned} \tag{3.23}$$

Taking into account that $\left|Y_t^{(n+1)} - Y_t^{(n)}\right|$ is a submartingale, we use the Doob inequality and Equation (3.22), in order to obtain

$$
\begin{aligned}
\mathbb{P}\left\{\sup_{a\le t\le b}\left|Y_t^{(n+1)} - Y_t^{(n)}\right| > \frac{1}{2n^2}\right\} &\le 4n^4\mathbb{E}\left(\left|Y_b^{(n+1)} - Y_b^{(n)}\right|^2\right)\\
&\le 4n^4\mathbb{E}\left(\left(\int_a^b \left(\sigma\left(s, X_s^{(n)}\right) - \sigma\left(s, X_s^{(n-1)}\right)\right) dB(s)\right)^2\right)\\
&\le 4n^4K^2\int_a^b \mathbb{E}\left(\left|X_s^{(n)} - X_t^{(n-1)}\right|^2\right) ds \qquad (3.24)\\
&\le 4n^4K^2\left(1+\mathbb{E}\left(\xi^2\right)\right)\left(2K^2(1+b)\right)^{n-1}\int_a^b \frac{s^{n-1}}{(n-1)!} ds\\
&= 4n^4K^2\left(1+\mathbb{E}\left(\xi^2\right)\right)\left(2K^2(1+b)\right)^{n-1}\frac{b^n}{n!}.
\end{aligned}
$$

Then, we take the supremum on both sides in Equation (3.18) to get

$$
\sup_{a\le t\le b}\left|Z_t^{(n+1)} - Z_t^{(n)}\right|^2 \le bK^2\int_a^b \left|X_s^{(n)} - X_s^{(n-1)}\right|^2 ds. \qquad (3.25)
$$

We apply *Chebyshev inequality* (see Appendix C) in Equation (3.22) and in Equation (3.25). Thus, we have

$$
\begin{aligned}
\mathbb{P}\left\{\sup_{a\le t\le b}\left|Z_t^{(n+1)} - Z_t^{(n)}\right| > \frac{1}{2n^2}\right\} &\le 4n^4\mathbb{E}\left(\left(\left|Z_t^{(n+1)} - Z_t^{(n)}\right|\right)^2\right) \qquad (3.26)\\
&\le 4n^4K^2b\left(1+\mathbb{E}\left(\xi^2\right)\right)\left(2K^2(1+b)\right)^{n-1}\frac{b^n}{n!}.
\end{aligned}
$$

Next, we substitute Equation (3.24) and Equation (3.26) into Equation (3.23) to get

$$
\mathbb{P}\left\{\sup_{a\le t\le b}\left|X_t^{(n+1)} - X_t^{(n)}\right| \le 4\left(1+\mathbb{E}\left(\xi^2\right)\right)\frac{n^4b^n\left(2K^2(1+b)\right)^n}{n!}, \qquad (3.27)
$$

such that, if we sum on both sides of the inequality, we have

$$
\sum_{n=1}^{\infty}\mathbb{P}\left\{\sup_{a\le t\le b}\left|X_t^{(n+1)} - X_t^{(n)}\right| \le 4\left(1+\mathbb{E}\left(\xi^2\right)\right)\sum_{n=1}^{\infty}\frac{n^4b^n\left(2K^2(1+b)\right)^n}{n!} < \infty. \qquad (3.28)
$$

Indeed, by the *Borel-Cantelli lemma* (see Appendix C), for almost all $\omega \in \Omega$, there exists $n_0(\omega) \in \mathbb{N}$, such that $\forall n \in \mathbb{N}, \ge n_0$

$$
\sup_{a\le t\le b}\left|X_t^{(n+1)}(\omega) - X_t^{(n)}(\omega)\right| \le \frac{1}{n^2}. \qquad (3.29)
$$

Hence, we have

$$
X_t^{(n+1)} = \xi + \sum_{j=1}^{n-1}\left(X_t^{(j+1)} - X_t^{(j)}\right).
$$

Then, the series on the right-hand side converges uniformly on t for almost all $\omega \in \Omega$. Therefore, the limit exists and is uniform on $t \in [a, b]$

$$
\lim_{n\to\infty} X_t^{(n)} = X_t,
$$

almost surely.

Step 3. $\{X_t, t \in [a,b]\}$ is the solution of the SDE (3.9).

Let us recall that $X_t^{(n)}$ is adapted to the filtration and has continuous sample paths almost surely. In addition, by convergence, X_t is also adapted to the filtration and has continuous sample paths almost surely. Next, we prove that $X_t \in L^2_{ad}$, such that

$$\begin{aligned}
||X_t||_{L^2(\Omega)} &= \left|\left|\xi + \sum_{j=1}^{\infty}\left(X_t^{(j+1)} - X_t^{(j)}\right)\right|\right|_{L^2(\Omega)} \\
&= ||\xi||_{L^2(\Omega)} + \sum_{j=1}^{\infty}\left|\left|X_t^{(j+1)} - X_t^{(j)}\right|\right|_{L^2(\Omega)} \\
&= ||\xi||_{L^2(\Omega)} + \sum_{j=1}^{\infty}\sqrt{\left(1+\mathbb{E}\left(\xi^2\right)\right)}\frac{b^{\frac{n}{2}}\left(2K^2(1+b)\right)^{\frac{n}{2}}}{\sqrt{n!}} < \infty.
\end{aligned}$$

Thus, we have

$$\int_a^b \mathbb{E}\left(X_t^2\right)dt < \infty. \tag{3.30}$$

Next, we prove that X_t satisfies the conditions of Definition 3.2. By Equation (3.30), we have

$$\mathbb{E}\left(\int_a^b \sigma(t,X_t)^2\,dt\right) \le K^2\int_a^b \left(1+\mathbb{E}\left(X_t^2\right)\right)dt \le K^2\left(b + \int_a^b \mathbb{E}\left(X_t^2\right)dt\right) < \infty.$$

Then, $\sigma(t,X_t) \in L^2_{ad}$. Then, we have that $\mu(t,X_t) \in L^1([a,b])$ almost surely. Next, we take limits on the expression below to prove the condition (iii) from Definition 3.2, such that

$$X_t^{(n)} = \xi + \int_a^t \sigma\left(s, X_s^{(n-1)}\right)dB(s) + \int_a^t \mu\left(s, X_s^{(n-1)}\right)ds. \tag{3.31}$$

We know from **Step 2**, on $t \in [a,b]$

$$\lim_{n\to\infty} X_t^{(n)} = X_t, \tag{3.32}$$

converges uniformly almost surely. Then, as $n \to \infty$, we have

$$\begin{aligned}
\left|\int_a^t \left(\mu\left(s, X_s^{(n-1)}\right) - \mu(s,X_s)\right)ds\right| &\le K\int_a^t \left|X_s^{(n-1)} - X_s\right|ds \\
&\le Kb \sup_{a\le t\le b}\left|X_t^{(n)} - X_t\right| \to 0,
\end{aligned} \tag{3.33}$$

almost surely. Then, for any $\epsilon > 0$ and n large enough, we get

$$\begin{aligned}
\mathbb{P}\left(\left|\int_a^t \left(\sigma\left(s,X_s^{(n)}\right) - \sigma(s,X_s)\right)dB(s)\right| > \epsilon\right) &\le \mathbb{P}\left(\int_a^t \left|\sigma\left(s,X_s^{(n)}\right) - \sigma(s,X_s)\right|^2 ds > \epsilon^2\right) + \epsilon \\
&\le \mathbb{P}\left(K^2\int_a^t \left|X_s^{(n)} - X_s\right|^2 ds > \epsilon^2\right) + \epsilon \\
&\le \mathbb{P}\left(K^2 t \sup_{a\le t\le b}\left|X_s^{(n)} - X_s\right|^2 ds > \epsilon^2\right) + \epsilon \\
&\le 2\epsilon.
\end{aligned} \tag{3.34}$$

We can assume that convergence is almost surely. Then, if we let $n \to \infty$ in Equation (3.31) and by Equation (3.32), Equation (3.33) and Equation (3.34), we can conclude that, for any $t \in [a,b]$

$$X_t = \xi + \int_a^t \sigma(s, X_s) dB(s) + \int_a^t \mu(s, X_s) ds,$$

holds almost surely.

□

3.3 Black-Scholes-Merton Model

F. Black, M. Scholes and R. C. Merton in [6, 30] in 1973 faced the problem of pricing a European option on a non-dividend paying stock. This method is based on the theory of stochastic process, such that, it models stock price variations as an Itô process.
The *Black-Scholes-Merton model* is a continuous-time model, which aim is to describe the behaviour of stock price, with one risky asset (a stock price S_t at time t) and a riskless asset (a bond with price S_t^0 at time t). We suppose the behaviour of S_t^0 to be described by the following (ordinary) differential equation

$$\begin{cases} dS_0(t) = \rho S_0(t) dt, \\ S_0(0) = M_0, \end{cases} \tag{3.35}$$

where ρ is a non-negative constant and M_0 is the initial amount of money invested in the bond. Moreover, let us assume that the behaviour of the stock price is described by the following stochastic differential equation

$$\begin{cases} dS_1(t) = \sigma S_1(t) dB(t) + \mu S_1(t) dt, \\ S_1(0) = M_1, \end{cases} \tag{3.36}$$

where μ, σ are two positive constants, the appreciation rate (drift term) and the volatility of the stock, respectively. The initial amount of money invested in the risky asset is denoted by M_1 and $B(t)$ is a standard Brownian motion.
The *investor* or *trader* has an initial amount $M > 0$ and invests some of it in the bond and the rest of it in the stock in order to maximize the average payoff, such that

$$M = M_0 + M_1.$$

Remark 3.1. Note that, we assume that $\mu > \rho$, because of the *risk-return binomial*, it means, the stock price must have a larger return because we take risk when we invest in stock instead of in the bond. Indeed, in financial markets, the more risk we take, the more return we can get.

Next, let us prove the following theorem, which establishes the solution to the ODE (3.35) and the SDE (3.36).

Theorem 3.2. *Consider the stochastic differential equations from ODE (3.35) and SDE (3.36). Hence, for $t \in [0,T]$, the solutions are*

$$\begin{cases} S_0(t) = M_0 e^{\rho t}, \\ S_1(t) = M_1 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B(t)}. \end{cases} \tag{3.37}$$

Proof. The computation of $S_0(t)$ follows immediately. On the other hand, we use the Itô formula to solve the stochastic differential equation. Let

$$Z(t) = \log S_1(t).$$

Then, we have

$$\begin{aligned} dZ_t &= \frac{1}{S_1(t)} dS_1(t) - \frac{1}{2}\frac{1}{S_1(t)^2}(dS_1(t))^2 \\ &= \mu dt + \sigma dB(t) - \frac{1}{2}\sigma^2 dt. \end{aligned}$$

Integrating on both sides of the equality from 0 to T, we have

$$Z_t = Z_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B(t).$$

Hence, removing the change of variables, for $t \in [0,T]$, we get

$$S_1(t) = M_1 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B(t)}.$$

Finally, we have that both solutions exist and are unique because of Theorem 3.1. □

Remark 3.2. The total wealth at any time $t \in [0,T]$ is given by

$$S(t) = S_0(t) + S_1(t).$$

Remark 3.3. Note that, the process S_t is a solution of the SDE if and only if the process $\log(S_t)$ is a Brownian motion. According to Definition 1.8, the process S_t satisfies the following properties

(i) The sample paths of the process S_t are continuous;

(ii) The increments are independent;

(iii) The increments are stationary.

Indeed, these properties describe the hypotheses of Black-Scholes-Merton on the behavior of the stock price.

CHAPTER 4

THE AYED-KUO STOCHASTIC INTEGRAL

In this chapter, we present the Ayed-Kuo anticipating stochastic integral, which was first studied by W. Ayed and H.-H Kuo in [1] in 2008. This new setting generalizes the Itô integral in the sense that it deals with the anticipating stochastic calculus.

In the Itô integration theory, the stochastic integral can be defined in terms of Riemann sums, which consists in evaluating the integrand at the left endpoints of the intervals of a partition. For the Ayed-Kuo integral, the integrand is assumed to be a product of an adapted stochastic process with respect to a Brownian filtration and an instantly independent stochastic process. Then, the adapted process is evaluated as in the Itô theory while the instantly independent process at the right endpoints.

This chapter is organized as follows. First we give a definition for the Ayed-Kuo integral and calculate some examples. We transpose some properties of the Itô integral for this new one, which have been proved in [2, 24, 27]. We study an Itô formula for the Ayed-Kuo integral, which has been proposed in [23, 28], and calculate some examples in order to show how it works. Finally, we study a general solution for stochastic differential equations with anticipating initial conditions, which has been discussed in [16, 21, 39], and a solution for the Black-Scholes-Merton model under Ayed-Kuo theory.

4.1 Definition of the Ayed-Kuo Stochastic Integral

Let $B(t)$ be a Brownian motion $\{B(t), t \geq 0\}$ and let $\{\mathcal{F}_t, t \geq 0\}$ be the associated filtration, i.e., $\mathcal{F}_t = \sigma\{B(s), t \geq s \geq 0\}$, such that

(i) For each $t \geq 0$, $B(t)$ is $\mathcal{F}_t$-measurable;

(ii) For any $0 \leq s \leq t$, $B(t) - B(s)$ is independent of $\mathcal{F}_s$.

Definition 4.1 (Instantly independent stochastic process)**.** A process $\{\varphi(t), a \leq t \leq b\}$, is said to be an *instantly independent stochastic process* with respect to $\{\mathcal{F}_t, a \leq t \leq b\}$, if and only if, $\varphi(t)$ and $\mathcal{F}_t$ are independent for each t.

Proposition 4.1. *If $\{\varphi(t), a \leq t \leq b\}$ is $\mathcal{F}_t$-adapted and an instantly independent process with respect to the filtration $\{\mathcal{F}_t, a \leq t \leq b\}$, then $\varphi(t)$ is a deterministic function.*

Proof. The process $\varphi(t)$ is $\mathcal{F}_t$-adapted. Hence, by the conditional expectation properties (see Appendix B), we get

$$\mathbb{E}\left(\varphi(t)|\mathcal{F}_t\right) = \varphi(t).$$

Moreover, $\varphi(t)$ is also an instantly independent process with respect to the filtration $\{\mathcal{F}_t, a \leq t \leq b\}$, such that

$$\mathbb{E}\left(\varphi(t)|\mathcal{F}_t\right) = \mathbb{E}\left(\varphi(t)\right).$$

Then, combining both statements, we have

$$\varphi(t) = \mathbb{E}\left(\varphi(t)\right).$$

Thus, $\varphi(t)$ is a deterministic function. □

In this sense, we can view instantly independent stochastic processes as a counterpart of the adapted stochastic processes for the Itô integral.

Definition 4.2 (Ayed-Kuo stochastic integral)**.** Let $\{f(t),\ a \leq t \leq b\}$ be a $\mathcal{F}_t$-adapted stochastic process and let $\{\varphi(t), a \leq t \leq b\}$ be an instantly independent stochastic process with respect to the filtration $\mathcal{F}_t$. The *Ayed-Kuo stochastic integral* of $f(t)\varphi(t)$ is defined by

$$I(f\varphi) = \int_a^b f(t)\varphi(t)dB(t) = \lim_{||\Delta_n||\to 0} \sum_{i=1}^{n} f(t_{i-1})\varphi(t_i)(B(t_i) - B(t_{i-1})), \tag{4.1}$$

provided that the limit in probability exists, where $\Delta = \{a = t_0, t_1, t_2, ..., t_n = b\}$ is a partition of the interval $[a, b]$ and $||\Delta_n|| = \max_{1\leq i\leq n}(t_i - t_{i-1})$.

As is explained in Chapter 2, the Itô integral can be defined in terms of Riemann sums by evaluating the integrand at the left endpoints of the intervals of the partition (see Theorem 2.1). The Definition 4.2 follows the same argument with the adapted process, while the instantly independent process is evaluated at the right endpoints of the intervals of the partition in order to take advantage of the independence property.

Remark 4.1. Note that if we consider $\varphi(t) = 1$ and $f(t)$ pathwise continuous. Then, by Theorem 2.1, the Ayed-Kuo integral coincides with the Itô integral.

Remark 4.2. By Equation (4.1), we can state that the Ayed-Kuo integral is linear. In addition, we see that many anticipating integrals can be written as a product of an adapted stochastic process and an instantly independent stochastic process.

Next, we calculate some stochastic processes in order to show that the Ayed-Kuo integral allows us to compute some anticipating integrals. In Section 4.3, we will check that the results obtained by the definition coincide with the ones calculated by the extension of the Itô formula.

