On Riemann surfaces of genus g with $4g$ automorphisms

Emilio Bujalance
Departamento de Matemáticas Fundamentales,
UNED,
Paseo Senda del Rey 9,
28040 Madrid, Spain

Antonio F. Costa
Departamento de Matemáticas Fundamentales,
UNED,
Paseo Senda del Rey 9,
28040 Madrid, Spain

Milagros Izquierdo
Matematiska institutionen
Linköpings universitet
581 83 Linköping, Sweden
miizq@mai.liu.se

Abstract

We determine, for all genus $g \geq 2$ the Riemann surfaces of genus g with exactly $4g$ automorphisms. For $g \neq 3, 6, 12, 15$ or 30, this surfaces form a real Riemann surface $\mathcal{F}_g$ in the moduli space $\mathcal{M}_g$: the Riemann sphere with three punctures. We obtain the automorphism groups and extended automorphism groups of the surfaces in the family. Furthermore we determine the topological types of the real forms of real Riemann surfaces in $\mathcal{F}_g$. The set of real Riemann surfaces in $\mathcal{F}_g$ consists of three intervals its closure in the Deligne-Mumford compactification of $\mathcal{M}_g$ is a closed Jordan curve. We describe the nodal surfaces that are limits of real Riemann surfaces in $\mathcal{F}_g$.

2000 Mathematics Subject Classification: Primary 30F10; Secondary 14H15, 30F60.

1 Introduction

Given a linear expression like $ag + b$, where a, b are fixed integers, it is very difficult to claim precise information on the (compact) Riemann surfaces of genus $g \geq 2$ with automorphism groups of order $ag + b$: i.e. are there Riemann surfaces in these conditions?, how many?, which are their automorphism groups? For instance, there are many works about Hurwitz surfaces, i. e. surfaces of genus g with group of automorphisms of order $84g-84$ (maximal order), but there is no a complete answer to the above questions. Surprisingly we shall give an almost complete answer (up to a finite number of genera g) to all questions on Riemann surfaces of genus g with $4g$ automorphisms.

For each integer $g \geq 2$ we find an equisymmetric (complex)-uniparametric family $\mathcal{F}_g$ of Riemann surfaces of genus g having (full) automorphism group of order $4g$. The maximal order of equisymmetric and uniparametric families of Riemann surfaces appearing in all genera is $4g+4$ and the second possible larger order is precisely $4g$ (this is a consequence of Riemann-Hurwitz formula). If $g \neq 3, 6, 15$ all surfaces with $4g$ automorphisms are in the family $\mathcal{F}_g$ with one or two more exceptional surfaces in a few genera: $g = 3, 6, 12, 30$. For genera $g = 3, 6$ and 15 it appears another exceptional uniparametric family. Finally for genera $3, 6, 12$ and 30 there are one or two exceptional surfaces with $4g$ automorphisms.

The automorphism group of the surfaces in $\mathcal{F}_g$ is D_{2g} and the quotient $X/\mathrm{Aut}(X)$ is the Riemann sphere $\widehat{\mathbb{C}}$, the meromorphic function $X \to X/\mathrm{Aut}(X) = \widehat{\mathbb{C}}$ have four singular values of orders $2, 2, 2, 2g$.

Ravi S. Kulkarni [15] showed that, for any genus $g \equiv 0, 1, 2 \bmod 4$, there is a unique surface of genus g with full automorphism group of order $8(g+1)$ (the family of Accola-Maclachan [1] and [18]), and for $g \equiv -1 \bmod 4$, there is just another surface of genus g (the Kulkarni surface [15]). In [16] Kulkarni shows that, if $g \neq 3$ there is a unique Riemann surface of genus g admitting an automorphism of order $4g$, while for $g = 3$ there are two such surfaces. The surfaces in this last family have exactly $8g$ automorphisms, except for $g = 2$, where the surface has 48 automorphisms. For cyclic groups there are some cases where the order of the group determines the Riemann surface (see [16], [19], [14]). Analogous results are known for Klein surfaces: [4], [7], [8] and [3].

The family $\mathcal{F}_g$ contains surfaces admitting anticonformal automorphisms, forming the subset $\mathbb{R}\mathcal{F}_g$. These points in the moduli space correspond to Riemann surfaces given by the complexification of real algebraic curves. The extended groups of automorphisms of the surfaces in $\mathbb{R}\mathcal{F}_g$ (including the anticonformal automorphisms) are isomorphic either to $D_{2g} \times C_2$ or D_{4g}, and such groups contain anticonformal involutions, so the surfaces in $\mathbb{R}\mathcal{F}_g$ are real Riemann surfaces. The topological types of conjugacy classes of anticonformal involutions (real forms) of the real Riemann surfaces in $\mathcal{F}_g$ are either $\{+2, 0, -2, -2\}$, $\{-1, -1, -g, -g\}$, $\{0, 0, -2, -2\}$ if g is odd or

$\{+1, 0, -1, -3\}$, $\{-1, -1, -g, -g\}$, $\{-2\}$ if g is even.