Example 4.1. Consider the stochastic process

$$\int_0^t B(1)dB(s), \qquad 0 \le t \le 1.$$

By linearity, we have

$$\int_0^t B(1)dB(s) = \int_0^t (B(1)-B(s))dB(s) + \int_0^t B(s)dB(s), \qquad 0 \le t \le 1. \tag{4.2}$$

The second integral on the right-hand side of Equation (4.2) is an adapted stochastic process, while the first integral is an instantly independent stochastic process with respect to the filtration $\mathcal{F}_t$, it means anticipating. We calculate the first integral of the right-hand side of Equation (4.2) as follows

$$\begin{aligned}
\boxed{\int_0^t (B(1)-B(s))dB(s)} &= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^n (B(1)-B(s_i))(B(s_i)-B(s_{i-1})) \\
&= \lim_{\|\Delta_n\|\to 0} \Bigg(B(1)\sum_{i=1}^n (B(s_i)-B(s_{i-1})) \\
&\quad - \sum_{i=1}^n (B(s_i)-B(s_{i-1})+B(s_{i-1}))(B(s_i)-B(s_{i-1}))\Bigg) \\
&= \lim_{\|\Delta_n\|\to 0} \Bigg(B(1)B(t) - \Bigg(\sum_{i=1}^n (B(s_i)-B(s_{i-1}))^2 \\
&\quad + \sum_{i=1}^n B(s_{i-1})(B(s_i)-B(s_{i-1}))\Bigg)\Bigg) \\
&= \boxed{B(1)B(t) - t - \int_0^t B(s)dB(s), \qquad 0 \le t \le 1.}
\end{aligned} \tag{4.3}$$

Hence, substituting Equation (4.3) into Equation (4.2), we get

$$\begin{aligned}
\boxed{\int_0^t B(1)dB(s)} &= \int_0^t (B(1)-B(s))dB(s) + \int_0^t B(s)dB(s) \\
&= B(1)B(t) - t - \cancel{\int_0^t B(s)dB(s)} + \cancel{\int_0^t B(s)dB(s)} \\
&= \boxed{B(1)B(t) - t, \qquad 0 \le t \le 1.}
\end{aligned} \tag{4.4}$$

Note that, when $t > 1$ we can write the integral $\int_0^t$ as $\int_0^1 + \int_1^t$ to obtain the equality, such that

$$\begin{aligned}
\boxed{\int_0^t B(1)dB(s)} &= \int_0^1 B(1)dB(s) + \int_1^t B(1)dB(s) \\
&= B(1)^2 - 1 + B(1)\int_1^t dB(s) \\
&= \cancel{B(1)^2} - 1 + B(1)B(t) - \cancel{B(1)^2} \\
&= \boxed{B(1)B(t) - 1, \qquad t > 1.}
\end{aligned}$$

Example 4.2. Consider the stochastic process

$$\int_0^t B(1)B(s)dB(s), \qquad 0 \le t \le 1.$$

Note that the integrand can be decompose as follows

$$B(1)B(s) = (B(1) - B(s) + B(s))B(s) = (B(1) - B(s))B(s) + B(s)^2.$$

By linearity, we have

$$\int_0^t B(1)B(s)dB(s) = \int_0^t B(s)(B(1) - B(s))dB(s) + \int_0^t B(s)^2 dB(s), \qquad 0 \le t \le 1. \tag{4.5}$$

The first integral on the right-hand side of Equation (4.5) is an instantly independent stochastic process with respect to the filtration $\mathcal{F}_t$, while the second is an adapted stochastic process. Therefore, we calculate the first integral of the right-hand side of Equation (4.5) as follows

$$\begin{aligned}
\int_0^t B(s)(B(1) - B(s))dB(s) &= \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^n B(s_{i-1})(B(1) - B(s_i))(B(s_i) - B(s_{i-1})) \\
&= \lim_{\|\Delta_n\| \to 0} \Big(B(1) \sum_{i=1}^n B(s_{i-1})(B(s_i) - B(s_{i-1})) \\
&\quad - \sum_{i=1}^n B(s_{i-1})(B(s_i) - B(s_{i-1}) + B(s_{i-1}))(B(s_i) - B(s_{i-1})) \Big) \\
&= \lim_{\|\Delta_n\| \to 0} \Big(B(1) \sum_{i=1}^n B(s_{i-1})(B(s_i) - B(s_{i-1})) \\
&\quad - \sum_{i=1}^n B(s_{i-1})(B(s_i) - B(s_{i-1}))^2 - \sum_{i=1}^n B(s_{i-1})^2 (B(s_i) - B(s_{i-1})) \Big) \\
&= \lim_{\|\Delta_n\| \to 0} \Big(B(1) \int_0^t B(s)dB(s) - \int_0^t B(s)ds - \int_0^t B(s)^2 dB(s) \Big).
\end{aligned}$$

By classical Itô integration theory (see Section 2.1), we know

$$\int_0^t B(s)dB(s) = \frac{1}{2}\left(B(t)^2 - t\right) \qquad \text{and} \qquad \int_0^t B(s)^2 dB(s) = \frac{1}{3}B(t)^3 - \int_0^t B(s)ds.$$

Thus, we have

$$\begin{aligned}
\boxed{\int_0^t B(s)(B(1) - B(s))dB(s)} &= \frac{1}{2}B(1)\left(B(t)^2 - t\right) - \cancel{\int_0^t B(s)ds} - \left(\frac{1}{3}B(t)^3 - \cancel{\int_0^t B(s)ds}\right) \\
&= \boxed{\frac{1}{2}B(1)\left(B(t)^2 - t\right) - \frac{1}{3}B(t)^3, \qquad 0 \le t \le 1.}
\end{aligned} \tag{4.6}$$

Hence, substituting Equation (4.6) into Equation (4.5), we get

$$\begin{aligned}
\boxed{\int_0^t B(1)B(s)dB(s)} &= \int_0^t B(s)(B(1) - B(s))dB(s) + \int_0^t B(s)^2 dB(s) \\
&= \frac{1}{2}B(1)\left(B(t)^2 - t\right) - \cancel{\frac{1}{3}B(t)^3} + \left(\cancel{\frac{1}{3}B(t)^3} - \int_0^t B(s)ds\right) \\
&= \boxed{\frac{1}{2}B(1)\left(B(t)^2 - t\right) - \int_0^t B(s)ds, \qquad 0 \le t \le 1.}
\end{aligned}$$

Note that, when $t > 1$ we get

$$\begin{aligned}
\boxed{\int_0^t B(1)B(s)dB(s)} &= \int_0^1 B(1)B(s)dB(s) + \int_1^t B(1)B(s)dB(s) \\
&= \cancel{\frac{1}{2}B(1)\left(B(1)^2 - 1\right)} - \int_0^1 B(s)ds + \frac{1}{2}B(1)\left(B(t)^2 - t\right) - \cancel{\frac{1}{2}B(1)\left(B(1)^2 - 1\right)} \\
&= \boxed{\frac{1}{2}B(1)\left(B(t)^2 - t\right) - \int_0^1 B(s)ds, \qquad t > 1.}
\end{aligned}$$

Example 4.3. Consider the stochastic process

$$\int_0^t e^{B(1)} dB(s), \qquad 0 \le t \le 1.$$

whose integrand can be written in the form

$$e^{B(1)} = e^{B(s)} e^{B(1) - B(s)}.$$

Thus, we have

$$\int_0^t e^{B(1)} dB(s) = \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^n e^{B(1)} e^{-(B(s_i) - B(s_{i-1}))} (B(s_i) - B(s_{i-1})).$$

Note that we can write the integrand as a first-order Taylor expansion because $e^x \in \mathcal{C}^1$-function on $\mathbb{R}$. Hence, we get

$$\begin{aligned}
\boxed{\int_0^t e^{B(1)} dB(s)} &= \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^n e^{B(1)} e^{-(B(s_i) - B(s_{i-1}))} (B(s_i) - B(s_{i-1})) \\
&= \lim_{\|\Delta_n\| \to 0} e^{B(1)} \sum_{i=1}^n \Big(1 - (B(s_i) - B(s_{i-1})) + o((B(s_i) - B(s_{i-1})))\Big)(B(s_i) - B(s_{i-1})) \qquad (4.7) \\
&= \boxed{e^{B(1)}(B(t) - t), \qquad 0 \le t \le 1,}
\end{aligned}$$

in probability, since the first two summands converge in $L^2(\Omega)$-norm and the error converges to 0. Note that, when $t > 1$ we get

$$\begin{aligned}
\boxed{\int_0^t e^{B(1)} dB(s)} &= \int_0^1 e^{B(1)} dB(s) + \int_1^t e^{B(1)} dB(s) \\
&= e^{B(1)}(B(1) - 1) + e^{B(1)} \int_1^t dB(s) \\
&= \cancel{e^{B(1)}B(1)} - e^{B(1)} + e^{B(1)}B(t) - \cancel{e^{B(1)}B(1)} \\
&= \boxed{e^{B(1)}(B(t) - 1), \qquad t > 1.}
\end{aligned}$$

Example 4.4. Consider the stochastic process

$$\int_0^t B(1)B(\tfrac{1}{2})dB(s).$$

First of all, we consider the integral for $0 \le t \le \frac{1}{2}$. Note that the integrand can be decompose as follows

$$B(1)B(\tfrac{1}{2}) = ((B(1)-B(s))) + B(s))\left(\left(B(\tfrac{1}{2})-B(s)\right)+B(s)\right).$$

By linearity, we have

$$\begin{aligned}\int_0^t B(1)B(\tfrac{1}{2})dB(s) &= \int_0^t ((B(1)-B(s))+B(s))((B(\tfrac{1}{2})-B(s))+B(s))dB(s)\\ &= \int_0^t B(s)^2 dB(s) + \underbrace{\int_0^t (B(1)-B(s))(B(\tfrac{1}{2})-B(s))dB(s)}_{(*^1)}\\ &\quad + \underbrace{\int_0^t B(s)(B(\tfrac{1}{2})-B(s))dB(s)}_{(*^2)} + \underbrace{\int_0^t B(s)(B(1)-B(s))dB(s)}_{(*^3)}.\end{aligned} \tag{4.8}$$

Note that the first integral of the right-hand side of Equation (4.8) is an adapted processes, while the three other integrals are instantly independent processes with respect to the filtration. We start calculating the first one of the anticipating processes $(*^1)$. Let $\delta B_i = (B(s_i)-B(s_{i-1}))$. Hence, by analogy with the previous examples, we have

$$\begin{aligned}\int_0^t (B(1)-B(s))(B(\tfrac{1}{2})-B(s))dB(s) &= \lim_{\|\Delta_n\|\to 0}\sum_{i=1}^n (B(1)-B(s_i))(B(\tfrac{1}{2})-B(s_i))\delta B_i\\ &= \lim_{\|\Delta_n\|\to 0}\left(B(1)B(\tfrac{1}{2})\sum_{i=1}^n \delta B_i - B(\tfrac{1}{2})\sum_{i=1}^n B(s_i)\delta B_i\right.\\ &\left. - B(1)\sum_{i=1}^n B(s_i)\delta B_i + \sum_{i=1}^n B(s_i)^2 \delta B_i\right)\\ &= \lim_{\|\Delta_n\|\to 0}\left(B(1)B(\tfrac{1}{2})B(t) - B(\tfrac{1}{2})\sum_{i=1}^n (B(s_i)-B(s_{i-1})+B(s_{i-1}))\delta B_i\right.\\ & - B(1)\sum_{i=1}^n (B(s_i)-B(s_{i-1})+B(s_{i-1}))\delta B_i\\ &\left. + \sum_{i=1}^n (B(s_i)-B(s_{i-1})+B(s_{i-1}))^2 \delta B_i\right)\\ &= \lim_{\|\Delta_n\|\to 0}\left(B(1)B(\tfrac{1}{2})B(t) - B(\tfrac{1}{2})\sum_{i=1}^n \delta B_i^2\right.\\ & + B(\tfrac{1}{2})\sum_{i=1}^n B(s_{i-1})\delta B_i - B(1)\sum_{i=1}^n \delta B_i^2\\ & - B(1)\sum_{i=1}^n B(s_{i-1})\delta B_i + \sum_{i=1}^n \delta B_i^3\\ &\left. + \sum_{i=1}^n B(s_{i-1})^2 \delta B_i + 2\sum_{i=1}^n B(s_{i-1})\delta B_i^2\right)\\ &= B(1)B(\tfrac{1}{2})B(t) - (B(\tfrac{1}{2})+B(1))\frac{B_t^2+t}{2}\\ & + \int_0^t B(s)^2 dB(s) + 2\int_0^t B(s)ds, \qquad 0 \le t \le \frac{1}{2}.\end{aligned}$$

Next, we calculate the second anticipating integral ($*^2$) of Equation (4.8) as follows

$$
\begin{aligned}
\int_0^t B(s)(B(\tfrac{1}{2})-B(s))dB(s) &= \lim_{||\Delta_n||\to 0}\left(\sum_{i=1}^{n} B(s_{i-1})(B(\tfrac{1}{2})-B(s_i))\delta B_i\right)\\
&= \lim_{||\Delta_n||\to 0}\left(B(\tfrac{1}{2})\sum_{i=1}^{n} B(s_{i-1})\delta B_i - \sum_{i=1}^{n} B(s_i)B(s_{i-1})\delta B_i\right)\\
&= \lim_{||\Delta_n||\to 0}\left(B(\tfrac{1}{2})\frac{B_t^2-t}{2} - \sum_{i=1}^{n}(B(s_i)-B(s_{i-1})+B(s_{i-1}))B(s_{i-1})\delta B_i\right)\\
&= \lim_{||\Delta_n||\to 0}\left(B(\tfrac{1}{2})\frac{B_t^2-t}{2} - \sum_{i=1}^{n} B(s_{i-1})\delta B_i^2 - \sum_{i=1}^{n} B(s_{i-1})^2\delta B_i\right)\\
&= B(\tfrac{1}{2})\frac{B_t^2-t}{2} - \int_0^t B(s)^2 dB(s) - \int_0^t B(s)ds, \qquad 0\le t\le \frac{1}{2}.
\end{aligned}
$$

By analogy with the previous processes, we calculate the third anticipating integral ($*^3$) of Equation (4.8) as follows

$$
\begin{aligned}
\int_0^t B(s)(B(1)-B(s))dB(s) &= \lim_{||\Delta_n||\to 0}\left(\sum_{i=1}^{n} B(s_{i-1})(B(1)-B(s_i))\delta B_i\right)\\
&= \lim_{||\Delta_n||\to 0}\left(B(1)\sum_{i=1}^{n} B(s_{i-1})\delta B_i - \sum_{i=1}^{n} B(s_i)B(s_{i-1})\delta B_i\right)\\
&= \lim_{||\Delta_n||\to 0}\left(B(1)\frac{B_t^2-t}{2} - \sum_{i=1}^{n} B(s_i - B(s_{i-1})+B(s_{i-1}))B(s_{i-1})\delta B_i\right)\\
&= \lim_{||\Delta_n||\to 0}\left(B(1)\frac{B_t^2-t}{2} - \sum_{i=1}^{n} B(s_{i-1})\delta B_i^2 - \sum_{i=1}^{n} B(s_{i-1})^2\delta B_i\right)\\
&= B(1)\frac{B_t^2-t}{2} - \int_0^t B(s)^2 dB(s) - \int_0^t B(s)ds, \qquad 0\le t\le \frac{1}{2}.
\end{aligned}
$$

Hence, substituting into Equation (4.8), we get

$$
\begin{aligned}
\boxed{\int_0^t B(1)B(\tfrac{1}{2})dB(s)} &= \int_0^t B(s)^2 dB(s) + \left(B(\tfrac{1}{2})\frac{B_t^2-t}{2} - \int_0^t B(s)^2 dB(s) - \int_0^t B(s)ds\right)\\
&+ \left(B(1)B(\tfrac{1}{2})B(t) - (B(\tfrac{1}{2})+B(1))\frac{B_t^2+t}{2} + \int_0^t B(s)^2 dB(s) + 2\int_0^t B(s)ds\right)\\
&+ \left(B(1)\frac{B_t^2-t}{2} - \int_0^t B(s)^2 dB(s) - \int_0^t B(s)ds\right)\\
&= \boxed{B(1)B(\tfrac{1}{2})B(t) - t\big(B(\tfrac{1}{2})+B(1)\big), \qquad 0\le t\le \frac{1}{2}.}
\end{aligned}
$$

Note that, when $\frac{1}{2} < t \le 1$ we get

$$\begin{aligned}
\boxed{\int_0^t B(1)B(\tfrac{1}{2})dB(s)} &= \int_0^{\frac{1}{2}} B(1)B(\tfrac{1}{2})dB(s) + \int_{\frac{1}{2}}^t B(1)B(\tfrac{1}{2})dB(s) \\
&= \left(B(1)B(\tfrac{1}{2})^2 - \frac{1}{2}\left(B(\tfrac{1}{2}) + B(1)\right)\right) + \int_{\frac{1}{2}}^t (B(1) - B(s) + B(s))B(\tfrac{1}{2})dB(s) \\
&= \left(B(1)B(\tfrac{1}{2})^2 - \frac{1}{2}B(\tfrac{1}{2}) - \frac{1}{2}B(1)\right) + \int_{\frac{1}{2}}^t (B(1) - B(s))B(\tfrac{1}{2})dB(s) + \int_{\frac{1}{2}}^t B(s)B(\tfrac{1}{2})dB(s) \\
&= \left(B(1)B(\tfrac{1}{2})^2 - \frac{1}{2}B(\tfrac{1}{2}) - \frac{1}{2}B(1)\right) + B(\tfrac{1}{2}) \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^n (B(1) - B(s_i))(B(s_i) - B(s_{i-1})) \\
&+ B(\tfrac{1}{2}) \lim_{\|\Delta_n\| \to 0} \sum_{i=1}^n B(s_{i-1})(B(s_i) - B(s_{i-1})) \\
&= \left(B(1)B(\tfrac{1}{2})^2 - \frac{1}{2}B(\tfrac{1}{2}) - \frac{1}{2}B(1)\right) + B(\tfrac{1}{2})\left[B(1)B(t) - t\right]_{\frac{1}{2}}^t \\
&= \cancel{B(1)B(\tfrac{1}{2})^2} - \cancel{\frac{1}{2}B(\tfrac{1}{2})} - \frac{1}{2}B(1) + B(\tfrac{1}{2})B(1)B(t) - B(\tfrac{1}{2})t - \cancel{B(\tfrac{1}{2})^2B(1)} + \cancel{B(\tfrac{1}{2})\frac{1}{2}} \\
&= \boxed{B(1)B(\tfrac{1}{2})B(t) - \frac{1}{2}B(1) - B(\tfrac{1}{2})t, \qquad \frac{1}{2} < t \le 1.}
\end{aligned}$$

Note also that, when $t > 1$ we have

$$\begin{aligned}
\boxed{\int_0^t B(1)B(\tfrac{1}{2})dB(s)} &= \int_0^{\frac{1}{2}} B(1)B(\tfrac{1}{2})dB(s) + \int_{\frac{1}{2}}^1 B(1)B(\tfrac{1}{2})dB(s) + \int_1^t B(1)B(\tfrac{1}{2})dB(s) \\
&= \left[B(1)B(\tfrac{1}{2})B(t) - t\left(B(\tfrac{1}{2}) + B(1)\right)\right]_0^{\frac{1}{2}} \\
&+ \left[B(1)B(\tfrac{1}{2})B(t) - \frac{1}{2}\left(B(\tfrac{1}{2}) + B(1)\right)\right]_{\frac{1}{2}}^1 + \int_1^t B(1)B(\tfrac{1}{2})dB(s) \\
&= \cancel{\left(B(1)B(\tfrac{1}{2})^2 - \frac{1}{2}(B(\tfrac{1}{2}) + B(1))\right)} + \left(B(1)^2B(\tfrac{1}{2}) - \frac{1}{2}B(1) - B(\tfrac{1}{2})\right. \\
&\left. \cancel{- B(1)B(\tfrac{1}{2})^2 + \frac{1}{2}(B(1) + B(\tfrac{1}{2}))}\right) + B(1)B(\tfrac{1}{2})\int_1^t dB(s) \\
&= \cancel{B(1)^2B(\tfrac{1}{2})} - \frac{1}{2}B(1) - B(\tfrac{1}{2}) + B(1)B(\tfrac{1}{2})B(t) - \cancel{B(1)^2B(\tfrac{1}{2})} \\
&= \boxed{B(1)B(\tfrac{1}{2})B(t) - \frac{1}{2}B(1) - B(\tfrac{1}{2}), \qquad t > 1.}
\end{aligned}$$

Hence, we can conclude

$$\boxed{\int_0^t B(1)B(\tfrac{1}{2})dB(s) = \begin{cases} B(\frac{1}{2})B(1)B(t) - tB(\frac{1}{2}) - tB(1) & , \quad 0 \le t \le \frac{1}{2}, \\ B(\frac{1}{2})B(1)B(t) - tB(\frac{1}{2}) - \frac{1}{2}B(1) & , \quad \frac{1}{2} < t \le 1, \\ B(\frac{1}{2})B(1)B(t) - B(\frac{1}{2}) - \frac{1}{2}B(1) & , \quad t > 1. \end{cases}}$$

4.2 Properties of the Ayed-Kuo Stochastic Integral

In this section, we study the Itô integral properties, which were introduced in Chapter 2, for the Ayed-Kuo integral. We prove that this new anticipating stochastic integral satisfies the zero mean property, the near-martingale property and we propose an extension of the Itô isometry.

4.2.1 Zero Mean Property

Note that, in Equation (4.1) the evaluation points for $f(t)$ and $\varphi(t)$ on $[t_{i-1}, t_i]$ are the left and right endpoints, respectively. Moreover, let us recall that, if $\varphi(t) = 1$, the Ayed-Kuo integral reduces to the Itô integral.
In the next theorem, we prove that the Ayed-Kuo integral satisfies the zero mean property, in the same form as the Itô integral does.

Theorem 4.1 (W. Ayed, H.-H Kuo, [1]). *Let $\{f(t), a \leq t \leq b\}$ be an $\{\mathcal{F}_t\}$-adapted stochastic process and let $\{\varphi(t), a \leq t \leq b\}$ be an instantly independent stochastic process with respect to the filtration $\mathcal{F}_t$. If the Ayed-Kuo integral $\int_a^b f(t)\varphi(t)dB(t)$ exists such that, for all $a \leq t \leq b$, $\mathbb{E}|f(t)| < \infty$ and $\mathbb{E}|\varphi(t)| < \infty$, then*

$$\mathbb{E}\left(\int_a^b f(t)\varphi(t)dB(t)\right) = 0.$$

Proof. Let Δ be a partition such that $\Delta = \{a = t_0 < t_1 < t_2 < ... < t_n = b\}$. By the conditional expectation properties (see Appendix B), we have

$$\begin{aligned}\mathbb{E}\left(f(t_{i-1})\varphi(t_i)(B(t_i) - B(t_{i-1}))\right) &= \mathbb{E}\left(\mathbb{E}\left(f(t_{i-1})\varphi(t_i)(B(t_i) - B(t_{i-1}))\middle|\mathcal{F}_t\right)\right) \\ &= \mathbb{E}\left(f(t_{i-1})(B(t_i) - B(t_{i-1}))\mathbb{E}\left(\varphi(t_i)\middle|\mathcal{F}_t\right)\right).\end{aligned}$$

Note that $\mathbb{E}\{\varphi(t_i)|\mathcal{F}_t\} = \mathbb{E}\{\varphi(t_i)\}$, since $\varphi(t)$ is instantly independent. Hence, we get

$$\begin{aligned}\mathbb{E}\left(f(t_{i-1})\varphi(t_i)(B(t_i) - B(t_{i-1}))\right) &= \mathbb{E}\left(\varphi(t_i)\right)\mathbb{E}(f(t_{i-1})(B(t_i) - B(t_{i-1}))) \\ &= \mathbb{E}\left(\varphi(t_i)\right)\mathbb{E}\left(\mathbb{E}\left(f(t_{i-1})(B(t_i) - B(t_{i-1}))\middle|\mathcal{F}_{t_{i-1}}\right)\right) \\ &= \mathbb{E}\left(\varphi(t_i)\right)\mathbb{E}\left(f(t_{i-1})\mathbb{E}\left((B(t_i) - B(t_{i-1}))\middle|\mathcal{F}_{t_{i-1}}\right)\right) \\ &= 0,\end{aligned}$$

since $\mathbb{E}\left((B(t_i) - B(t_{i-1}))\middle|\mathcal{F}_{t_{i-1}}\right) = 0$.

By Definition 4.1, we have

$$\int_a^b f(t)\varphi(t)dB(t) = \lim_{||\Delta_n||\to 0}\sum_{i=1}^n f(t_{i-1})\varphi(t_i)(B(t_i) - B(t_{i-1})),$$

in probability. By taking subsequence, we have that the limit holds in $L^1(\Omega)$. We denote by S_n the partial sums and by I the integral. We have already proved that $\mathbb{E}(S_n) = 0$. Then, we have

$$|\mathbb{E}(I)| = |\mathbb{E}(I - S_n) + \mathbb{E}(S_n)| \leq \mathbb{E}|I - S_n| \to 0,$$

as $n \to \infty$. □

4.2.2 Near-Martingale Property

In this section, we show with a counterexample that the Ayed-Kuo integral does not satisfy the martingale property, because of the evaluation points of the integrand. However, we introduce the notion of near-martingale, which is satisfied by this new setting.

Example 4.5. Consider the stochastic process introduced in Example 4.1

$$X_t = \int_0^t B(1)dB(s) = B(1)B(t) - t, \qquad 0 \le t \le 1. \tag{4.9}$$

Let $s \le t$, then

$$\begin{aligned}\mathbb{E}(X_t|\mathcal{F}_s) &= \mathbb{E}\left(B(1)B(t) - t\middle|\mathcal{F}_s\right)\\ &= \mathbb{E}\left(((B(1)-B(t)) + (B(t)-B(s)) + B(s))((B(t)-B(s)) + B(s))\middle|\mathcal{F}_s\right) - t\\ &= t - s + B(s)^2 - t\\ &= B(s)^2 - s.\end{aligned} \tag{4.10}$$

Hence, $\{X_t\}$ is not a martingale with respect to the filtration $\{\mathcal{F}_t\}$, since $X_s \ne B(s)^2 - s$. Note that $\{X_s\}$ is not $\mathcal{F}_t$-measurable. Moreover, by taking $t = s$ in Equation (4.10), we get

$$\mathbb{E}\left(X_s\middle|\mathcal{F}_s\right) = B(s)^2 - s. \tag{4.11}$$

Then, by Equation (4.10) and Equation (4.11), for any $s \le t$, we have

$$\mathbb{E}\left(X_s\middle|\mathcal{F}_s\right) = \mathbb{E}\left(X_t\middle|\mathcal{F}_s\right).$$

This equality is the motivation for the near-martingale definition.

Definition 4.3 (Near-martingale property)**.** A stochastic process $\{X_t\}$ is said to be a *near-martingale* with respect to the filtration $\{\mathcal{F}_t\}$ if

(i) For all $0 \le t$, $\mathbb{E}|X_t| < \infty$;

(ii) For all $0 \le s \le t$, $\mathbb{E}\left(X_t - X_s\middle|\mathcal{F}_s\right) = 0$, or equivalently, $\mathbb{E}\left(X_t\middle|\mathcal{F}_s\right) = \mathbb{E}\left(X_s\middle|\mathcal{F}_s\right)$.

Example 4.6. In Example 4.5, we show that the stochastic process introduced in Equation (4.9) is not a martingale. Let us check that the process does satisfy the near-martingale property. From Equation (4.10), for $0 \le s \le t \le 1$, we get

$$\mathbb{E}\left(X_t\middle|\mathcal{F}_s\right) = B(s)^2 - s.$$

Furthermore, we have

$$\begin{aligned}\mathbb{E}\left(X_s\middle|\mathcal{F}_s\right) &= \mathbb{E}\left(B(1)B(s) - s\middle|\mathcal{F}_s\right)\\ &= B(s)\mathbb{E}\left(B(1)\middle|\mathcal{F}_s\right) - s\\ &= B(s)^2 - s.\end{aligned}$$

Hence, $\{X_t\}$ is a near-martingale with respect to the filtration $\{\mathcal{F}_t\}$.

Remark 4.3. Note that when the near-martingale process $\{X_t\}$ is adapted to the filtration, $\{X_t\}$ is also a martingale. Moreover, for any $0 \le s \le t$, we have

$$\mathbb{E}(X_t) = \mathbb{E}\left(X_t \middle| \mathcal{F}_s\right) = \mathbb{E}\left(X_s \middle| \mathcal{F}_s\right) = \mathbb{E}(X_s),$$

such that, for all $t \ge 0$, we get

$$\mathbb{E}(X_t) = \mathbb{E}(X_0).$$

Thus, the near-martingale property implies the fair game property (see Remark 1.2).

The next theorem proves that the near-martingale property is the analogue of the martingale property in the Itô integral for the Ayed-Kuo integral.

Theorem 4.2 (H.-H. Kuo et al., [24])**.** *Let $\{f(t), a \le t \le b\}$ be a $\{\mathcal{F}_t\}$-adapted stochastic process and let $\{\varphi(t), a \le t \le b\}$ be an instantly independent stochastic process with respect to the filtration. Consider the stochastic process $\{X_t\}$, such that*

$$X_t = \int_a^t f(B(s))\varphi(B(b) - B(s))dB(s), \qquad a \le t \le b,$$

and assume that $\mathbb{E}|X_t| < \infty$ for all $a \le t \le b$. Then, the stochastic process $\{X_t\}$ is a near-martingale with respect to the filtration $\{\mathcal{F}_t\}$.

Proof. We aim to check that $\mathbb{E}(X_t - X_s|\mathcal{F}_s) = 0$ for $a \le s \le t$. By the properties of the Riemann sums, we have

$$X_t - X_s = \int_s^t f(B(u))\varphi(B(b) - B(u))dB(u).$$

Let $\Delta = \{s = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = t\}$ be a partition of the interval $[s,t]$. Moreover, let $\delta B_i = (B(t_i) - B(t_{i-1}))$. By the definition of the Ayed-Kuo integral, we have

$$\begin{aligned}
\mathbb{E}(X_t - X_s|\mathcal{F}_s) &= \mathbb{E}\left(\int_s^t f(B(u))\varphi(B(b) - B(u))dB(u) \middle| \mathcal{F}_s\right) \\
&= \mathbb{E}\left(\lim_{||\Delta_n|| \to 0} \sum_{i=1}^n f(B(t_{i-1}))\varphi(B(b) - B(t_i))\delta B_i \middle| \mathcal{F}_s\right) \\
&= \lim_{||\Delta_n|| \to 0} \sum_{i=1}^n \mathbb{E}\left(f(B(t_{i-1}))\varphi(B(b) - B(t_i))\delta B_i \middle| \mathcal{F}_s\right).
\end{aligned}$$

Thus, it is enough to show that every component of the last sum is equal to zero. By the conditional expectation properties (see Appendix B), and since $f(B(t_{i-1}))$ is $\mathcal{F}_{t_i}$-measurable and

$\varphi(B(b)-B(t_i))$ is independent of $\mathcal{F}_{t_i}$, we have

$$\begin{aligned}
\mathbb{E}\left(f(B(t_{i-1}))\varphi(B(b)-B(t_i))\Delta B_i \middle| \mathcal{F}_s\right) &= \mathbb{E}\left(\mathbb{E}\left(f(B(t_{i-1}))\varphi(B(b)-B(t_i))\delta B_i \middle| \mathcal{F}_{t_{i-1}}\right) \middle| \mathcal{F}_s\right) \\
&= \mathbb{E}\left(f(B(t_{i-1}))\mathbb{E}\left(\varphi(B(b)-B(t_i))\delta B_i \middle| \mathcal{F}_{t_{i-1}}\right) \middle| \mathcal{F}_s\right) \\
&= \mathbb{E}\left(f(B(t_{i-1}))\mathbb{E}\left(\mathbb{E}\left(\varphi(B(b)-B(t_i))\delta B_i \middle| \mathcal{F}_{t_i}\right) \middle| \mathcal{F}_{t_{i-1}}\right) \middle| \mathcal{F}_s\right) \\
&= \mathbb{E}\left(f(B(t_{i-1}))\mathbb{E}\left(\delta B_i \mathbb{E}\left(\varphi(B(b)-B(t_i)) \middle| \mathcal{F}_{t_i}\right) \middle| \mathcal{F}_{t_{i-1}}\right) \middle| \mathcal{F}_s\right) \\
&= \mathbb{E}\left(\varphi(B(b)-B(t_i))\right)\mathbb{E}\left(f(B(t_{i-1}))\mathbb{E}\left(\delta B_i \middle| \mathcal{F}_{t_{i-1}}\right) \middle| \mathcal{F}_s\right) \\
&= \mathbb{E}\left(\varphi(B(b)-B(t_i))\right)\mathbb{E}(\delta B_i)\mathbb{E}\left(f(B(t_{i-1})) \middle| \mathcal{F}_s\right) \\
&= 0.
\end{aligned}$$

Finally, we get

$$\begin{aligned}
\mathbb{E}(X_t - X_s|\mathcal{F}_s) &= \mathbb{E}\left(\int_s^t f(B(u))\varphi(B(b)-B(u))dB(u) \middle| \mathcal{F}_s\right) \\
&= \mathbb{E}\left(\lim_{||\Delta_n||\to 0} \sum_{i=1}^n f(B(t_{i-1}))\varphi(B(b)-B(t_i))\delta B_i \middle| \mathcal{F}_s\right) \\
&= \lim_{||\Delta_n||\to 0} \sum_{i=1}^n \mathbb{E}\left(f(B(t_{i-1}))\varphi(B(b)-B(t_i))\delta B_i \middle| \mathcal{F}_s\right) \\
&= 0.
\end{aligned}$$

Hence, $\{X_t\}$ is a near-martingale with respect to the filtration $\{\mathcal{F}_t\}$. □

4.2.3 An extension of the Itô Isometry

In this section, we propose an extension of the Itô isometry for the Ayed-Kuo stochastic integral. The identity is for a specific type of stochastic processes, including polynomial and exponential functions of $B(t)$.

Theorem 4.3 (H.-H. Kuo et al., [27]). *Let f and φ be $\mathcal{C}^1$-functions on $\mathbb{R}$. Then*

$$\begin{aligned}
\mathbb{E}\left(\left(\int_a^b f(B(t))\varphi(B(b)-B(t))dB(t)\right)^2\right) &= \int_a^b \mathbb{E}\left(f(B(t))^2\varphi(B(b)-B(t))^2\right)dt \\
+2\int_a^b\int_0^t \mathbb{E}\left(f(B(s))\varphi'(B(b)-B(s))f'(B(s))\varphi(B(b)-B(s))\right)dsdt.
\end{aligned} \tag{4.12}$$

This result is based on McLaurin expansions and the conditional expectation properties. We do not prove it as it is highly extensive. The proof can be found in [27].

Remark 4.4. Note that if $\varphi(x) = 1$, we have the Itô isometry property from Itô stochastic integration theory. If $f(x) = 1$, we do also have this isometry.

Next, we apply the result from Theorem 4.3 to some of the processes introduced in Section 4.1, in order to show how it works.

Example 4.7. Consider the stochastic process introduced in Example 4.2 (see Equation (4.6) for detail)

$$\int_0^T B(t)(B(T)-B(t))\,dB(t) = \frac{1}{2}B(T)\left(B(T)^2-T\right) - \frac{1}{3}B(T)^3.$$

First, we begin with the left brace of Equation (4.12)

$$\begin{aligned}
\mathbb{E}\left(\left(\int_0^T f(B(t))\varphi(B(b)-B(t))\,dB(t)\right)^2\right) &= \mathbb{E}\left(\left(\int_0^T B(t)(B(T)-B(t))\,dB(t)\right)^2\right)\\
&= \mathbb{E}\left(\left(\frac{1}{6}B(T)^3 - \frac{1}{2}TB(T)\right)^2\right)\\
&= \frac{1}{36}\mathbb{E}\left(B(T)^6\right) - \frac{1}{6}T\mathbb{E}\left(B(T)^4\right) + \frac{1}{4}T^2\mathbb{E}\left(B(T)^2\right)\\
&= \frac{1}{36}5!!T^3 - \frac{1}{6}3!!T^3 + \frac{1}{4}T^3\\
&= \frac{1}{6}T^3.
\end{aligned}$$

According to Theorem 4.3, we consider the function $f(x) = \varphi(x) = x$, such that $f'(x) = \varphi'(x) = 1$. Then, we have

$$\begin{aligned}
\mathbb{E}\left(\left(\int_0^T B(t)(B(T)-B(t))\,dB(t)\right)^2\right) &= \int_0^T \mathbb{E}\left(\left(B(t)^2(B(T)-B(t))\right)^2\right)dt\\
&\quad + 2\int_0^T\int_0^t \mathbb{E}(B(s)(B(T)-B(t)))\,ds\,dt.
\end{aligned} \tag{4.13}$$

Note that $B(T)-B(t)$ is independent of $B(s)$ and since Brownian motion fulfills the zero mean property, we have $\mathbb{E}(B(s)(B(T)-B(t))) = 0$. Hence, we get

$$\begin{aligned}
\mathbb{E}\left(\left(\int_0^T B(t)(B(T)-B(t))\,dB(t)\right)^2\right) &= \int_0^T \mathbb{E}\left(B(t)^2\right)\mathbb{E}\left((B(T)-B(t))^2\right)dt\\
&= \int_0^T t(T-t)dt\\
&= \frac{1}{2}T^3 - \frac{1}{3}T^3\\
&= \frac{1}{6}T^3.
\end{aligned}$$

Finally, we conclude that the Ayed-Kuo isometry property is satisfied for this example.