The family $\mathcal{F}_g$ is the Riemann sphere with three punctures, having an anticonformal involution whose fixed point set consists of three arcs a_1, a_2, b. Each one of these arcs is formed by the real Riemann surfaces in $\mathbb{R}\mathcal{F}_g$ with a different set of topological types of real forms. Adding three points to the surface $\mathcal{F}_g$ we obtain a compact Riemann surface $\overline{\mathcal{F}_g} \subset \widehat{\mathcal{M}}_g$, where $\overline{a_1 \cup a_2 \cup b}$ (the closure of $a_1 \cup a_2 \cup b$ in $\widehat{\mathcal{M}}_g$) is a closed Jordan curve. The space $\widehat{\mathcal{M}}_g$ is the Mumford-Deligne compactification of $\mathcal{M}_g$. As a consequence we have that $\overline{\mathbb{R}\mathcal{F}_g} \cap \mathcal{M}_g$ has two connected components.

Acknowledgement. All authors partially supported by the project MTM2014-55812-P.

2 Preliminaries

2.1 Non-Euclidean crystallographic groups

A *non-Euclidean crystallographic group* (or *NEC group*) Γ is a discrete group of isometries of the hyperbolic plane $\mathbb{D}$. We shall assume that an NEC group has a compact orbit space. If Γ is such a group then its algebraic structure is determined by its signature

$$(h; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}). \qquad (1)$$

The orbit space $\mathbb{D}/\Gamma$ is a surface, possibly with boundary. The number h is called the *genus* of Γ and equals the topological genus of $\mathbb{D}/\Gamma$, while k is the number of the boundary components of $\mathbb{D}/\Gamma$, and the sign is $+$ or $-$ according to whether the surface is orientable or not. The integers $m_i \geq 2$, called the *proper periods*, are the branch indices over interior points of $\mathbb{D}/\Gamma$ in the natural projection $\pi : \mathbb{D} \to \mathbb{D}/\Gamma$. The bracketed expressions $(n_{i1}, \ldots, n_{is_i})$, some or all of which may be empty (with $s_i = 0$), are called the *period cycles* and represent the branchings over the i^{th} boundary component of the surface. Finally the numbers $n_{ij} \geq 2$ are the *link periods*.

Associated with each signature there exists a *canonical presentation* for the group Γ. If the signature (2.1) has sign $+$ then Γ has the following generators:

$x_1, \ldots, x_r$ (elliptic elements),
$c_{10}, \ldots, c_{1s_1}, \ldots, c_{k0}, \ldots, c_{ks_k}$ (reflections),
$e_1, \ldots, e_k$ (boundary transformations),
$a_1, b_1, \ldots, a_g, b_g$ (hyperbolic elements);

these generators satisfy the defining relations

$x_i^{m_i} = 1$ (for $1 \leq i \leq r$),

$c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1$, $c_{is_i} = e_i^{-1}c_{i0}e_i$ (for $1 \leq i \leq k, 0 \leq j \leq s_i$),

$x_1 \ldots x_r e_1 \ldots e_k a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_h b_h a_h^{-1} b_h^{-1} = 1$.

If the sign is $-$ then we just replace the hyperbolic generators a_i, b_i by glide reflections $d_1, \ldots, d_h$, and the last relation by $x_1 \ldots x_r e_1 \ldots e_k d_1^2 \ldots d_h^2 = 1$.

The hyperbolic area of an arbitrary fundamental region of an NEC group Γ with signature (2.1) is given by

$$\mu(\Gamma) = 2\pi \left(\varepsilon h - 2 + k + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}}\right) \right) \quad (2)$$

where $\varepsilon = 2$ if the sign is $+$, and $\varepsilon = 1$ if the sign is $-$. Furthermore, any discrete group Λ of isometries of $\mathbb{D}$ containing Γ as a subgroup of finite index is also an NEC group, and the hyperbolic area of a fundamental region for Λ is given by the Riemann-Hurwitz formula:

$$[\Lambda : \Gamma] = \mu(\Gamma)/\mu(\Lambda). \quad (3)$$

The NEC groups with signature of the form $(h; +; [m_1, \ldots, m_r]; \{-\})$ are Fuchsian groups. For any NEC group Λ, let Λ^+ denote the subgroup of orientation-preserving elements of Λ, called the *canonical Fuchsian subgroup* of Λ. If $\Lambda^+ \neq \Lambda$ then Λ^+ has index 2 in Λ and we say that Λ is a *proper* NEC group (see [6]).