Example 4.8. Consider the stochastic process introduced in Example 4.3

$$\int_0^1 e^{B(1)}dB(t) = e^{B(1)}(B(1)-1).$$

First, we begin with the left brace of the identity

$$\begin{aligned}
\mathbb{E}\left(\left(\int_0^1 e^{B(1)}dB(t)\right)^2\right) &= \mathbb{E}\left(\left(e^{B(1)}(B(1)-1)\right)^2\right)\\
&= \int_{-\infty}^{+\infty} e^{2x}(x-1)^2\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = 2e^2.
\end{aligned}$$

According to Theorem 4.3, we consider the functions $f(x) = \varphi(x) = e^x$, such that $f'(x) = \varphi'(x) = e^x$. Then, we have

$$\int_0^t \mathbb{E}\left(f(B(s))^2\varphi(B(1)-B(s))\right)^2 ds = e^2,$$
$$\int_0^t \int_0^b \mathbb{E}\left(f(B(s))\varphi'(B(b)-B(s))f'(B(s))\varphi(B(b)-B(s))\right) dbdt = \frac{1}{2}e^2.$$

Hence, we get

$$\mathbb{E}\left(\left(\int_a^b f(B(t))\varphi(B(b)-B(t))dB(t)\right)^2\right) = 2e^2.$$

Finally, we conclude that the Ayed-Kuo isometry property is satisfied for this example.

4.3 An extension of the Itô formula for the Ayed-Kuo Stochastic Integral

In this section, we extend the Itô formula established in Section 2.3 for the Ayed-Kuo integral. The formula is derived by H.-H. Kuo et al. in [23, 28]. After we obtain the formula, we show how it works for the same examples that were calculated in Section 4.1 and we check that the results coincide.

Consider an Itô process of the form

$$X_t = X_a + \int_a^t g(s,\omega)dB(s) + \int_a^t \gamma(s,\omega)ds, \tag{4.14}$$

where X_a is a $\mathcal{F}_a$-measurable random variable, $g \in L^2_{ad}(\Omega \times [a,b])$ and $\gamma \in L^1([a,b])$ almost surely. Consider the process

$$Y^{(t)} = Y_a + \int_t^b h(s)dB(s) + \int_t^b \chi(s)ds, \tag{4.15}$$

where Y_a is a random variable independent of $\mathcal{F}_t$, $h \in L^2([a,b])$ and $\chi \in L^1([a,b])$ are two deterministic functions.

By Itô theory, we have that the stochastic process X_t is adapted to the filtration $\mathcal{F}_t$. In the next results, we show that Y_t is an instantly independent stochastic process.

Proposition 4.2 (H.-H. Kuo et al., [23]). *Let $h \in L^2([a,b])$, $\chi \in L^1([a,b])$ be two deterministic functions and Y_a a random variable independent of $\mathcal{F}_t$. Then,*

$$Y^{(t)} = Y_a + \int_t^b h(s)dB(s) + \int_t^b \chi(s)ds, \quad t \in [a,b],$$

is an instantly independent stochastic process with respect to the Brownian filtration.

Proof. Our aim is to check that $Y^{(t)}$ is independent of the filtration $\mathcal{F}_t$ for $a \le t \le b$. Note that Y_a in independent of the filtration $\mathcal{F}_t$, for all $a \le t \le b$. Then, by assumption and the Lebesgue integral $Y^{(t)}$ is also independent of $\mathcal{F}_t$ because it is deterministic. Hence, we need to study the stochastic integral. By definition of the Ayed-Kuo stochastic integral, we have

$$\int_t^b h(s)dB(s) = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^n h(s_i)(B(s_i) - B(s_{i-1})),$$

where $\Delta = \{t = s_0 < s_1 < s_2 < ... < s_n = b\}$ is a partition of $[t,b]$. The evaluation points of h are not as important as it is the deterministic function. By the Brownian motion properties, $h(s_i)(B(s_i)-B(s_{i-1}))$ is independent of $\mathcal{F}_t$, for all $1 \le i \le n$, because $t \le s_{i-1}$. Then, the sum

$$\sum_{i=1}^n h(s_i)(B(s_i) - B(s_{i-1})),$$

is also independent of the filtration $\mathcal{F}_t$. We can conclude that the integral is also independent of the filtration $\mathcal{F}_t$ as a limit of independent random variables. □

In the following result, we present the Itô formula, which allows us to compute Itô stochastic processes. This method is fundamental in the theory of stochastic differential equations.

Theorem 4.4 (H.-H. Kuo et al., [23]). *Let $\theta(x,y) = f(x)\varphi(y)$ be a function such that $f,\varphi \in \mathcal{C}^2(\mathbb{R})$. Let X_t and $Y^{(t)}$, $a \le t \le b$, be two stochastic processes as in Equation (4.14) and Equation (4.15), respectively. The following equality holds almost surely for $a \le t \le b$,*

$$\begin{aligned}\theta(X_t, Y^{(t)}) = \theta\left(X_a, Y^{(a)}\right) &+ \int_a^t \frac{\partial\theta}{\partial x}\left(X_s, Y^{(s)}\right)dX_s + \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial x^2}\left(X_s, Y^{(s)}\right)(dX_s)^2 \\ &+ \int_a^t \frac{\partial\theta}{\partial y}\left(X_s, Y^{(s)}\right)dY^{(s)} - \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial y^2}\left(X_s, Y^{(s)}\right)\left(dY^{(s)}\right)^2.\end{aligned} \tag{4.16}$$

In differential form,

$$\begin{aligned}d\theta(X_t, Y^{(t)}) = &\frac{\partial\theta}{\partial x}\left(X_s, Y^{(s)}\right)dX_s + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}\left(X_s, Y^{(s)}\right)(dX_s)^2 \\ &+ \frac{\partial\theta}{\partial y}\left(X_s, Y^{(s)}\right)dY^{(s)} - \frac{1}{2}\frac{\partial^2\theta}{\partial y^2}\left(X_s, Y^{(s)}\right)\left(dY^{(s)}\right)^2.\end{aligned} \tag{4.17}$$

Proof. Let $\Delta = \{t = s_0 < s_1 < s_2 < ... < s_n = b\}$ be a partition of the interval $[a,t]$. Moreover, let $\delta X_i = X_{t_i} - X_{t_{i-1}}$. Let us express $F\left(X_t, Y^{(t)}\right) - F\left(X_a, Y^{(a)}\right)$ as a telescoping sum

$$\begin{aligned}\theta\left(X_t, Y^{(t)}\right) - \theta\left(X_a, Y^{(a)}\right) &= \sum_{i=1}^n \left(\theta\left(X_{t_i}, Y^{(t_i)}\right) - \theta\left(X_{t_{i-1}}, Y^{(t_{i-1})}\right)\right) \\ &= \sum_{i=1}^n \left(f\left(X_{t_i}\right)\varphi\left(Y^{(t_i)}\right) - f\left(X_{t_{i-1}}\right)\varphi\left(Y^{(t_{i-1})}\right)\right).\end{aligned} \tag{4.18}$$

In Equation (4.18), in order to get an Ayed-Kuo integral, we have to take the left endpoints of the intervals $[t_{i-1}, t_i]$ to evaluate every occurrence of f and the right endpoints of the intervals to evaluate every occurrence of φ. We proceed in the same form as in the proof of the classical Itô formula. Then, we use Taylor expansion up to second order. The restriction to second order is enough, since for $k > 2$, we have

$$o\left((\delta X_i)^k\right) > o(\delta t) \qquad and \qquad o\left((\delta Y_i)^k\right) > o(\delta t).$$

Then, both $(\delta X_i)^k$ and $(\delta Y_i)^k$ tend to zero as $||\Delta_n|| \to 0$. Thus, we expand $f\left(X_{t_i}\right)$ around the point $X_{t_{i-1}}$ with $1 \le i \le n$,

$$f\left(X_{t_i}\right) \approx f\left(X_{t_{i-1}}\right) + f'\left(X_{t_{i-1}}\right)\delta X_i + \frac{1}{2} f''\left(X_{t_{i-1}}\right)(\delta X_i)^2. \tag{4.19}$$

Next, we expand $\varphi\left(Y^{(t_{i-1})}\right)$ around the point $Y^{(t_i)}$ for $1 \le i \le n$,

$$\varphi\left(Y^{(t_{i-1})}\right) \approx \varphi\left(Y^{(t_i)}\right) + \varphi'\left(Y^{(t_i)}\right)(-\delta Y_i) + \frac{1}{2}\varphi''\left(Y^{(t_i)}\right)(-\delta Y_i)^2. \tag{4.20}$$

Substituting Equation (4.19) and Equation (4.20) into Equation (4.18), we get

$$\begin{aligned}
\theta\left(X_t, Y^{(t)}\right) - \theta\left(X_a, Y^{(a)}\right) &\approx \sum_{i=1}^{n}\Bigg(\left(f\left(X_{t_{i-1}}\right) + f'\left(X_{t_{i-1}}\right)\delta X_i + \frac{1}{2} f''\left(X_{t_{i-1}}\right)(\delta X_i)^2\right)\varphi\left(Y^{(t_i)}\right) \\
&\quad - f\left(X_{t_{i-1}}\right)\left(\varphi\left(Y^{(t_i)}\right) + \varphi'\left(Y^{(t_i)}\right)(-\delta Y_i) + \frac{1}{2}\varphi''\left(Y^{(t_i)}\right)(-\delta Y_i)^2\right)\Bigg) \\
&= \sum_{i=1}^{n}\Bigg(f'\left(X_{t_{i-1}}\right)\varphi\left(Y^{(t_i)}\right)\delta X_i + \frac{1}{2} f''\left(X_{t_{i-1}}\right)\varphi\left(Y^{t_i}\right)(\delta X_i)^2 \\
&\quad + f\left(X_{t_{i-1}}\right)\varphi'\left(Y^{(t_i)}\right)\delta Y_i - \frac{1}{2} f\left(X_{t_{i-1}}\right)\varphi''\left(Y^{(t_i)}\right)(\delta Y_i)^2\Bigg) \\
&\to \int_a^t \frac{\partial\theta}{\partial x}\left(X_s, Y^{(s)}\right) dX_s + \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial x^2}\left(X_s, Y^{(s)}\right)(dX_s)^2 \\
&\quad + \int_a^t \frac{\partial\theta}{\partial y}\left(X_s, Y^{(s)}\right) dY^{(s)} - \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial y^2}\left(X_s, Y^{(s)}\right)\left(dY^{(s)}\right)^2,
\end{aligned}$$

as $||\Delta_n|| \to 0$. □

Corollary 4.1 (H.-H. Kuo et al., [23]). *Consider a function $\theta(t,x,y) = \tau(t)f(x)\varphi(y)$ such that $f, \varphi \in \mathcal{C}^2(\mathbb{R})$ and $\tau \in \mathcal{C}^1([a,b])$. Let X_t and $Y^{(t)}$, $a \le t \le b$, be two stochastic processes as in Equation (4.14) and Equation (4.15), respectively. The following equality holds almost surely for $a \le t \le b$,*

$$\begin{aligned}
\theta\left(t, X_t, Y^{(t)}\right) &= \theta\left(a, X_a, Y^{(a)}\right) + \int_A^t \frac{\partial\theta}{\partial s}\left(s, X_s, Y^{(s)}\right) ds \\
&\quad + \int_a^t \frac{\partial\theta}{\partial x}\left(s, X_s, Y^{(s)}\right) dX_s + \frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial x^2}\left(s, X_s, Y^{(s)}\right)(dX_s)^2 \\
&\quad + \int_a^t \frac{\partial\theta}{\partial y}\left(s, X_s, Y^{(s)}\right) dY^{(s)} - \frac{1}{2}\int_a^t \frac{\partial\theta^2}{\partial y^2}\left(s, X_s, Y^{(s)}\right)\left(dY^{(s)}\right)^2.
\end{aligned} \tag{4.21}$$

In differential form,

$$\begin{aligned}
d\theta\left(X_t, Y^{(t)}\right) &= \frac{\partial\theta}{\partial s}\left(s, X_s, Y^{(s)}\right) ds + \frac{\partial\theta}{\partial x}\left(s, X_s, Y^{(s)}\right) dX_s + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}\left(s, X_s, Y^{(s)}\right)(dX_s)^2 \\
&\quad + \frac{\partial\theta}{\partial y}\left(s, X_s, Y^{(s)}\right) dY^{(s)} - \frac{1}{2}\frac{\partial^2\theta}{\partial y^2}\left(s, X_s, Y^{(s)}\right)\left(Y^{(s)}\right)^2.
\end{aligned} \tag{4.22}$$

Remark 4.5. If $\varphi(t) = 1$ in Equation (4.22), we get the classical Itô formula.

Next, we derive a particular case of the formula obtained in Corollary 4.1, which will be useful in the calculus of some processes.

Corollary 4.2 (H.-H. Kuo et al., [23]). *Let $\theta(t,x,y) = \tau(t)f(x)\varphi(y)$ be a function, such that $f,\varphi \in \mathcal{C}^2(\mathbb{R})$ and $\tau \in \mathcal{C}^1([a,b])$. The following equality holds*

$$d\theta(t,B(t),B(b)) = \left(\frac{\partial\theta}{\partial t} + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial xy}\right)dt + \frac{\partial\theta}{\partial x}dB(t). \tag{4.23}$$

Proof. Note that $X_t = B(t)$ is an adapted process. However, $Y_t = B(b)$ is not an instantly independent process. Then, we have

$$B(b) = B(b) - B(t) + B(t).$$

Let us define a function ω, such that $\omega(t,x,y) = \theta(t,x,x+y)$. Thus, we get

$$d\omega(t,B(t),B(b)-B_t) = d\theta(t,B(t),B(b)).$$

Thus, $d\omega(t,B(t),B(b)-B(t))$ can be calculated using Corollary 4.1. Hence, its partial derivatives are

$$\frac{\partial\omega}{\partial t} = \theta_1, \qquad \frac{\partial\omega}{\partial x} = \theta_2 + \theta_3, \qquad \frac{\partial\omega}{\partial y} = \theta_3, \qquad \frac{\partial^2\omega}{\partial x^2} = \theta_{22} + 2\theta_{23} + \theta_{33}, \qquad \frac{\partial^2\omega}{\partial y^2} = \theta_{33},$$

where the indexes 1,2,3 are referred to derivatives with respect to the first, second and third variables of θ, respectively. By Equation (4.22), we have

$$\begin{aligned}
d\theta(t,B(t),B(b)) &= \frac{\partial\omega}{\partial t}dt + \frac{\partial\omega}{\partial x}dB(t) + \frac{1}{2}\frac{\partial^2\omega}{\partial x^2}(dB(t))^2 + \frac{\partial\omega}{\partial y}(-dB(t)) - \frac{1}{2}\frac{\partial^2\omega}{\partial y^2}(-dB(t))^2 \\
&= \theta_1 dt + (\theta_2+\theta_3)dB(t) + \frac{1}{2}(\theta_{22}+2\theta_{23}+\theta_{33})dt - \theta_3 dB(t) - \frac{1}{2}\theta_{33}dt \\
&= \theta_1 dt + \theta_2 dB(t) + \frac{1}{2}\theta_{22}dt + \theta_{23}dt \\
&= \left(\frac{\partial\theta}{\partial t} + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial xy}\right)dt + \frac{\partial\theta}{\partial x}dB(t).
\end{aligned}$$

□

Next, we apply the obtained results to the stochastic processes introduced in Section 4.1 and check that the results obtained by the definition and the formula coincide.

Example 4.9. Consider the stochastic process introduced in Example 4.2

$$X_t = \int_0^T B(T)B(t)dB(t).$$

According to Corollary 4.2, we consider the function $\theta(t,x,y)$, such that

$$\frac{\partial\theta}{\partial x}(t,x,y) = xy.$$

Hence, we take $\theta(t,x,y) = x^2/2y$, whose partial derivatives are

$$\frac{\partial\theta}{\partial t} = 0, \qquad \frac{\partial\theta}{\partial x} = xy, \qquad \frac{\partial^2\theta}{\partial x^2} = y, \qquad \frac{\partial^2\theta}{\partial xy} = x.$$

Then, we get

$$d\left(\frac{1}{2}B(t)^2B(T)\right) = B(T)B(t)dB(t) + \left(\frac{1}{2}B(T) + B(t)\right)dt.$$

Integrating in both sides of the equality from 0 to T, we have

$$\frac{1}{2}B(T)^3 = \int_0^T B(T)B(t)dB(t) + \int_0^T \frac{1}{2}B(T) + B(t)dt.$$

Thus, we get

$$\boxed{\int_0^T B(T)B(t)dB(t)} = \frac{1}{2}B(T)^3 - \int_0^T \frac{1}{2}B(T) - B(t)dt$$
$$= \boxed{\frac{1}{2}B(T)^3 - \frac{1}{2}TB(T) - \int_0^T B(t)dt,}$$

which coincides with the result obtained in Example 4.2.

Example 4.10. Consider the stochastic process introduced in Example 4.3

$$X_t = \int_0^T e^{B(T)}dB(t).$$

According to Corollary 4.2, we consider the function $\theta(t,x,y)$, such that

$$\frac{\partial\theta}{\partial x}(t,x,y) = e^y.$$

Hence, we take $\theta(t,x,y) = xe^y$, whose partial derivatives are

$$\frac{\partial\theta}{\partial t} = 0, \qquad \frac{\partial\theta}{\partial x} = e^y, \qquad \frac{\partial^2\theta}{\partial x^2} = 0, \qquad \frac{\partial^2\theta}{\partial xy} = e^y.$$

Then, we get

$$d\left(B(t)e^{B(T)}\right) = e^{B(T)}dB(t) + e^{B(t)}dt.$$

Integrating on both sides of the equality from 0 to T, we have

$$\int_0^T d\left(B(t)e^{B(T)}\right)dB(t) = \int_0^T e^{B(t)}dB(t) + \int_0^T e^{B(T)}dt.$$

Hence, we get

$$\boxed{\int_0^T e^{B(t)}dB(t)} = B(t)e^{B(T)} - \int_0^T e^{B(T)}dt$$
$$= \boxed{e^{B(T)}(B(T) - T),}$$

which coincides with the result obtained in Example 4.3.

Example 4.11. Consider the stochastic process introduced in Example 4.3 (see Equation (4.6) for detail)

$$X_t = \int_0^T B(t)(B(T) - B(t))\, dB(t).$$

According to Corollary 4.2, we consider the function $\theta(t,x,y)$, such that

$$\frac{\partial\theta}{\partial x}(t,x,y) = x(y-x).$$

Hence, we take $\theta(t,x,y) = yx^2/2 - x^3/3$, whose partial derivatives are

$$\frac{\partial\theta}{\partial t} = 0, \qquad \frac{\partial\theta}{\partial x} = x(y-x), \qquad \frac{\partial^2\theta}{\partial x^2} = y - 2x, \qquad \frac{\partial^2\theta}{\partial xy} = x.$$

Then, we get

$$d\left(B(T)B(t)^2/2 - B(t)^3/3\right) = B(t)(B(T)-B(t))\, dB(t) + \left(\frac{1}{2}(B(T) - 2B(t)) + B(t)\right) dt.$$

Integrating on both sides of the equality from 0 to T, we have

$$\int_0^T d\left(B(T)B(t)^2/2 - B(t)^3/3\right) dB(t) = \int_0^T B(t)(B(T)-B(t))\, dB(t) + \int_0^T \frac{1}{2}B(T)dt.$$

Hence, we get

$$\boxed{\int_0^T B(t)(B(T)-B(t))\, dB(t)} = B(T)B(t)^2/2 - B(t)^3/3 - \int_0^T \frac{1}{2}B(T)dt$$
$$= \boxed{\frac{1}{2}B(T)\left(B(t)^2 - T\right) - \frac{1}{3}B(t)^3,}$$

which coincides with the result obtained in Example 4.2 (see Equation (4.6) for detail).

Remark 4.6. In addition, we would like to mention that an extension of the Girsanov theorem for the Ayed-Kuo stochastic integral is proved in [25, 26] by H.-H. Kuo, Y. Peng and B. Szozda. As we have discussed in Section 2.4, this result plays a fundamental role in the theory of stochastic processes, as well as in its applications, for example in financial modeling.

4.4 Stochastic Differential Equations with Anticipating Initial Conditions for the Ayed-kuo Stochastic Integral

In this final section, our aim is to study a theorem that establishes a general solution for the stochastic differential equations with anticipating initial conditions for the Ayed-Kuo integral. This result has been proved in [21] by N. Khalifa, H.-H. Kuo, H. Ouerdiane and B. Szozda. Then, we propose an example of a linear stochastic differential equation for the financial classical problem under the Ayed-Kuo integration theory. Aditional results for stochastic differential equations with anticipating initial conditions under the Ayed-Kuo theory can be found in [16, 39].

4.4.1 A general solution for Stochastic Differential Equations with Anticipating Initial Conditions for the Ayed-Kuo Stochastic Integral

Theorem 4.5 (N. Khalifa et al., [21]). *Let* $\alpha(t) \in L^2([a,b])$ *and* $\beta(t) \in L^2_{ad}(\Omega \times [a,b])$*. Consider* $\rho \in \mathcal{M}^\infty \cap \mathcal{S}(\mathbb{R})$*. Then, the stochastic differential equation*

$$\begin{cases} dX_t = \alpha(t)X_t dB(t) + \beta(t)X_t dt, & a \le t \le b, \\ X_a = \rho(B(b) - B(a)), \end{cases} \tag{4.24}$$

has a unique solution given by

$$X_t = \big(\rho(B(b) - B(a)) - \xi(t, B(b) - B(a))\big) Z_t, \tag{4.25}$$

where

$$\xi(t,y) = \int_a^t \alpha(s)\rho'\left(y - \int_s^t \alpha(u)du\right) ds,$$

and

$$Z_t = \exp\left(\int_a^t \alpha(s)dB(s) + \int_a^t \left(\beta(s) - \frac{1}{2}\alpha(s)^2\right) ds\right).$$

We do not prove this results as it is highly extensive. The proof of the existence and uniqueness of the solution of Theorem 4.5 can be found in [21].

Remark 4.7. Note that, if $a = 0$, $\alpha(t) = \alpha$ and $\beta(t) = \beta$, the coefficients are constants and the evolution starts at 0. Hence, the solution to Equation (4.24) has the form

$$X_t = \omega(t, B(T)) \exp\left(\alpha B(t) + \left(\beta - \frac{1}{2}\alpha^2\right) t\right), \tag{4.26}$$

where $\omega(t,x)$ is the solution of the following partial differential equation

$$\begin{cases} \frac{\partial \omega}{\partial t}(t,x) = -\alpha \frac{\partial \omega}{\partial x}(t,x), & 0 \le t \le b, \\ \omega(0,x) = \rho(x). \end{cases} \tag{4.27}$$

Hence, in order to show that the solution from Theorem 4.5 coincides with Equation (4.26), it is enough to show that $\omega(t,x) = \rho(x) - \xi(t,x)$ solves Equation (4.27). Note that, in the case of constant coefficients, we have $\omega(t,x) = \rho(x - \alpha t)$.

4.4.2 Black-Scholes-Merton Model under Ayed-Kuo Theory

Consider the following stochastic differential equation for the Black-Scholes-Merton model with a slightly modification, whose financial sense will be explained in Chapter 6

$$\begin{cases} dS_t = \sigma S_t dB(t) + \mu S_t dt, \\ S_0 = B(T). \end{cases} \tag{4.28}$$

As it has been already explained, the Black-Scholes-Merton model is a continuous-time model which aim is to describe the behaviour of the prices of one risky asset (a stock with price S_t at time t) and a riskless asset (with price S_t^0 at time t), where μ are σ are two constants, such that

$$\begin{cases} \mu \equiv \text{appreciation rate of the stock } S_t; \\ \sigma \equiv \text{volatility of the asset } S_t; \\ B(t) \equiv \text{standard brownian motion;} \\ S_0 \equiv \text{spot price observed at time } t = 0. \end{cases}$$

Remark 4.8. If we consider that the solution of the SDE (4.28) is the same as the solution of the classical SDE from Itô theory, we check that it does not work, as we get an extra term because of the anticipating initial condition.

By the result obtained in Theorem 4.5, we are able to find a solution for the Black-Scholes-Merton model with an anticipating condition under Ayed-Kuo theory. Note that, for the SDE (4.28), we have

$$\alpha(t) = \sigma, \qquad \beta(t) = \mu, \qquad \rho(x) = x.$$

Then, we consider the solution

$$S_t = (B(T) - \sigma t)\, e^{\sigma B(t) + (\mu - \frac{1}{2}\sigma^2)t}. \tag{4.29}$$

where

$$\omega(t,x) = \rho(x - \sigma t) \qquad \text{and} \qquad \omega(0,x) = x.$$

Remark 4.9. Hence, according to Theorem 4.5, we conclude that the Equation (4.29) is the solution for the SDE (4.28), whose existence and uniqueness has already been proved.

Now, let us check that this result is accomplished by the extension of the Itô formula for the Ayed-Kuo integral, which has been studied in Section 4.3. According to Corollary 4.2, we consider the function $\theta(t,x,y)$, such that

$$\theta(t,x,y) = (y - \xi(t))\, e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma x},$$

whose partial derivatives are

$$\begin{cases} \frac{\partial \theta}{\partial t} = -\xi'(t) e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x} + \left(\mu - \frac{1}{2}\sigma^2\right)(y - \xi(t))\, e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x}, \\ \frac{\partial \theta}{\partial x} = \sigma (y - \xi(t))\, e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x}, \\ \frac{\partial^2 \theta}{\partial x^2} = \sigma^2 (y - \xi(t))\, e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x}, \\ \frac{\partial^2 \theta}{\partial xy} = \sigma e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x}. \end{cases}$$

Hence, we have

$$
\begin{aligned}
dS_t &= \frac{\partial\theta}{\partial t}dt + \frac{\partial\theta}{\partial x}dB(t) + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial xy}dt \\
&= \left(-\xi'(t)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma B(t)} + \left(\mu - \frac{1}{2}\sigma^2\right)S_t\right)dt + \sigma S_t dB(t) + \frac{1}{2}\sigma^2 S_t dt + \sigma e^{(\mu-\frac{1}{2}\sigma^2)+\sigma B(t)}dt \\
&= -\xi'(t)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma B(t)}dt + \mu S_t dt - \cancel{\frac{1}{2}\sigma^2 S_t dt} + \sigma S_t dB(t) + \cancel{\frac{1}{2}\sigma^2 S_t dt} + \sigma e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma B(t)}dt \\
&= -\xi'(t)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma B(t)}dt + \mu S_t dt + \sigma S_t dB(t) + \sigma e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma B(t)}dt.
\end{aligned}
$$

Note that, $\xi(t)$ is a deterministic function, whose value is determined by imposing that Equation (4.29) is the solution of the SDE (4.28), such that, $\xi(t)$ must satisfy the conditions

$$
\begin{cases}
\xi'(t) = \sigma, & 0 \le t \le T, \\
\xi(0) = 0.
\end{cases}
$$

Thus, we get

$$
\xi(t) = \sigma t, \quad 0 \le t \le T. \tag{4.30}
$$

Finally, substituting Equation (4.30) into Equation (4.29), we have

$$
\boxed{S_t = (B(T) - \sigma t)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma B(t)},}
$$

is the solution of the SDE (4.28).

CHAPTER 5

The Russo-Vallois Stochastic Integral

The Russo-Vallois integral was first introduced by F. Russo and P. Vallois in [36] in 1993. This integration encompasses three different stochastic processes: forward, backward and symmetric integration. Since the forward integral is the only one that generalizes the Itô integral, we would always refer with Russo-Vallois integral to the forward integral.
This setting can be defined in terms of Riemann sums, as well as the Itô integral and the Ayed-Kuo integral, where the integrand is assumed to be a product of an adapted stochastic process with respect to a Brownian filtration and an anticipating stochastic process. This setting is characterized by not having the analytical structure of the Ayed-Kuo one, as it does not satisfy any of the properties studied for the other integrals. However, it has a more desirable behaviour in financial modeling, while the Ayed-Kuo integral does not, as we will analyze in Chapter 6.
This chapter is organized as follows. First we give a definition for the Russo-Vallois integral and calculate some examples. We prove that this new setting does not satisfy the martingale property, the near-martingale property or the zero mean property. Then, we study an Itô formula for the Russo-Vallois integral, which has been proved in [37], and calculate some examples in order to show how the formula works. Finally, we study a solution for the Black-Scholes-Merton model under Russo-Vallois theory.

5.1 Definition of the Russo-Vallois Stochastic Integral

Let $B(t)$ be a Brownian motion $\{B(t), t \geq 0\}$ and let $\{\mathcal{F}_t, t \geq 0\}$ be the associated filtration, i.e., $\mathcal{F}_t = \sigma\{B(s), t \geq s \geq 0\}$, such that

(i) For each $t \geq 0$, $B(t)$ is $\mathcal{F}_t$-measurable;

(ii) For any $t \geq s \geq 0$, $B(t) - B(s)$ is independent of $\mathcal{F}_s$.

Definition 5.1 (Forward integrable stochastic process)**.** A stochastic process $\varphi = \varphi(t)$, $t \in [a,b]$, is said to be a *forward integrable stochastic process* (in the weak sense) over the interval $[a,b]$ with respect to the Brownian motion $\{B(t), t \in [a,b]\}$ if there exists a process $\{I(t), t \in [a,b]\}$, such that

$$\sup_{t\in[a,b]} \left| \int_a^t \frac{\varphi(s)B(s+\epsilon)-B(s)}{\epsilon} ds - I(t) \right| \to 0, \qquad \epsilon \to 0^+,$$

in probability. In this case, the forward integral of $\varphi(t)$ can defined by

$$I(t) = \int_a^t \varphi(s) d^- B(s), \qquad t \in [a,b],$$

with respect to $B(t)$ on $[a,b]$.

We can also define the Russo-Vallois integral in terms of Riemann sums, as F. Biagini and B. Øksendal proved in [5].

Lemma 5.1 (Russo-Vallois stochastic integral)**.** *Let φ be a càglàd and forward integrable stochastic process. Then, the Russo-Vallois integral can be defined by*

$$\int_a^b \varphi(s) d^- B(s) = \lim_{||\Delta_n|| \to 0} \sum_{i=1}^n \varphi(t_{i-1})(B(t_i) - B(t_{i-1})), \tag{5.1}$$

provided that the limit in probability exists, where $\Delta = \{a = t_0 < t_1 < t_2 < \dots < t_n = b\}$ is a partition of the interval $[a,b]$ and $||\Delta_n|| = \max_{1\le i\le n}(t_i - t_{i-1})$.

Proof. Let us assume that φ is a simple stochastic process. Hence, we get

$$\varphi(t) = \sum_{i=1}^n \varphi(t_{i-1}) \chi_{(t_{i-1},t_i]}(t), \qquad t \in [a,b].$$

By Fubini theorem, we have

$$\begin{aligned}
\int_a^b \varphi(s) d^- B(s) &= \lim_{\epsilon \to 0^+} \int_a^b \varphi(s) \frac{B(s+\epsilon) - B(s)}{\epsilon} ds \\
&= \sum_{i=1}^n \varphi(t_{i-1}) \lim_{\epsilon \to 0^+} \int_{t_{i-1}}^{t_i} \frac{B(s+\epsilon) - B(s)}{\epsilon} ds \\
&= \sum_{i=1}^n \varphi(t_{i-1}) \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_{t_{i-1}}^{t_i} \int_s^{s+\epsilon} dB(u) ds \\
&= \sum_{i=1}^n \varphi(t_{i-1}) \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_{t_{i-1}}^{t_i} \int_{u-\epsilon}^{u} ds dB(u) \\
&= \sum_{i=1}^n \varphi(t_{i-1})(B(t_i) - B(t_{i-1})).
\end{aligned}$$

□

Remark 5.1. Note that, in Lemma 5.1 the Riemann sums are an approximation to the Itô integral of φ with respect to the Brownian motion, if the integrand φ is $\mathcal{F}_t$-adapted to the filtration. Hence, in this case the Russo-Vallois, Itô and Ayed-Kuo integrals coincide.

Remark 5.2. By Equation (5.1), we can state that the Russo-Vallois integral is linear.

The following result is an immediate consequence of the Definition 5.1. It is a useful property in order to calculate forward stochastic processes.

Lemma 5.2 (G. Di Nunno, B. Øksendal, [8])**.** *Let φ be a forward integrable stochastic process and G a random variable. Then, the product $G\varphi$ is a forward integrable stochastic process and*

$$\int_a^b G\varphi(t)d^-B(t) = G\int_a^b \varphi(t)d^-B(t).$$

Next, we calculate some stochastic processes in order to show that the Russo-Vallois integral allows us to compute some anticipating integrals and to compare them with the results obtained with the Ayed-Kuo integral (see Section 4.1). In Section 5.3, we will also check that the results obtained by the definition coincide with the ones calculated by the extension of the Itô formula for the Russo-Vallois integral.

Example 5.1. Consider the stochastic process

$$\int_0^T B(T)d^-B(t).$$

By Lemma 5.2, we have

$$\begin{aligned} \boxed{\int_0^T B(T)d^-B(t)} &= B(T)\int_0^T dB(t) \\ &= \boxed{B(T)^2,} \end{aligned} \tag{5.2}$$

which does not coincide with the result calculated by the Ayed-Kuo integral (see Example 4.1).

Example 5.2. Consider the stochastic process

$$\int_0^T B(T)B(t)d^-B(t).$$

By Lemma 5.2, we have

$$\begin{aligned} \boxed{\int_0^T B(T)B(t)d^-B(t)} &= B(T)\int_0^T B(t)dB(t) \\ &= \frac{1}{2}B(T)\left(B(T)^2 - T\right) \\ &= \boxed{\frac{1}{2}B(T)^3 - \frac{1}{2}TB(T),} \end{aligned} \tag{5.3}$$

which does not coincide with the result calculated by the Ayed-Kuo integral (see Example 4.2).

Example 5.3. Consider the stochastic process

$$\int_0^t e^{B(T)} d^- B(t).$$

By Lemma 5.2, we have

$$\boxed{\int_0^T e^{B(T)} d^- B(t)} = e^{B(T)} \int_0^T dB(t) \tag{5.4}$$
$$= \boxed{e^{B(T)} B(T),}$$

which does not coincide with the result calculated by the Ayed-Kuo integral (see Example 4.3).

Example 5.4. Consider the stochastic process

$$\int_0^T B(t)(B(T)-B(t))\, d^- B(t).$$

By Lemma 5.2, we have

$$\begin{aligned}
\boxed{\int_0^T B(t)(B(T)-B(t))\, d^- B(t)} &= \int_0^T B(t)B(T) d^- B(t) - \int_0^T B(t)^2 dB(t) \\
&= B(T)\int_0^T B(t)dB(t) - \left(\frac{1}{3}B(T)^3 - \int_0^T B(t)dt\right) \\
&= \frac{1}{2}B(T)\left(B(T)^2 - T\right) - \left(\frac{1}{3}B(T)^3 - \int_0^T B(t)dt\right) \\
&= \boxed{\frac{1}{6}B(T)^3 - \frac{1}{2}TB(T) + \int_0^T B(t)dt,}
\end{aligned} \tag{5.5}$$

which does not coincide with the result calculated by the Ayed-Kuo integral (see Example 4.2, Equation (4.6) for detail).

5.2 Properties of the Russo-Vallois Stochastic Integral

In this section, our aim is to check if the Russo-Vallois integral satisfies some of the properties studied before for the Itô integral and the Ayed-Kuo integral, as zero mean property and the near-martingale property.

As we have proved in Section 4.2, the anticipating stochastic integral of Ayed-Kuo satisfies the near-martingale property, the zero mean property and we are also able to establish an extension of the Itô isometry for this setting. However, in this section we prove that the Russo-Vallois integral does not satisfy any of this properties, neither the martingale property. Indeed, it does not have the analytical structure of the Ayed-Kuo one.

5.2.1 Zero Mean Property

In Subsection 2.2.1 and in Subsection 4.2.1, it has been shown that the Itô integral and the Ayed-Kuo integral, respectively, satisfy the zero mean property.
Let us show a counterexample in order to illustrate that the Russo-Vallois integral does not have zero mean for every process.

Example 5.5. Consider the stochastic process from Example 5.1

$$\int_0^T B(T)d^-B(t) = B(T)^2.$$

We can verify

$$\begin{aligned}\mathbb{E}\left(\int_0^T B(T)d^-B(t)\right) &= \mathbb{E}\left(B(T)^2\right)\\ &= T,\end{aligned}$$

which is clearly not equal to 0.

Remark 5.3. Hence, we conclude that the Russo-Vallois integral does not satisfy the zero mean property.