2.2 Riemann surfaces, automorphisms and uniformization groups

A Riemann surface is a surface endowed with a complex analytical structure. Let X be a compact Riemann surface of genus $g > 1$. Then there is a surface Fuchsian group Γ (that is, an NEC group with signature $(g; +; [-]; \{-\}))$ such that $X = \mathbb{D}/\Gamma$, and if G is a group of automorphisms of X there is a Fuchsian group Δ, containing Γ, and an epimorphism $\theta : \Delta \to G$ such that $\ker\theta = \Gamma$. If G^* is a group of conformal and anticonformal automorphism then there is an NEC group Λ, and an epimorphism $\theta^* : \Lambda \to G$ such that $\ker\theta^* = \Lambda$. In particular the full automorphism group $\mathrm{Aut}(X)$ of X is isomorphic to Δ/Γ, where Δ is a Fuchsian group containing Γ. The extended (full) automorphism group $\mathrm{Aut}^{\pm}(X)$ of X (including anticonformal automorphisms) is isomorphic to Λ/Γ, where Λ is an NEC group such that $\Lambda^+ = \Delta$.

2.3 Topological types of anticonformal involutions

Given a Riemann surface X of genus g, the topological type of the action of an anticonformal involution $\sigma \in \mathrm{Aut}(X)$ is determined by the number of connected components, called *ovals*, of its fixed point set $Fix(\sigma)$ and the orientability of the Klein surface $X/\langle\sigma\rangle$. We say that σ has *species* $+k$ if $Fix(\sigma)$ consists of k ovals and $X/\langle\sigma\rangle$ is orientable, and $-k$ if $Fix(\sigma)$ consists of k ovals and $X/\langle\sigma\rangle$ is nonorientable (i. e. two surfaces with symmetries of the same species have topologically conjugate quotient orbifolds and vice versa). The set $Fix(\sigma)$ corresponds to the real part of a complex algebraic curve representing X, which admits an equation with real coefficients. The "+" sign in the species of σ means that the real part disconnects its complement in the complex curve and then we say that σ separates. By a classical theorem of Harnack the possible values of species run between $-g$ and $+(g+1)$, where $+k \equiv g+1 \bmod 2$ (see [10] for a geometrical proof).

A Riemann surface with an anticonformal involution is said to be a real Riemann surface. The type of symmetry of a Riemann surface X is the set of topological types of anticonformal involutions of X.

There is a categorical equivalence between compact Riemann surfaces and complex projective smooth algebraic curves. The conjugacy classes of anticonformal involutions of Riemann surfaces correspond to the real forms of the corresponding algebraic curve: i. e. real algebraic curves (see [20]). The topological type of an anticonformal involutions gives us important information about the real points of a real algebraic curve, the number of connected components of the real points of the algebraic curve and the separability character of the real points inside the complex algebraic curve.

2.4 Teichmüller and moduli spaces

Here we follow reference [17] on moduli spaces of Riemann and Klein surfaces.

Let s be a signature of NEC groups and let $\mathcal{G}$ be an abstract group isomorphic to the NEC groups with signature s. We denote by $\mathbf{R}(s)$ the set of monomorphisms $r : \mathcal{G} \to Aut^{\pm}(\mathbb{D})$ such that $r(\mathcal{G})$ is an NEC group with signature s. The set $\mathbf{R}(s)$ has a natural topology given by the topology of $Aut^{\pm}(\mathbb{D})$. Two elements r_1 and $r_2 \in \mathbf{R}(s)$ are said to be equivalent, $r_1 \sim r_2$, if there exists $g \in Aut^{\pm}(\mathbb{D})$ such that for each $\gamma \in \mathcal{G}$, $r_1(\gamma) = g r_2(\gamma) g^{-1}$. The space of classes $\mathbf{T}(s) = \mathbf{R}(s)/\sim$ is called the *Teichmüller space* of NEC groups with signature s. If the signature s is given in section 2.1, the

Teichmüller space $\mathbf{T}(s)$ is homeomorphic to $\mathbb{R}^{d(s)}$, where

$$d(s) = 3(\varepsilon h - 1 + k) - 3 + (2r + \sum_{i=1}^{k} r_i).$$

The modular group $\mathrm{Mod}(\mathcal{G})$ of $\mathcal{G}$ is the quotient $\mathrm{Mod}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})/\mathrm{Inn}(\mathcal{G})$, where $\mathrm{Inn}(\mathcal{G})$ denotes the inner automorphisms of $\mathcal{G}$. The *moduli space* of NEC groups with signature s is the quotient $\mathcal{M}_s = \mathbf{T}(s)/\mathrm{Mod}(\mathcal{G})$ endowed with the quotient topology. Hence $\mathcal{M}_s$ is an orbifold with fundamental orbifold group $\mathrm{Mod}(\mathcal{G})$.