5.2.2 Martingale and Near-Martingale Property

Next, we prove that the Russo-Vallois integral does not satisfy the near-martingale property, and consequently the martingale property is not satisfied either.

Remark 5.4. If the stochastic process $\{X_t\}$ is a martingale, then it is a near-martingale. On the other hand, if the stochastic process $\{X_t\}$ does not satisfy the near-martingale property, it does not fulfill the martingale property either.

As it was shown in Subsection 4.2.2, a stochastic process $\{X_t\}$ is said to be a near-martingale with respect to the filtration $\{\mathcal{F}_t\}$ if the mean is constant, it means

$$\mathbb{E}(X_t|\mathcal{F}_s) = \mathbb{E}(X_s|\mathcal{F}_s) \implies \mathbb{E}(\mathbb{E}(X_t|\mathcal{F}_s)) = \mathbb{E}(\mathbb{E}(X_s|\mathcal{F}_s)) \implies \mathbb{E}(X_t) = \mathbb{E}(X_s).$$

Example 5.6. Consider the stochastic process

$$\int_0^t B(T)d^-B(t), \quad 0 \le t \le T.$$

Following the above argument, we have

$$\begin{aligned}\mathbb{E}(B(T)B(t)) &= \mathbb{E}((B(T) - B(t) + B(t))B(t))\\ &= \mathbb{E}((B(T) - B(t))B(t)) + \mathbb{E}\left(B(t)^2\right)\\ &= t,\end{aligned}$$

which is clearly not constant.

Remark 5.5. Hence, we conclude that the Russo-Vallois integral does not satisfy the near-martingale property. Therefore, the martingale property is not satisfied either.

5.3 An extension of the Itô formula for the Russo-Vallois Stochastic Integral

In this section, we extend the Itô formula established in Section 2.3 for the Russo-Vallois integral. This formula is proposed in [37, 38].
First, it is convenient to introduce an analogous notation to the classical one for Itô processes.

Definition 5.2 (Forward process). A forward process with respect to $B(t)$ is a stochastic process of the form

$$X_t = X_a + \int_a^t u(s)ds + \int_a^t v(s)d^-B(s), \qquad t \in [a,b], \tag{5.6}$$

where

$$\int_a^b |u(s)|ds < \infty,$$

and v is a forward integrable stochastic process. In differential form,

$$d^-X_t = u(t)dt + v(t)d^-B(t). \tag{5.7}$$

Next, we present the extension of the Itô formula for the Russo-Vallois integral. The proof can be found in [38]. We only make a brief sketch of the proof as it is highly extensive.

Theorem 5.1 (F. Russo, P. Vallois, [38]). *Let*

$$d^-X_t = u(t)dt + v(t)d^-B(t),$$

be a forward process. Let $\theta \in \mathcal{C}^{1,2}([a,b]\times\mathbb{R})$ *and define*

$$Y^{(t)} = \theta(t, X_t), \qquad t \in [a,b].$$

Then, $Y^{(t)}$*, for* $t \in [a,b]$*, is a forward process and*

$$d^-Y^{(t)} = \frac{\partial\theta}{\partial t}(t,X_t)dt + \frac{\partial\theta}{\partial x}(t,X_t)d^-X_t + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}(t,X_t)v^2(t)dt. \tag{5.8}$$

Sketch of proof. Let $\theta(t,x) = \theta(x)$ for $t \in [a,b]$ and $x \in \mathbb{R}$. Let $\Delta = \{a = t_0 < t_1 < ... < t_n = b\}$ be a partition of the interval $[a,b]$. Then, by Taylor expansion, we have, for some point $\bar{X}_i \in [X_{t_{i-1}}, X_{t_i}]$

$$\begin{aligned} \theta(X_t) - \theta(X_0) &= \sum_{i=1}^n \theta\left(X_{t_i}\right) - \theta\left(X_{t_{i-1}}\right) \\ &= \sum_{i=1}^n \theta'\left(X_{t_{i-1}}\right)\left(X_{t_i} - X_{t_{i-1}}\right) \\ &+ \frac{1}{2}\sum_{i=1}^n \theta''\left(\bar{X}_i\right)\left(X_{t_i} - X_{t_{i-1}}\right)^2. \end{aligned} \tag{5.9}$$

By Lemma 5.1, we have

$$
\begin{aligned}
\sum_{i=1}^{n}\theta'\left(X_{t_{i-1}}\right)\left(X_{t_i}-X_{t_{i-1}}\right) &= \sum_{i=1}^{n}\theta'\left(X_{t_{i-1}}\right)\left(\int_{t_{i-1}}^{t_i}u(s)ds+\int_{t_{i-1}}^{t_i}v(s)d^-B(s)\right)\\
&=\sum_{i=1}^{n}\left(\int_{t_{i-1}}^{t_i}\theta'\left(X_{t_{i-1}}\right)u(s)ds+\int_{t_{i-1}}^{t_i}\theta'\left(X_{t_{i-1}}\right)v(s)d^-B(s)\right)\\
&=\int_a^b\left(\sum_{i=1}^{n}\theta'\left(X_{t_{i-1}}\right)\chi_{(t_{i-1},t_i]}(s)\right)u(s)ds\\
&\quad+\int_a^b\left(\sum_{i=1}^{n}\theta'\left(X_{t_{i-1}}\right)\chi_{(t_{i-1},t_i]}(s)\right)v(s)d^-B(s)\\
&\to\int_a^b\theta'(X_s)u(s)ds+\int_a^b\theta'(X_s)v(s)d^-B(s)\\
&=\int_a^b\theta'(X_s)d^-X_s,
\end{aligned}
\tag{5.10}
$$

as $||\Delta_n||\to 0$, with convergence in probability. As in the classical case, one can also prove

$$
\sum_{i=1}^{n}f''\left(\bar{X}_i\right)\left(X_{t_i}-X_{t_{i-1}}\right)^2\to\int_a^b f''(X_s)v^2(s)ds,
\tag{5.11}
$$

$||\Delta_n||\to 0$, in probability. Combining Equation (5.9), Equation (5.10) and Equation (5.11), we obtain the result from Equation (5.8). □

Next, we apply the obtained formula to the stochastic processes previously introduced in Section 5.1. We check that the results calculated by the definition and the formula coincide.

Example 5.7. Consider the stochastic process introduced in Example 5.2

$$\int_0^T B(T)B(t)d^-B(t).$$

According to Theorem 5.1, we consider the function $\theta(t,x)=B(T)x^2/2$, whose partial derivatives are

$$\frac{\partial\theta}{\partial t}=0,\qquad\frac{\partial\theta}{\partial x}=B(T)x,\qquad\frac{\partial^2\theta}{\partial x^2}=B(T).$$

Thus, we get

$$d^-\left(B(T)B(t)\right)=B(T)B(t)d^-B(t)+\frac{1}{2}B(T)dt.$$

Integrating in both sides of the equality from 0 to T, we have

$$\int_0^T d^-\left(B(T)B(t)^2/2\right)d^-B(t)=\int_0^T B(T)B(t)d^-B(t)+\frac{1}{2}\int_0^T B(T)dt.$$

Hence, we get

$$
\begin{aligned}
\boxed{\int_0^T B(T)B(t)d^-B(t)} &= \int_0^T d^-\left(B(T)B(t)^2/2\right)d^-B(t)-\frac{1}{2}\int_0^T B(T)dt\\
&=\frac{1}{2}B(T)^3-\frac{1}{2}TB(T)\\
&=\boxed{\frac{1}{2}B(T)\left(B(T)^2-T\right),}
\end{aligned}
$$

which coincides with the result obtained in Example 5.2.

Example 5.8. Consider the stochastic process introduced in Example 5.1

$$\int_0^T e^{B(T)} d^- B(t).$$

According to Theorem 5.1, we consider the function $\theta(t,x) = e^{B(T)}x$, whose partial derivatives are

$$\frac{\partial \theta}{\partial t} = 0, \qquad \frac{\partial \theta}{\partial x} = e^{B(T)}, \qquad \frac{\partial^2 \theta}{\partial x^2} = 0.$$

Thus, we get

$$d^- \left(e^{B(T)} B(t) \right) = e^{B(T)} d^- B(t).$$

Integrating in both sides of the equality from 0 to T, we have

$$\int_0^T d^- \left(e^{B(T)} B(t) \right) d^- B(t) = \int_0^T e^{B(T)} d^- B(t).$$

Hence, we get

$$\boxed{\int_0^T e^{B(T)} d^- B(t)} = \int_0^T d^- \left(e^{B(T)} B(t) \right) d^- B(t)$$
$$= \boxed{e^{B(T)} B(T),}$$

which coincides with the result obtained in Example 5.3.

Example 5.9. Consider the stochastic process introduced in Example 5.4

$$\int_0^T B(t)(B(T) - B(t)) d^- B(t).$$

By linearity, we have

$$\int_0^T B(t)(B(T) - B(t)) d^- B(t) = \int_0^T B(t)B(T) d^- B(t) - \int_0^T B(t)^2 d^- B(t),$$

where the first integral of the right-hand side of the equality is the stochastic process from the Example 5.7 and we have already calculated the solution. For the second integral, according to Theorem 5.1, we consider the function $\theta(t,x) = x^3/3$, whose partial derivatives are

$$\frac{\partial \theta}{\partial t} = 0, \qquad \frac{\partial \theta}{\partial x} = x^2, \qquad \frac{\partial^2 \theta}{\partial x^2} = 2x.$$

Thus, we get

$$d^- \left(B(t)^3/3 \right) = B(t)^2 d^- B(t) + B(t) dt.$$

Integrating in both sides of the equality from 0 to T, we have

$$\int_0^T d^- \left(B(t)^3/3 \right) d^- B(t) = \int_0^T B(t)^2 d^- B(t) + \int_0^T B(t) dt.$$

Then, we get

$$\boxed{\int_0^T B(t)^2 d^- B(t)} = \int_0^T d^-\left(B(t)^3/3\right) d^- B(t) - \int_0^T B(t)dt$$
$$= \boxed{\frac{1}{3}B(T)^3 - \int_0^T B(t)dt.}$$

Hence, combining both integrals we have

$$\boxed{\int_0^T B(t)(B(T)-B(t))d^- B(t)} = \int_0^T B(t)B(T)d^- B(t) - \int_0^T B(t)^2 d^- B(t)$$
$$= \frac{1}{2}B(T)\left(B(T)^2 - T\right) - \frac{1}{3}B(T)^2 + \int_0^T B(t)dt$$
$$= \boxed{\frac{1}{6}B(T)^3 - \frac{1}{2}TB(T) + \int_0^T B(t)dt,}$$

which coincides with the result obtained in Example 5.4 (see Equation (5.5) for detail).

5.4 Stochastic Differential Equations with Anticipating Initial Conditions for the Russo-Vallois Stochastic Integral

In this final section, our aim is to study a solution for a particular linear stochastic differential equation, which has already been introduced in Section 3.3 and in Subsection 4.4.2. We study the Black-Scholes-Merton model under Russo-Vallois theory.

5.4.1 Black-Scholes-Merton Model under Russo-Vallois Theory

Consider the linear stochastic differential equation for the Black-Scholes-Merton model, with a slightly modification, as in the Ayed-Kuo setting

$$\begin{cases} d^- S_t = \sigma S_t d^- B(t) + \mu S_t dt, \\ S_0 = B(T). \end{cases} \tag{5.12}$$

Let us also consider the same solution as for the classical linear stochastic differential equation from Itô theory

$$S_t = B(T)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B(t)}. \tag{5.13}$$

Then, the function $\theta(t,x)$ chosen is

$$\theta(t,x) = B(T)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma x},$$

whose partial derivatives are

$$\begin{cases} \frac{\partial\theta}{\partial t} = \left(\mu - \frac{1}{2}\sigma^2\right) B(T) e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x}, \\ \frac{\partial\theta}{\partial x} = \sigma B(T) e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x}, \\ \frac{\partial^2\theta}{\partial x^2} = \sigma^2 B(T) e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x}. \end{cases}$$

According to Theorem 5.1, we have

$$\begin{aligned} d^-\left(B(T)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma B(t)}\right) &= \left(\mu - \frac{1}{2}\sigma^2\right) B(T)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x}dt + \sigma^2 B(T)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x}dt \\ &+ \sigma B(T)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x} d^- B(t) \\ &= \mu B(T)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x}dt \cancel{- \frac{1}{2}\sigma^2 B(T)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x}dt} \\ &\cancel{+ \sigma^2 B(T)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x}}dt + \sigma B(T)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x} d^- B(t) \\ &= \mu B(T)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x}dt + \sigma B(T)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x} d^- B(t). \end{aligned}$$

Finally, we check whether Equation (5.13) is a solution of the stochastic differential equation (5.12)

$$d^-\left(B(T)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma B(t)}\right) = \mu B(T)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x}dt + \sigma B(T)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x} d^- B(t),$$

which yields, according to the results studied in this chapter, that

$$\begin{aligned} d^-(S_t) &= B(T)\mu e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x}dt + \sigma B(T)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma x} d^- B(t) \\ &= \mu S_t dt + \sigma S_t d^- B(t). \end{aligned}$$

Remark 5.6. Note that, in spite of the anticipating initital condition, the solution obtained for the stochastic differential equation is equivalent to the one calculated by the classical Itô theory.

Now, let us prove the existence and uniqueness of the solution proposed. The existence has been proved by a guess based on Itô calculus. Then, we have left to check that S_t in Equation (5.13) is the unique solution of the stochastic differential equation in SDE (5.12).

By reducing to absurd, let X_t be another solution to the SDE (5.12). Hence, we have

$$S_t = B(T) + \mu \int_0^t S_u d(u) + \sigma \int_0^t S_u d^- B(u),$$

and

$$X_t = B(T) + \mu \int_0^t X_u d(u) + \sigma \int_0^t X_u d^- B(u).$$

If we define $Z_t = S_t - X_t$, for all $0 \leq t \leq T$, we have that Z_t is a stochastic process satisfying

$$\begin{cases} Z_t = \mu \int_0^t Z_u d(u) + \sigma \int_0^t Z_u d^-(u), \qquad 0 \leq t \leq T, \\ Z_0 = 0. \end{cases}$$

This linear stochastic differential equation is a non-anticipating Black-Scholes-Merton type of equation, whose unique solution is given by

$$Z_t = Z_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B(t)}.$$

The uniqueness of this solution is guaranteed by Theorem 3.1. Since $Z_0 = 0$, we have

$$\mathbb{P}(Z_t = 0, \text{ for all } 0 \le t \le T) = 1,$$

such that

$$\mathbb{P}(S_t = X_t, \text{ for all } 0 \le t \le T) = 1.$$

Hence, we conclude that S_t from Equation (5.13) is the same solution as X_t, and the unique solution of the SDE (5.12).

CHAPTER 6

FINANCIAL MODELING

The aim of this final chapter is to transpose the results studied among this dissertation into an specific problem of financial modeling. As we have discussed in Chapter 3, we are able model stock price behaviour with the stochastic differential equations theory. Then, in Subsection 4.4.2 and in Subsection 5.4.1, we propose an extension of the Black-Scholes-Merton model, which deals with the anticipating stochastic calculus, for the Ayed-Kuo and the Russo-Vallois stochastic integrals respectively.

The problem we are presenting in this chapter is called the insider trading. Consider a trader who has privileged information from the financial markets, where our aim is to model this approach with stochastic calculus. The logic encourages us to related this idea to the anticipating stochastic theory explained among this thesis.

This chapter is organized as follows. First, we introduce a simplified version of the insider trading. Then, we propose a solution for this problem under the Ayed-Kuo and the Russo-Vallois stochastic integration theories, and we compare both alternatives. We discuss that the Russo-Vallois integral has a more desirable solution in the financial sense, while the Ayed-Kuo setting does not, at least for this version of the insider trading problem. Finally, we propose two new theorems that we have proved in this work. In these, we establish the optimal investment strategy for both integrals according to their solutions for the general version of insider trading.

6.1 The Insider Trading Problem

The *insider trading* is the trading of stocks by individuals with access to privileged information from those securities. This idea is clearly related to the anticipating stochastic theory that we have already discussed among this dissertation.

Let us consider a simplified version of the problem of insider trading in the financial market. We approach it by means of anticipating stochastic calculus. Let us start from the classical financial model (see Section 3.3 for detail) with one asset free of risk, the bond

$$\begin{cases} dS_t^0 = \rho S_t^0 dt, \\ S_0^0 = M_0, \end{cases} \tag{6.1}$$

and a risky asset, the stock

$$\begin{cases} dS_t^1 = \mu S_t^1 dt + \sigma S_t^1 dB(t), \\ S_0^1 = M_1, \end{cases} \tag{6.2}$$

where M_0, M_1, ρ, μ and σ are constants. We have the following financial meaning to all the variables that make up the model

$$\begin{cases} M_0 \equiv \text{initial wealth invested in the bond;} \\ M_1 \equiv \text{initial wealth invested in the stock;} \\ \rho \equiv \text{interest rate of the bond } S_t^0; \\ \mu \equiv \text{appreciation rate of the stock } S_t^1; \\ \sigma \equiv \text{ volatility of the stock } S_t^1; \\ B(t) \equiv \text{standard brownian motion;} \\ S_0^1 \equiv \text{spot price observed at time t=0.} \end{cases}$$

Let us assume that $\mu > \rho$ because of the risk-return binomial. Also, we consider that the trader has a fixed total wealth M at the initial time $t = 0$ and is free to choose what fraction is invested in each asset. Then, the total initial wealth invested by the trader is

$$M = M_0 + M_1.$$

Clearly, at any time $t > 0$, the total wealth is given by

$$S_t = S_t^0 + S_t^1.$$

We consider this financial market on $[0, T]$ for a fixed future time $T > 0$. Then, we have the following results for the classical financial theory.

Theorem 6.1 (J. Bastons, C. Escudero, [4]). *The expected value of the total wealth at time $t = T$ is*

$$\mathbb{E}(S_T) = M_0 e^{\rho T} + M_1 e^{\mu T},$$

for ODE (6.1) and SDE (6.2).