If s is the signature of a surface group uniformizing surfaces of topological type $t = (g, \pm, k)$, then we denote by $\mathbf{T}(s) = \mathbf{T}_t$ and $\mathcal{M}_s = \mathcal{M}_t$ the Teichmüller and the moduli space of Klein surfaces of topological type t.

Let $\mathcal{G}$ and $\mathcal{G}'$ be abstract groups isomorphic to NEC groups with signatures s and s' respectively. Given an inclusion mapping $\alpha : \mathcal{G} \to \mathcal{G}'$ there is an induced embedding $\mathbf{T}(\alpha) : \mathbf{T}(s') \to \mathbf{T}(s)$ defined by $[r] \mapsto [r \circ \alpha]$.

If a finite group G is isomorphic to a group of automorphisms of Klein surfaces with topological type $t = (g, \pm, k)$, then the action of G is determined by an epimorphism $\theta : \mathcal{D} \to G$, where $\mathcal{D}$ is an abstract group isomorphic to NEC groups with a given signature s and $\ker(\theta) = \mathcal{G}$ is a group isomorphic to NEC surface groups uniformizing Klein surfaces of topological type t. Then there is an inclusion $\alpha : \mathcal{G} \to \mathcal{D}$ and an embedding $\mathbf{T}(\alpha) : \mathbf{T}(s) \to \mathbf{T}_t$. The continuous map $\mathbf{T}(\alpha)$ induces a continuous map $\mathcal{M}_s \to \mathcal{M}_t$ and as a consequence:

Proposition 1 *[17] The set $\mathcal{B}_t^{G,\theta}$ of points in $\mathcal{M}_t$ corresponding to surfaces having a group of automorphisms isomorphic to G, with action determined by θ, is a connected set.*

2.5 Compactification of moduli spaces

A *Riemann surface with nodes* is a connected complex analytic space S if and only if (see [2]):

1. there are $k = k(S) \geq 0$ points $p_1, ..., p_k \in S$ called nodes such that every node p_j has a neighborhood isomorphic to the analytic set $\{z_1 z_2 = 0 : \|z_1\| < 1,\ \|z_2\| < 1\}$ with p_j corresponding to $(0, 0)$.
2. the set $S \setminus \{p_1, ..., p_k\}$ has $r \geq 1$ connected components $\Sigma_1, ..., \Sigma_r$ called components of S, each of them is a Riemann surface of genus g_i, with n_i punctures with $3g_i - 3 + n_i \geq 0$ and $n_1 + ... + n_r = 2k$.

3. we denote $g = (g_1 - 1) + ... + (g_r - 1) + k + 1$

If $k = k(S) = 0$, S is called *non singular* and if $k = k(S) = 3g - 3$, S is called *terminal*.

To a Riemann surface with nodes S we can associate a weighted graph, *the graph of* S, $\mathcal{G}(S) = (V_S, E_S, w)$, where V_S is the set of vertices, E_S is the set of edges, and w is a function on the set V_S with non-negative integer values. This triple is defined in the following way:

1. To each component Σ_i corresponds a vertex in V_S.

2. To each node joining the components Σ_i and Σ_j corresponds and edge in E_S connecting the corresponding vertices. Multiple edges between the same pair of vertices and loops are allowed in $\mathcal{G}(S)$.

3. The function $w : V(\mathcal{G}(S)) \to \mathbb{Z}_{\geq 0}$ associates to any vertex of $\mathcal{G}(S)$ the genus g_i of Σ_i.

Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g. A well known result of Deligne and Mumford states that the set $\widehat{\mathcal{M}_g}$ of Riemann surfaces with nodes of genus g can be endowed with a structure of projective complex variety and contains $\mathcal{M}_g$ as a dense open subvariety [11]. If $g \geq 2$ then $\widehat{\mathcal{M}_g}$ is an irreducible complex projective variety of dimension $3g - 3$.

3 Riemann surfaces of genus g with $4g$ automorphisms

Lemma 2 *Let X be a Riemann surface of genus g and let Γ be a surface Fuchsian group of genus g uniformizing X. If G is an automorphism group of X, then $G \cong \Gamma'/\Gamma$ where Γ' is a Fuchsian group. If $|G| = 4g$, $g \neq 3, 6, 15$ and X is not in a finite set of exceptional Riemann surfaces whose genera are $3, 6, 12$ or 30, then the signature of Γ' must be:*

1. $(0; +; [2, 4g, 4g])$

2. $(0; +; [3, 6, 2g])$

3. $(0; +; [4, 4, 2g])$

4. $(0; +; [2, 2, 2, 2g])$

Proof. Let Γ' have signature:

$$(g'; +; [m_1, ..., m_r])$$

By Riemann-Hurwitz formula we have:

$$\frac{2g-2}{2g'-2+\sum_{i=1}^{r}(1-\frac{1}{m_i})} = 4g$$

then

$$2g'-2+\sum_{i=1}^{r}(1-\frac{1}{m_i}) = \frac{1}{2} - \frac{1}{2g} \tag{4}$$

where we may assume that $m_{i-1} \leq m_i$, $i = 2, ..., r$. It is important to note that m_i divides $4g$ (m_i is the order of a cyclic subgroup of G). Hence $g' = 0$ and $r \leq 4$, and formula (4) becomes:

$$\sum_{i=1}^{r}(1-\frac{1}{m_i}) = \frac{5}{2} - \frac{1}{2g}, r \leq 4$$

For $r = 4$ we have $\sum_{i=1}^{4}\frac{1}{m_i} = \frac{3}{2} + \frac{1}{2g}$, then if $g \neq 3, 6, 15$ we have only a solution $m_1 = m_2 = m_3 = 2, m_4 = 2g$, that is case 4 (note that for $g = 3, 6, 15$ we have the solutions $(2, 2, 3, 3)$, $(2, 2, 3, 4)$ and $(2, 2, 3, 5)$ respectively).

If $r = 3$ we have

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = \frac{1}{2} + \frac{1}{2g} \tag{5}$$

From the formula (5) we have that $m_1 \leq 5$. If $m_1 = 2$, using the formula and that m_i divides $4g$ we have a unique solution $m_2 = m_3 = 4g$ (case 1).

For $m_1 = 3, 4, 5$ it is possible to make a case by case analysis giving for each value a bound for m_2 and for each possible value of m_2 a finite set of solutions if $m_3 \neq 2g$. The solutions with $m_3 \neq 2g$ correspond to following set of values of g:

$$\{3, 6, 9, 10, 12, 14, 15, 18, 20, 21, 24, 28, 30, 33, 36, 40, 42,$$
$$45, 60, 66, 72, 84, 90, 105, 126, 132, 153, 190, 273, 276, 420, 429, 861\}$$

Using finite group theory and the algebra symbolic package MAGMA one shows that there exist one or two exceptional surfaces exactly for genera $g = 3, 6, 12$ or 30. We thank Professor Marston Conder for helping us with these calculations with MAGMA.

For $m_3 = 2g$ there are only two infinite set of solutions:

$$m_1 = 3, m_2 = 6, m_3 = 2g \text{ and } m_1 = 4, m_2 = 4, m_3 = 2g$$

that are cases 2 and 3. ■

Remark 3 *See [16], section 2.3, for related results.*

In the next proposition we shall eliminate the cases 1, 2 and 3 of the preceding Lemma using group theory and the fact that the order of $\mathrm{Aut}(X)$ is exactly $4g$.

Proposition 4 *Let X be a Riemann surface of genus g, uniformized by a surface Fuchsian group Γ and with full automorphism group $\mathrm{Aut}(X) = G$ of order $4g$. If Γ' is a Fuchsian group such that $\Gamma \leq \Gamma'$ and $X/Aut(X) = \mathbb{D}/\Gamma'$ then the signature of Γ' is different from*

1. $(0; +; [2, 4g, 4g])$
2. $(0; +; [3, 6, 2g])$
3. $(0; +; [4, 4, 2g])$

Proof. Case 1. Assume that the signature of Γ' is $(0; +; [2, 4g, 4g])$. Then there is a natural epimorphism $\theta : \Gamma' \to G \cong \Gamma'/\Gamma$. If Γ' has a canonical presentation $\left\langle x_1, x_2, x_3 : x_1^2 = x_2^{4g} = x_3^{4g} = x_1x_2x_3 = 1 \right\rangle$ thus $\theta(x_2)$ and $\theta(x_3)$ have order $4g$, since Γ is a surface Fuchsian group. Then G is a cyclic group generated by $\theta(x_3) = C$. We have $\theta(x_1) = C^{2g}, \theta(x_2) = C^{2g-1}, \theta(x_3) = C$.