Proof. By Itô theory, we have the following solutions to ODE (6.1) and SDE (6.2) respectively

$$\begin{cases} S_t^0 = M_0 e^{\rho t}, \\ S_t^1 = M_1 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma B(T)\right). \end{cases}$$

Hence, the expectation of S_t at time $t = T$ is

$$\begin{aligned} \mathbb{E}(S_T) &= \mathbb{E}\left(S_T^0\right) + \mathbb{E}\left(S_T^1\right) \\ &= M_0 \mathbb{E}\left(e^{\rho t}\right) + M_1 \mathbb{E}\left(\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma B(T)\right)\right) \\ &= M_0 e^{\rho T} + M_1 e^{\mu T}. \end{aligned}$$

□

Corollary 6.1. *The optimal investment strategy for ODE (6.1) and SDE (6.2) is*

$$M_0 = 0 \qquad and \qquad M_1 = M.$$

Remark 6.1. The aim of the trader is to maximize the expected wealth at time $t = T$. As we have assumed that $\mu > \rho$, the maximal expected wealth is

$$\mathbb{E}(S_T) = M e^{\mu T}, \tag{6.3}$$

which is the one obtained by the *investment strategy* established in Corollary 6.1.

Remark 6.2. Combining the martingale property for the Itô integral and the assumption of μ being the expected rate of return of the risky asset, we have that SDE (6.2) is an Itô stochastic differential equation.

According to Remark 6.2, things should be different under the assumption of the trader possessing privileged information with respect to the one contained in the filtration generated by $B(T)$ at time $t = 0$. The honest trader will choose the strategy proposed in Remark 6.2, while the dishonest trader, it means the insider trader, will take advantage of privileged information.

Let us consider the following anticipating situation. Our trader is an insider who has some privileged information on the future price of the stock. Specifically, the trader knows at the initial time $t = 0$ the value $B(T)$. Therefore, the value S_T. However, in our simplification of the problem we assume that the trader does not fully trust this information. Then, we use an adjustment of the information for the initial condition, in order to take advantage of the privilege of the trader. The strategy assumed for the bond is

$$\begin{cases} dS_t^0 = \rho S_t^0 dt, \\ S_0^0 = M\left(\frac{\sigma^2 T/2 - \sigma B(T)}{2(\mu-\rho)T}\right), \end{cases} \tag{6.4}$$

and, for the stock

$$\begin{cases} dS_t^1 = \mu S_t^1 dt + \sigma S_t^1 dB(t), \\ S_0^1 = M\left(1 + \frac{\sigma B(T) - \sigma^2 T/2}{2(\mu-\rho)T}\right). \end{cases} \tag{6.5}$$

This strategy is linear in $B(T)$. However, it is also a form to introduce some trust of the insider trader in the privileged information that the trader possesses.

Remark 6.3. The modulation of the strategy imposes the following assumptions:

- The amount invested at the initial time $t = 0$ is the same for the bond and the stock, if their values at time $t = T$ are equal for a given common initial investment.
- The amount invested in the bond is null whenever the realization of the Brownian motion yields the average result of Equation (6.3).
- The strategy allows negative values for the investment, which means that the trader can borrow money.

Remark 6.4. Note that, the problem formulated in SDE (6.5) is ill-posed, while problem described in ODE (6.4) can be regarded as an ordinary differential equation subject to a random initial condition. The anticipating initial condition makes the stochastic differential equation ill-defined in the Itô sense. If we change the notion of Itô stochastic integration to another one that considers anticipating integrands, the problem from SDE (6.5) might be well-posed.

In the following section, we discuss how the anticipating settings studied among this dissertation, the Ayed-Kuo stochastic integral and the Russo-Vallois stochastic integral, find a solution for this problem, and we compare the results obtained for each one. Both of these anticipating stochastic integrals guarantee the well-posednesss of this version of the insider trading problem. However, in the financial sense, the solution given by each integral might be different. In particular, we will check that the Russo-Vallois integral gives a more desirable solution, while the Ayed-Kuo solution seems to be counterintuitive in the financial sense.

6.2 Comparison between Ayed-Kuo and Russo-Vallois integration for Insider Trading

Let us consider the notation and results from Chapter 4. For the Ayed-Kuo stochastic integral, we arrive at the initial value problem

$$\begin{cases} dS_t^1 = \mu S_t^1 dt + \sigma S_t^1 dB(t), \\ S_0^1 = M\left(1 + \frac{\sigma B(T) - \sigma^2 T/2}{2(\mu-\rho)T}\right), \end{cases} \tag{6.6}$$

for a Ayed-Kuo stochastic differential equation. The existence and uniqueness of solutions for linear stochastic differential equations has been already proved in Section 4.4.

Theorem 6.2 (J. Bastons, C. Escudero, [4])**.** *The expected value of the total wealth of the insider at time $t = T$ under Ayed-Kuo theory is*

$$\mathbb{E}\left(S_T^{(AK)}\right) = M\left(\frac{\sigma^2}{4(\mu-\rho)}e^{\rho T} + \left(1 - \frac{\sigma^2}{4(\mu-\rho)}\right)e^{\mu T}\right),$$

for ODE (6.4) and SDE (6.6).

Proof. According to Theorem 4.5, we have that the solution of SDE (6.6) is

$$S_t^1 = M\left(1 + \frac{\sigma B(T) - \sigma^2 t - \sigma^2 T/2}{2(\mu-\rho)T}\right)\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B(t)\right). \tag{6.7}$$

Let us recall that the total wealth of the insider trader is

$$S_T^{(AK)} = S_t^0 + S_t^1,$$

such that, for the expected wealth at time $t = T$ we have

$$\mathbb{E}\left(S_T^{(AK)}\right) = \mathbb{E}\left(S_T^0\right) + \mathbb{E}\left(S_T^1\right).$$

Hence, at maturity time $t = T$, we get

$$\begin{aligned}
\mathbb{E}\left(S_T^{(AK)}\right) &= \mathbb{E}\left(S_T^0\right) + \mathbb{E}\left(S_T^1\right) \\
&= \mathbb{E}\left(M\frac{\sigma^2 T/2 - \sigma B(T)}{2(\mu-\rho)T}\right)e^{\rho T} \\
&+ \mathbb{E}\left(M\left(1 + \frac{\sigma B(T) - 3\sigma^2 T/2}{2(\mu-\rho)T}\right)\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma B(T)\right)\right) \\
&= M\frac{\sigma^2 T/2 - \sigma\mathbb{E}(B(T))}{2(\mu-\rho)T}e^{\rho T} \\
&+ M\frac{\sigma}{2(\mu-\rho)T}\mathbb{E}\left(B(T)\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma B(T)\right)\right) \\
&+ M\left(1 - \frac{3\sigma^2}{4(\mu-\rho)}\right)\mathbb{E}\left(\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma B(T)\right)\right) \\
&= M\frac{\sigma^2}{4(\mu-\rho)}e^{\rho T} + M\frac{\sigma^2}{2(\mu-\rho)}e^{\mu T} + M\left(1 - \frac{3\sigma^2}{4(\mu-\rho)}\right)e^{\mu T} \\
&= M\left(\frac{\sigma^2}{4(\mu-\rho)}e^{\rho T} + \left(1 - \frac{\sigma^2}{4(\mu-\rho)}\right)e^{\mu T}\right),
\end{aligned}$$

where $B(T) \sim \mathcal{N}(0, T)$. □

Corollary 6.2 (J. Bastons, C. Escudero, [4])**.** *The expected value of the total wealth of the insider at time $t = T$ is strictly smaller than that of the honest trader*

$$\mathbb{E}\left(S_T^{(AK)}\right) < \mathbb{E}\left(S_T^{(IT\hat{O})}\right).$$

Remark 6.5. The statement from Corollary 6.2 implies that the Ayed-Kuo stochastic integration does not take advantage of the anticipating condition. Hence, in the financial sense, we might say that the Ayed-Kuo integral does not work, at least for this version of insider trading.

Next, let us consider the notation and results from Chapter 5. For the Russo-Vallois stochastic integral, we arrive at the initial value problem

$$\begin{cases} d^- S_t^1 = \mu S_t^1 dt + \sigma S_t^1 d^- B(t), \\ S_0^1 = M\left(1 + \frac{\sigma B(T) - \sigma^2 T/2}{2(\mu-\rho)T}\right). \end{cases} \tag{6.8}$$

Theorem 6.3 (J. Bastons, C. Escudero, [4])**.** *The expected value of the total wealth of the insider at time $t = T$ under Russo-Vallois theory is*

$$\mathbb{E}\left(S_T^{(RV)}\right) = M\left(\frac{\sigma^2}{4(\mu-\rho)} e^{\rho T} + \left(1 + \frac{\sigma^2}{4(\mu-\rho)}\right) e^{\mu T}\right),$$

for ODE (6.4) and SDE (6.8).

Proof. The Russo-Vallois integral preserves Itô calculus. Hence, using the classical stochastic calculus, we have that the solution of SDE (6.8) is

$$S_t^1 = M\left(1 + \frac{\sigma B(T) - \sigma^2 T/2}{2(\mu-\rho)T}\right) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B(t)\right). \tag{6.9}$$

Let us remind that the total wealth of the insider trader is

$$S_T^{(RV)} = S_t^0 + S_t^1,$$

such that, for the expected wealth at time $t = T$, we have

$$\begin{aligned}
\mathbb{E}\left(S_T^{(RV)}\right) &= \mathbb{E}\left(S_T^0\right) + \mathbb{E}\left(S_T^1\right) \\
&= \mathbb{E}\left(M \frac{\sigma^2 T/2 - \sigma B(T)}{2(\mu-\rho)T}\right) e^{\rho T} \\
&\quad + \mathbb{E}\left(M\left(1 + \frac{\sigma B(T) - \sigma^2 T/2}{2(\mu-\rho)T}\right) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma B(T)\right)\right) \\
&= M \frac{\sigma^2 T/2 - \sigma \mathbb{E}(B(T))}{2(\mu-\rho)T} e^{\rho T} \\
&\quad + M \frac{\sigma}{2(\mu-\rho)T} \mathbb{E}\left(B(T) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma B(T)\right)\right) \\
&\quad + M\left(1 - \frac{\sigma^2}{4(\mu-\rho)}\right) \mathbb{E}\left(\exp\left(\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma B(T)\right)\right) \\
&= M \frac{\sigma^2}{4(\mu-\rho)} e^{\rho T} + M \frac{\sigma^2}{2(\mu-\rho)} e^{\mu T} + M\left(1 - \frac{\sigma^2}{4(\mu-\rho)}\right) e^{\mu T} \\
&= M\left(\frac{\sigma^2}{4(\mu-\rho)} e^{\rho T} + \left(1 + \frac{\sigma^2}{4(\mu-\rho)}\right) e^{\mu T}\right),
\end{aligned}$$

where $B(T) \sim \mathcal{N}(0, T)$. □

Corollary 6.3 (J. Bastons, C. Escudero, [4])**.** *The expected value of the total wealth of the insider at time $t = T$ is strictly larger than that of the honest trader*

$$\mathbb{E}\left(S_T^{(RV)}\right) > \mathbb{E}\left(S_T^{(IT\hat{O})}\right).$$

Remark 6.6. The statement from Corollary 6.3 implies that the Russo-Vallois stochastic integration does take advantage of the anticipating condition. Hence, in the financial sense, we might say that the Russo-Vallois integral works, at least for this version of insider trading.

In the next theorem, we show that the expected value of the wealth of the Ayed-Kuo insider is always strictly smaller than that of the expected value of the wealth of the Russo-Vallois insider.

Theorem 6.4 (J. Bastons, C. Escudero, [4])**.** *The respective solutions to the initial value problems*

$$\begin{cases} dS_t^{(AK)} = \mu S_t^{(AK)} dt + \sigma S_t^{(AK)} dB(t), \\ S_0^{(AK)} = \mathcal{C}(B(T)), \end{cases} \tag{6.10}$$

and

$$\begin{cases} d^- S_t^{(RV)} = \mu S_t^{(RV)} dt + \sigma S_t^{(RV)} d^- B(t), \\ S_0^{(RV)} = \mathcal{C}(B(T)), \end{cases} \tag{6.11}$$

where $\mathcal{C}(\cdot)$ denotes an arbitrary monotonically increasing function that is both non-constant and continuous, satisfy

$$\mathbb{E}\left(S_T^{(AK)}\right) < \mathbb{E}\left(S_T^{(RV)}\right).$$

Proof. The solutions of SDE (6.10) and SDE (6.11) can be computed by the calculus rules for the Ayed-Kuo integral and the Russo-Vallois integral respectively. Hence, we get

$$S_t^{(AK)} = \mathcal{C}(B(T) - \sigma t) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B(t)\right),$$

and

$$S_t^{(RV)} = \mathcal{C}(B(T)) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B(t)\right).$$

By monotonicity, we have, for all $t > 0$

$$\mathcal{C}(B(T) - \sigma t) \le \mathcal{C}(B(T)),$$

with the inequality being strict for $B(T)$ taking values in at least some interval of $\mathbb{R}$. Hence, we get

$$\begin{aligned} \mathbb{E}\left(S_t^{(AK)}\right) &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} \mathcal{C}(B(T) - \sigma T) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma B(T)\right) \exp\left(-\frac{B(T)^2}{2T}\right) dB(T) \\ &< \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} \mathcal{C}(B(T)) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma B(T)\right) \exp\left(-\frac{B(T)^2}{2T}\right) dB(T) \\ &= \mathbb{E}\left(S_t^{(RV)}\right). \end{aligned}$$

□

Corollary 6.4. *According to the results from Corollary 6.2, Corollary 6.3 and Theorem 6.4, we have*

$$\mathbb{E}\left(S_T^{(AK)}\right) < \mathbb{E}\left(S_T^{(IT\hat{O})}\right) < \mathbb{E}\left(S_T^{(RV)}\right).$$

Remark 6.7. Note that, we can conclude that the Ayed-Kuo integral underestimates the expected wealth of the insider, while the Russo-Vallois has a more desirable behaviour in the financial sense, at least for this simplified version of the insider trading problem.

6.3 Optimal Investment Strategy for Insider Trading

In this final section, we discuss about the investment strategies for the general version of the insider trading problem. Indeed, the aim of any trader is to maximize the expected wealth at maturity time $t = T$. As a result of the work done in this thesis, we are able to prove two theorems that state which is the *optimal investment strategy* for the Ayed-Kuo and the Russo-Vallois integration theories.

Remark 6.8. Note that, we consider a insider trader who knows at the initial time $t = 0$ the value $B(T)$. Hence, the value S_T. In the financial sense, the optimal investment strategy is clear for an anticipating initial condition $f(B(T))$. The insider trader should invest all the amount M in the asset whose expected wealth is larger at maturity time $t = T$.

Let us consider the more general version of the insider trading problem, it means, without making adjustments on the anticipating initial condition. Let us also recall that we assume to be $\mu > \rho$, because of the risk-return binomial. Hence, we have the following strategy for the bond

$$\begin{cases} dS_t^0 = \rho S_t^0 dt, \\ S_0^0 = M(1 - f(B(T))), \end{cases} \tag{6.12}$$

and, for the stock

$$\begin{cases} dS_t^1 = \mu S_t^1 dt + \sigma S_t^1 dB(t), \\ S_0^1 = M(f(B(T))), \end{cases} \tag{6.13}$$

where f is a function of $B(T)$, such that $f \in L^\infty(\mathbb{R})$ and $0 \le f \le 1$.

In the next theorem, we prove that the investment strategy that maximizes the expected wealth for the Ayed-Kuo integration is to invest all the amount M into the stock asset. Indeed, we have the same strategy as in the Itô classical model without privileged information (see Corollary 6.1).

Theorem 6.5. *Let f be a function of $B(T)$ such that $f \in \mathcal{C}(\mathbb{R})$ and $0 \le f \le 1$. The optimal investment strategy for ODE (6.12) and SDE (6.13) under Ayed-Kuo integration is*

$$f(B(T)) = 1.$$

Proof. The solutions of ODE (6.12) and SDE (6.13) can be computed by the calculus rules for the Ayed-Kuo integral. Hence, we get

$$\begin{cases} S_t^0 = M(1-f(B(T)))e^{\rho T}, \\ S_t^1 = Mf(B(T)-\sigma T)e^{(\mu-\sigma^2/2)T+\sigma B(T)}. \end{cases} \tag{6.14}$$

The aim is to find the strategy f, such that $\mathbb{E}(M_t)$ is maximized. Indeed, we have

$$\begin{aligned}
\mathbb{E}(M_t) &= \mathbb{E}\left(S_t^0\right) + \mathbb{E}\left(S_t^1\right)\\
&= M(1-\mathbb{E}(f(B(T))))e^{\rho T} + M\mathbb{E}\left(f(B(T)-\sigma T)e^{\sigma B(T)}\right)e^{(\mu-\sigma^2/2)T}\\
&= Me^{\rho T}\left(1-\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi T}}f(x)e^{-\frac{x^2}{2T}}dx\right) + Me^{(\mu-\sigma^2/2)T}\left(\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi T}}f(x-\sigma T)e^{-\frac{x^2}{2T}}e^{\sigma x}dx\right).
\end{aligned}$$

Let us consider the change of variable

$$y = x - \sigma T,$$

and

$$\bar{M}(f(x)) = \frac{\mathbb{E}(M_t)}{M}.$$

Hence, we get

$$\begin{aligned}
\bar{M}(f(x)) &= e^{\rho T}\left(1-\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi T}}f(x)e^{-\frac{x^2}{2T}}dx\right) + e^{(\mu-\sigma^2/2)T}\left(\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi T}}f(x-\sigma T)e^{-\frac{x^2}{2T}}e^{\sigma x}dx\right)\\
&= e^{\rho T} - e^{\rho T}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi T}}f(x)e^{-\frac{x^2}{2T}}dx + \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi T}}f(y)e^{-\frac{(y+\sigma T)^2}{2T}}e^{(\mu-\sigma^2/2)T}e^{\sigma(y+\sigma T)}dy\\
&= e^{\rho T} - e^{\rho T}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi T}}f(x)e^{-\frac{x^2}{2T}}dx + \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi T}}f(y)e^{-\frac{(y^2+\sigma^2T^2+2y\sigma T)}{2T}}e^{\mu T-\sigma^2/2T}e^{\sigma y+\sigma^2 T}dy\\
&= e^{\rho T} - e^{\rho T}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi T}}f(x)e^{-\frac{x^2}{2T}}dx + e^{\mu T}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi T}}f(y)e^{-\frac{y^2}{2T}}dy.
\end{aligned}$$

Note that, we have

$$\mathbb{E}\left(\bar{M}_t\right) = e^{\rho T} - e^{\rho T}\mathbb{E}(f(B(T))) + e^{\mu T}\mathbb{E}(f(B(T))).$$

By assumption, f is a function of $B(T)$ such that $0 \le f \le 1$, and we have that $\mu > \rho$. Hence, since the exponential function is strictly monotone, we have that $\mathbb{E}(M_t) \in \left[e^{\rho T}, e^{\mu T}\right]$ and, in order to maximize $\mathbb{E}(M_t)$, we get

$$\mathbb{E}(f(B(T))) = 1.$$

Then, we have

$$\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty}f(x)e^{-\frac{x^2}{2T}}dx = 1,$$

such that, $f(B(T)) = 1$. □

Remark 6.9. Let us consider f such that $f \in L^\infty(\mathbb{R})$ and $0 \le f \le 1$. Let us also consider f_n to be a sequence of functions, such that $f_n \in \mathcal{C}(\mathbb{R})$, $n \in \mathbb{N}$, and $0 \le f_n \le 1$. Then, we have

$$\begin{aligned}
\left|\mathbb{E}(f(B(T))) - \mathbb{E}(f_n(B(T)))\right| &= \left|\mathbb{E}(f(B(T)) - f_n(B(T)))\right|\\
&\le \mathbb{E}\left(\left|f(B(T)) - f_n(B(T))\right|\right)\\
&= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty}\left|f(x)-f_n(x)\right|e^{-\frac{x^2}{2T}}dx.
\end{aligned}$$

By *Lusin theorem*, there exists a family of f_n such that

$$\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty}\left|f(x)-f_n(x)\right|e^{-\frac{x^2}{2T}}dx \to 0,$$

as $n \to \infty$. Hence, we have

$$\mathbb{E}(f_n(B(T))) \to \mathbb{E}(f(B(T))),$$

and if f is such that $f \in L^{\infty}(\mathbb{R})$, instead of being $f \in \mathcal{C}(\mathbb{R})$, the solution does not get better.