The group Γ' is included in a Fuchsian group Δ of signature $(0; +; [2, 4, 4g])$ (see [21]). Let

$$\left\langle x_1', x_2', x_3' : x_1'^2 = x_2'^4 = x_3'^{4g} = x_1'x_2'x_3' = 1 \right\rangle$$

be a canonical presentation of Δ. We have $x_1 = x_2'^2$, $x_2 = x_2'^{-1}x_3'x_2'$, $x_3 = x_3'$ and an epimorphism $\theta' : \Delta \to G'$, where

$$G' = \left\langle B, C : B^2 = C^{2g}, C^{4g} = 1, B^{-1}CB = C^{2g-1} \right\rangle,$$

θ' is defined by $\theta'(x_1') = C^{-1}B^{-1}, \theta'(x_2') = B, \theta'(x_3') = C$. Now $\theta' |_\Delta = \theta$ and then the automorphism group of X has order $> 4g$.

Case 2. Assume that the signature of Γ' is $(0; +; [3, 6, 2g])$. Then there is a natural epimorphism: $\theta : \Gamma' \to G \cong \Gamma'/\Gamma$. If

$$\left\langle x_1, x_2, x_3 : x_1^3 = x_2^6 = x_3^{2g} = x_1x_2x_3 = 1 \right\rangle$$

is a canonical presentation of Γ' then G has a presentation with generators $\theta(x_1) = A, \theta(x_2) = B, \theta(x_3) = C$ and some of the relations are:

$$A^3 = B^6 = C^{2g} = ABC = 1$$

Hence G is generated by A and C.

Since Γ' is a surface group the order of C is $2g$, then $\langle C \rangle$ is an index two subgroup of G and $A \notin \langle C \rangle$. Hence $A^2 \in \langle C \rangle$, so $A^2 = C^t$, and then $A = (A^2)^{-1} = C^{2g-t}$, in contradiction with $A \notin \langle C \rangle$. ∎

For the Case 3 we need a Lemma:

Lemma 5 *Let Δ be a Fuchsian group with signature $(0; +; [4, 4, 2g])$ and let*

$$\left\langle x_1, x_2, x_3 : x_1^4 = x_2^4 = x_3^{2g} = x_1 x_2 x_3 = 1 \right\rangle$$

be a canonical presentation of Δ. Let $\theta : \Delta \to G = \langle A, B \rangle$ be an epimorphism with kernel a surface Fuchsian group and $\theta(x_1) = A$, $\theta(x_2) = B$.

There is a Fuchsian group Δ' of signature $(0; +; [2, 4, 4g])$ with $\Delta \leq \Delta'$, $[\Delta : \Delta'] = 2$, a group G' with $G \leq G'$, $[G : G'] = 2$, and an epimorphism $\theta' : \Delta' \to G'$, such that $\theta' \mid_\Delta = \theta$ if and only if the group G admits an automorphism α such that $\alpha(A) = B$, $\alpha(B) = A$.

Proof. If G admits such an automorphism α, then we can construct the semidirect product $G' = G \rtimes_\alpha C_2$, which is generated by $G = \langle A, B \rangle$ and an order two element D, conjugation by which induces the automorphism α on G. The group Δ is contained in an NEC group Δ' with signature $(0; +; [2, 4, 4g])$ and having canonical generators x'_1, x'_2, x'_3. Define an epimorphism $\theta' : \Delta' \to G' = G \rtimes_\alpha C_2$ by setting

$$\theta'(x'_1) = D, \quad \theta'(x'_2) = B, \quad \theta'(x'_3) = DA^{-1}.$$

Note that G' is isomorphic to $C_{4g} \rtimes C_2 = \left\langle DA^{-1} \right\rangle \rtimes \langle D \rangle$ and to $C_{4g} \rtimes C_4 = \left\langle DA^{-1} \right\rangle \rtimes \langle B \rangle$.

Conversely, if such an extension $\theta' : \Delta' \to G'$ of θ exists and Δ' has canonical generators x'_1,x'_2, x'_3, then the embedding of Δ in Δ' is given by

$$x_1 \mapsto x'_1 x'_2 x'_1, \quad x_2 \mapsto x'_2, \quad x_3 \mapsto x'^2_3;$$

hence if D is the order two element $\theta'(x'_1)$, then

$$DAD = \theta'(x'_1 x_1 x'_1) = \theta'(x'_2) = \theta(x_2) = B$$

and

$$DBD = \theta'(x_1' x_2 x_1') = \theta'(x_1' x_2' x_1') = \theta(x_1) = A,$$

so conjugation by D gives the required automorphism. ■

Now we continue the proof of the Proposition:

Proof. Case 3. Assume that the signature of Γ' is $(0; +; [4, 4, 2g])$. Then there is a natural epimorphism $\theta : \Gamma' \to G \cong \Gamma'/\Gamma$. If

$$\left\langle x_1, x_2, x_3 : x_1^4 = x_2^4 = x_3^{2g} = x_1 x_2 x_3 = 1 \right\rangle$$

is a canonical presentation of Γ' then G has a presentation with generators $\theta(x_1) = A, \theta(x_2) = B, \theta(x_3) = C$ and some of the relations are $A^4 = B^4 = C^{2g} = ABC = 1$. Hence G is generated by A and C.

Since the order of $\langle C \rangle$ is $2g$ then $A^2 \in \langle C \rangle$ and since $\langle C \rangle \lhd G$ then $ACA^{-1} = C^t$. As Γ is a surface Fuchsian group, A^2 has order two and $A^2 = C^g$. We have that $A^{-1}C^{-1}$ has order four, then:

$$(A^{-1}C^{-1})^4 = 1, ACA^{-1} = C^t, A^2 = C^g$$

From the above relations we have that $2(t+1) \equiv 0 \bmod 2g$, then either $t = g - 1$ or $t = 2g - 1$.

If $t = g - 1$, then $A^{-1}C^{-1}$ has order two but as Γ is a surface Fuchsian group, then $A^{-1}C^{-1}$ must have order four, so this case is not possible.

If $t = 2g - 1$ we have the relation $ACA^{-1} = C^{-1}$. The group G has presentation:

$$\left\langle A, C : A^4 = C^{2g} = 1; ACA^{-1} = C^{-1}; A^2 = C^g \right\rangle$$

The assignation $A \to A^{-1}C^{-1}$ and $C \to C^{-1}$ defines an automorphism such that $A \to A^{-1}C^{-1} = B$ and $B = A^{-1}C^{-1} \to A$. By the preceding Lemma the automorphism group contains properly G and then $|Aut(X)| > 4$. ■

Remark 6 *For all $g \geq 2$, there is a Riemann surface $X_{8g} = \mathbb{D}/\Gamma$, the Wiman curve of type II: $w^2 = z(z^{2g} - 1)$ (see [22]), with $8g$ automorphisms (except for $g = 2$) and such that $X_{8g}/\mathrm{Aut}(X_{8g})$ is uniformized by a group of signature $(0; +; [2, 4, 4g])$ containing Γ. The groups G' in cases 1 and 3 are isomorphic to $\mathrm{Aut}(X_{8g})$. The full automorphism group of Wiman's curve of genus 2 is $GL(2.3)$, of order 48.*

Theorem 7 *Let X be a Riemann surface of genus g uniformized by a surface Fuchsian group Γ and with (full) automorphism group G of order $4g$. Assume that $g \neq 3, 6, 15$ and X is not in the finite set of exceptional Riemann surfaces in Lemma 2. If Γ' is a Fuchsian group such that $\Gamma \leq \Gamma'$ and $X/Aut(X) = \mathbb{D}/\Gamma'$ then the signature of Γ' is $(0; +; [2, 2, 2, 2g])$ and $G \cong D_{2g}$ (the dihedral group of $4g$ elements).*

Proof. Let X be a Riemann surface of genus g, uniformized by a surface Fuchsian group Γ and with automorphism group G of order $4g$. If Γ' is a Fuchsian group with $\Gamma \leq \Gamma'$ and $X/Aut(X) = \mathbb{D}/\Gamma'$ then, by Lemma 2 and Proposition 4 the signature of Γ' is $(0; +; [2, 2, 2, 2g])$.

There is a canonical presentation of Γ':

$$\left\langle x_1, x_2, x_3, x_4 : x_i^2 = x_4^{2g} = x_1 x_2 x_3 x_4 = 1, i = 1, 2, 3 \right\rangle$$

and an epimorphism:

$$\theta : \Gamma' \to G \cong \Gamma'/\Gamma$$

If $\theta(x_4) = D$, we have that the order of D is $2g$. Some of the $\theta(x_i)$, $i = 1, 2, 3$, does not belong to $\langle D \rangle$, using, if necessary, an automorphism of Γ' we may suppose that is $\theta(x_1) = A \notin \langle D \rangle$. Then $A^2 = 1$ and since $\langle D \rangle \lhd G$, $ADA^{-1} = D^t$, with $t^2 \equiv 1 \bmod 2g$.