Remark 6.10. The Theorem 6.5 reaffirms the argument that the Ayed-Kuo theory does not take advantage of the anticipating condition, as its optimal investment strategy is the same as the Itô one, which suggests to invest the whole amount M in the stock, such that

$$\mathbb{E}(M_t) = e^{\mu T}.$$

As we have discussed in Section 6.2, the behaviour of the Ayed-Kuo integral seems to be counter-intuitive from the financial point of view.

In the next theorem, we prove that the optimal investment strategy for the Russo-Vallois integration is to invest all the amount M in the asset whose expected wealth is larger at maturity time $t = T$, as Remark 6.8 states.

Theorem 6.6. *Let f be a function of $B(T)$ such that $f \in L^{\infty}(\mathbb{R})$ and $0 \le f \le 1$. The optimal investment strategy for ODE (6.12) and SDE (6.13) under Russo-Vallois integration is*

$$f(B(T)) = \mathbb{1}_{\left\{B(T) > \frac{T}{\sigma}\left(\rho - \mu + \frac{1}{2}\sigma^2\right)\right\}}.$$

Proof. The solutions of ODE (6.12) and SDE (6.13) can be computed by the calculus rules for the Russo-Vallois integral. Hence, we get

$$\begin{cases} S_t^0 = M(1-f(B(T)))e^{\rho t}, \\ S_t^1 = Mf(B(T))e^{(\mu-\sigma^2/2)T+\sigma B(T)}. \end{cases}$$

The aim is to find the strategy f, such that $\mathbb{E}(M_t)$ is maximized. Indeed, we have

$$\begin{aligned} \mathbb{E}(M_t) &= \mathbb{E}\left(S_t^0\right) + \mathbb{E}\left(S_t^1\right) \\ &= M(1-\mathbb{E}(f(B(T))))e^{\rho T} + M\mathbb{E}\left(f(B(T))e^{\sigma B(T)}\right)e^{(\mu-\sigma^2/2)T} \\ &= Me^{\rho T}\left(1-\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi T}}f(x)e^{-\frac{x^2}{2T}}dx\right) + Me^{(\mu-\sigma^2/2)T}\left(\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi T}}f(x)e^{-\frac{x^2}{2T}}e^{\sigma x}dx\right). \end{aligned}$$

Let us consider

$$\bar{M}(f(x)) = \frac{\mathbb{E}(M_t)}{M}.$$

Hence, we get

$$\begin{aligned}\bar{M}(f(x)) &= e^{\rho T}\left(1-\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi T}}f(x)e^{-\frac{x^2}{2T}}dx\right)+e^{(\mu-\sigma^2/2)T}\left(\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi T}}f(x)e^{-\frac{x^2}{2T}}e^{\sigma x}dx\right)\\ &= e^{\rho T}+\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi T}}f(x)e^{-\frac{x^2}{2T}}\left(-e^{\rho T}+e^{(\mu-\sigma^2/2)T}e^{\sigma x}\right)dx.\end{aligned}$$

By assumption, f is a function of $B(T)$ such that $0 \le f \le 1$, and we have that $\mu > \rho$. Hence, the sign of this integrand is determined by the value of x, such that the critical point x_c is

$$x_c = \frac{T}{\sigma}\left(\rho-\mu+\frac{\sigma^2}{2}\right).$$

Note that, if $x > x_c$, it means

$$B(T) > \frac{T}{\sigma}\left(\rho-\mu+\frac{\sigma^2}{2}\right),$$

the integrand is positive and we should take f as large as possible in order to maximize $\mathbb{E}(M_t)$. On the other hand, if $x < x_c$ the integrand is negative and we should take f as small as possible for the same reason. Hence, the optimal investment strategy is

$$f(B(T)) = \mathbb{1}_{\left\{B(T)>\frac{T}{\sigma}(\rho-\mu+\frac{1}{2}\sigma^2)\right\}}.$$

□

Remark 6.11. The function f from Theorem 6.6 implies that the trader should invest the whole amount M in the bond or the stock according to the value of $B(T)$, it means, in the asset whose expected value is larger at maturity time $t = T$. Hence, this investment strategy maximizes the expected value $\mathbb{E}(M_T)$, as it does take advantage of the anticipating condition. The Russo-Vallois integral works as one expects from the financial point of view.

Conclusions

This thesis gives a review of the main results of the classical stochastic integration theory. We study some of the most remarkable notions and results of Brownian motion, the Itô stochastic integration and the theory of stochastic differential equations.

Likewise, we study two extensions of the Itô classical stochastic integration theory, the Ayed-Kuo and the Russo-Vallois stochastic integrals, which generalize the Itô one in the sense that they deal with anticipating stochastic calculus, it means, with stochastic processes that are anticipating, and consequently non-adapted. Among this dissertation, we discuss some of the most important notions and results of both of them.

Finally, we introduce the insider trading problem, in which a trader is considered to have privileged information about future prices of assets. This idea is clearly related to the anticipating condition. Then, we study some of the most notorious and novel results about it. For this final point, we propose two new theorems that we have proved in this thesis, which deal with the optimal investment strategy for the insider trading problem under Ayed-Kuo and Russo-Vallois theories.

APPENDIX A

Normal Random Variables

In this appendix our aim is to give some elementary notions of the Normal random variables or Gaussian processes, which are used among this dissertation. In the probability theory, the normal distribution $\mathcal{N}(\mu,\sigma)$ is a very usual continuous probability distribution.

Definition A.1 (Univariate Normal random variable)**.** Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. The Normal random variable $X:\Omega\to\mathbb{R}$, denoted by $\mathcal{N}(\mu,\sigma)$, has a density function of the form

$$\phi_{\mu,\sigma}(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$

where μ and σ are constants, the mean and the standard deviation respectively. Moreover, the distribution is of the form

$$\Phi_{\mu,\sigma}(x)\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{x}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dy.$$

Remark A.1. Note that, by Definition A.1 we have

$$\mathbb{E}(X)=\int_{-\infty}^{\infty}x\phi_{\mu,\sigma}(x)dx=\mu,$$

and

$$V(X)=\int_{-\infty}^{\infty}x^2\phi_{\mu,\sigma}(x)dx-\mu^2=\sigma^2,$$

such that, the parameter μ moves the center of the distribution and the parameter σ widens or narrows it.

Definition A.2 (Multivariate Normal random variable)**.** The multivariate Normal random variable $X=(X_1,...X_n)$, denoted by $\mathcal{N}_p(\mu,\Sigma)$, has a density function of the form

$$f(x)=\frac{1}{(2\pi)^{n/2}\det(\Sigma)^{1/2}}\exp\left(-\frac{1}{2}(x-\mu)\Sigma^{-1}(x-\mu)^T\right),$$

where $\mu=(\mu_1,...,\mu_n)$ is the mean vector and Σ is a symmetric and positive definite matrix.

Remark A.2. Note that, by Definition A.1 we have

- The variables X_i are Normal random variables;
- The mean vector μ, is such that $\mathbb{E}(X_i) = \mu_i$ for each $i = 1, ..., n$;
- The matrix Σ is the variance-covariance matrix of the X_i variables.

Proposition A.1. *Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then, the random variable X can be written as*

$$X = \mu + \sigma Y,$$

where $Y \sim \mathcal{N}(0, 1)$.

Proposition A.2. *Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then, the four first moments of X are*

$$\begin{aligned}
\textit{(Mean)} \quad & \mathbb{E}(X) = \mathbb{E}(\mu + \sigma Y) = \mu + \sigma \mathbb{E}(Y) = \mu; \\
\textit{(Variance)} \quad & \mathbb{E}(X^2) = \mathbb{E}((\mu + \sigma Y)^2) = \mu^2 + \sigma^2; \\
\textit{(Skewness)} \quad & \mathbb{E}(X^3) = \mathbb{E}((\mu + \sigma Y)^3) = \mu^3 + 3\mu\sigma^2; \\
\textit{(Kurtosis)} \quad & \mathbb{E}(X^4) = \mathbb{E}((\mu + \sigma Y)^4) = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4.
\end{aligned}$$

Appendix B

Conditional Expectation

The concept of conditional expectation plays a fundamental role in modern Probability and the Theory of Stochastic Processes. In this appendix, our aim is to define this concept and study some of its main properties, which are used among this dissertation.

Theorem B.1. *Let us consider a sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$ in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable Y, such that $\mathbb{E}(Y) < \infty$. Then, there exists a random variable $X \in L^1(\Omega)$, such that*

(i) X is measurable respect to $\mathcal{G}$;

(ii) For all $G \in \mathcal{G}$, we have

$$\int_G Y(\omega)\mathbb{P}(d\omega) = \int_G X(\omega)\mathbb{P}(d\omega).$$

Moreover, the random variable X is unique up to a set of probability zero.

Remark B.1. The Radon-Nikodym theorem guarantees the existence of the random variable X.

The Theorem B.1 lead us to the following definition for the conditional expectation, which is well-defined and it is unique up to a set of probability zero.

Definition B.1 (Conditional expectation)**.** Let $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ be a random variable and let $\mathcal{G}$ be a sub-σ-algebra. We define the *conditional expectation* of Y given $\mathcal{G}$, denoted by $\mathbb{E}(Y|\mathcal{G})$, to any random variable $X : \Omega \to \mathbb{R}$, measurable respect to $\mathcal{G}$, satisfying

$$\int_G X(\omega)P(d\omega) = \int_G Y(\omega)P(d\omega),$$

for any $G \in \mathcal{G}$.

Remark B.2. Any $\mathbb{P}$-equivalent random variable satisfying the previous conditions, is called a version of $\mathbb{E}(Y|\mathcal{G})$.

The next theorem provides some important properties of the conditional expectation, which are used among this dissertation.

Theorem B.2. *Let X and Y be two integrable random variables and let $\mathcal{G}$ be a sub-σ-algebra. The following properties hold*

(i) If Y is $\mathcal{G}$-measurable, then $\mathbb{E}(Y|\mathcal{G}) = Y$.

(ii) $\mathbb{E}(\mathbb{E}(Y|\mathcal{G})) = \mathbb{E}(Y)$.

(iii) For any a, $b \in \mathbb{R}$, we have

$$\mathbb{E}(aX + bY|\mathcal{G}) = a\mathbb{E}(X|\mathcal{G}) + b\mathbb{E}(Y|\mathcal{G}).$$

(iv) If $\mathcal{E} \subset \mathcal{G}$ are sub-σ-algebras of $\mathcal{F}$, we get

$$\mathbb{E}(\mathbb{E}(Y|\mathcal{E})|\mathcal{G}) = \mathbb{E}(Y|\mathcal{G}) \qquad \text{and} \qquad \mathbb{E}(\mathbb{E}(Y|\mathcal{G})|\mathcal{E}) = \mathbb{E}(Y|\mathcal{G}).$$

(v) If $Y \geq 0$ almost surely. Then, $\mathbb{E}(Y|\mathcal{G}) \geq 0$ almost surely.

(vi) If the integrable random variable Z is $\mathcal{G}$-measurable. Then, we have

$$\mathbb{E}(YZ|\mathcal{G}) = Z\mathbb{E}(Y|\mathcal{G}).$$

(vii) If the integrable random variable Y and the sub-σ-algebra $\mathcal{G}$ are independent. Then, we get

$$\mathbb{E}(Y|\mathcal{G}) = \mathbb{E}(Y).$$

(viii) If the integrable random variable Z is $\mathcal{G}$-measurable and $h : \mathbb{R}^2 \to \mathbb{R}$ is measurable such that $\mathbb{E}(h(Y,Z)) < \infty$. Hence, with probability one, we have

$$\mathbb{E}(h(Y,Z)|\mathcal{G})(\omega) = \mathbb{E}(h(Y,Z(\omega))|\mathcal{G})(\omega).$$

In the following results, we provide the monotone convergence theorem and the dominated convergence theorem.

Theorem B.3 (Monotone convergence theorem)**.** *Let Y_n be a sequence of random variables, such that $Y_n \geq 0$ and $Y_n \uparrow Y$, where Y is an integrable random variable. Hence, we have*

$$\mathbb{E}(Y_n|\mathcal{G}) \uparrow \mathbb{E}(Y|\mathcal{G}).$$

According to property (v) of Theorem B.2, with probability one, $\mathbb{E}(Y_n|\mathcal{G})$ is an increasing and an upper bounded sequence of positive random variables by $\mathbb{E}(Y|\mathcal{G})$. Therefore, the sequence convergences almost surely to a limit lower than $\mathbb{E}(Y|\mathcal{G})$.

Theorem B.4 (Dominated convergence theorem). *Let Y_n be a sequence of random variables, such that $|Y_n| < X$ for each n, where X is an integrable random variable and $Y_n \to Y$ almost surely. Hence, we have*

$$\mathbb{E}(Y_n|\mathcal{G}) \to \mathbb{E}(Y|\mathcal{G}),$$

almost surely.

Theorem B.5. *Let Y be an integrable random variable and let $\mathcal{G}$ and $\mathcal{E}$ sub-σ-algebras of $\mathcal{F}$. The σ-algebra of $\mathcal{F}$ produce by $\mathcal{G} \cup \mathcal{E}$ is denoted by $\sigma(\mathcal{G}, \mathcal{E})$, and in the same form, $\sigma(Y, \mathcal{G})$ denotes the σ-algebra produce by $\mathcal{F}(Y)$ and $\mathcal{G}$. Then, if $\sigma(Y, \mathcal{G})$ and $\mathcal{E}$ are independent, we have*

$$\mathbb{E}(Y|\sigma(\mathcal{D}, \mathcal{E})) = \mathbb{E}(Y|\mathcal{D}),$$

almost surely.

The Theorem B.5 establishes that, by conditioning on $\mathcal{G}$ any expression, the $\mathcal{G}$-measurable variables can be consider constants and can be replace by their value. In this sense, by conditioning on the σ-algebra $\mathcal{G}$, makes all $\mathcal{G}$-measurable random variables become constants, it means, it supposes having the information of the value of any $\mathcal{G}$-measurable random variable. This interpretation of the σ-algebras as an expression of the available information seems useful in many circumstances.
In the following theorem, we give the Jensen inequality for conditional expectations.

Theorem B.6 (Jensen inequality). *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function and Y an integrable random variable. If $\mathbb{E}|f(Y)| < \infty$. Then, we have*

$$f(\mathbb{E}(Y|\mathcal{G})) \leq \mathbb{E}(f(Y)|\mathcal{G}),$$

with probability one.

APPENDIX C

Borel-Cantelli Lemma and Chebyshev Inequality

In this appendix, we study the Borel-Cantelli lemma and the Chebysev inequality. These two are seemingly disparate results from probability theory. However, they combine well in order to demonstrate some of the statements proposed among this dissertation.

Let $\{\mathcal{A}_n\}_{n=1}^{\infty}$ be a sequence of events in some probability space. Consider the event $\mathcal{A}$ given by

$$\mathcal{A} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \mathcal{A}_k.$$

It is easy to see that $\omega \in \mathcal{A}$ if and only if $\omega \in \mathcal{A}_n$ for infinitely many n's. Thus, we can think of the event $\mathcal{A}$ as the event that $\mathcal{A}_n$'s occur infinitely often. Let us use the following notation

$$\{\mathcal{A}_n, \text{ infinitely often}\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \mathcal{A}_k.$$

Theorem C.1 (Borel-Cantelli lemma)**.** *Let $\{\mathcal{A}_n\}_{n=1}^{\infty}$ be a sequence of events, such that*

$$\sum_{n=1}^{\infty} \mathbb{P}(\mathcal{A}_n) < \infty.$$

Then, we have

$$\mathbb{P}(\mathcal{A}_n, \textit{ infinitely often}) = 0.$$

Remark C.1. The Theorem C.1 is often called the first part of the Borel-Cantelli lemma. The second part of it states that if $\sum_{n=1}^{\infty} \mathbb{P}(\mathcal{A}_n) < \infty$ and the events $\mathcal{A}_n$ are independent. Then, we have

$$\mathbb{P}(\mathcal{A}_n, \text{ infinitely often}) = 1.$$

However, for our purposes we only use the first part of the lemma.

Theorem C.2 (Chebyshev inequality)**.** *Let X be a random variable, such that $\mathbb{E}|X| < \infty$. Then, for any $a > 0$, we have*

$$\mathbb{P}(|X| \geq a) \leq \frac{1}{a}\mathbb{E}|X|, \quad \forall a > 0.$$

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