The elements $\theta(x_2)$ and $\theta(x_3)$ have order 2 and all order two elements in $G = \langle A, D \rangle$ are A, D^g and $D^r A$ with $r(t+1) \equiv 0 \bmod 2g$. Since $x_1 x_2 x_3 x_4 = 1$, $\theta(x_2 x_3) = A^{-1} D^{-1}$, therefore either $\theta(x_2) = D^g$ and $\theta(x_3) = D^r A$ or $\theta(x_2) = D^r A$ and $\theta(x_3) = D^g$. Using if necessary an automorphism of Γ' we may assume $\theta(x_2) = D^r A$ and $\theta(x_3) = D^g$. Finally using $\theta(x_2 x_3) = A^{-1} D^{-1}$ we obtain $D^r A D^g = A^{-1} D^{-1}$ from where we have $rt + g + 1 \equiv 0 \bmod (2g)$. As $r(t+1) \equiv 0 \bmod 2g$, we have $g + 1 - r \equiv 0 \bmod (2g)$ and $r \equiv g + 1 \bmod 2g$, then $r = g + 1$ and $t = -1$. Hence $ADA = D^{-1}$, $A^2 = D^{2g} = 1$, the group G is D_{2g} and the epimorphism is unique (up to automorphisms of Γ' and G):

$$\theta(x_1) = A, \theta(x_2) = D^{g+1} A, \theta(x_3) = D^g, \theta(x_4) = D$$

■

Remark 8 *Note that the epimorphism $\theta : \Gamma' \to G \cong \Gamma'/\Gamma$ of the Theorem 7 is unique up to automorphisms of Γ' and G. So the surfaces of genus g having automorphism group of order $4g$ with $g \geq 31$ or in the conditions of the Theorem, form a connected equisymmetric uniparametric family.*

Remark 9 *The surfaces in the above theorem are hyperelliptic. The hyperelliptic involution corresponds to the element D^g of D_{2g}, since $\theta^{-1}(D)$ has signature $(0; +; [2, \overset{2g+2}{\ldots}, 2])$ (see [6]).*

4 Conformal and anticonformal automorphism groups

In this section we shall obtain the groups of conformal and anticonformal automorphisms of Riemann surfaces with automorphism group of order $4g$.

Theorem 10 *Let X be a Riemann surface of genus g, uniformized by a surface Fuchsian group Γ and with automorphism group G of order $4g$. If Γ' is a Fuchsian group with $\Gamma \leq \Gamma'$ and $X/Aut(X) = \mathbb{D}/\Gamma'$, we assume that the signature of Γ' is $(0; +; [2,2,2,2g])$. Let $Aut^{\pm}(X) = G^*$ be the group of conformal and anticonformal automorphisms of X and Γ^* be an NEC group such that $G^* \cong \Gamma^*/\Gamma$. If $G^* \gneq G$ then the signature of Γ^* is*

a. *$(0; +; [-]; (2,2,2,2g))$ then $G^* \cong D_{2g} \times C_2$ and there are two epimorphisms $\Gamma^* \to D_{2g} \times C_2$ (up to automorphisms of Γ^*).*

b. *$(0; +; [2]; (2,2g))$ then G^* has presentation:*

$$\langle x, z, w : x^2 = z^2 = w^2 = (zw)^{2g} = 1, xzx = (zw)^{g-1}z, xwx = (zw)^g z \rangle$$
$$= D_{2g} \rtimes_{\varphi} C_2$$

where $\varphi(z) = (zw)^{g-1}z$, $\varphi(w) = (zw)^g z$. Then $G^* \cong D_{4g}$ if g is even and $G^* \cong D_{2g} \times C_2$ if g is odd.

Proof. Since the signature of Γ' is $(0; +; [2,2,2,2g])$ and Γ' is an index two subgroup of the NEC group Γ^*, the signature of Γ^* must be either:
a. $(0; +; [-]; (2,2,2,2g))$ or
b. $(0; +; [2]; (2,2g))$.
Case a. Γ^* has signature $(0; +; [-]; (2,2,2,2g))$. Let

$$\langle c_0, c_1, c_2, c_3 : c_i^2 = (c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^2 = (c_3c_0)^{2g} = 1 \rangle$$

be a canonical presentation of Γ^*. Assume that the epimorphism $\theta^* : \Gamma^* \to G^* \cong \Gamma^*/\Gamma$, is given by

$$\theta^*(c_0) = x, \theta^*(c_1) = y, \theta^*(c_2) = z, \theta^*(c_3) = w,$$

then we have that $\{x, y, z, w\}$ is a set of generators of G^* and, since Γ is a surface Fuchsian group, x, y, z, w, xy, yz, zw have order 2 and wx has order $2g$. Since $|G^*| = 8g$ and $\langle x, w \rangle$ has order $4g$ then $\langle x, w \rangle \lhd G^*$. Note that either y or z is not in $\langle x, w \rangle$, assume that $y \notin \langle x, w \rangle$ (the argument assuming $z \notin \langle x, w \rangle$ is annalogous). Hence $y(wx)y = (wx)^t$, with $(t, 2g) = 1$.

The elements y and z are not the same, because yz has order 2. Since $G^* = G^* \cup yG^*$ we have two possibilities either $z \in G^*$ or $z \in yG^*$.

Case 1. $z \in G^*$, then either $z = (wx)^g$ or $z = (wx)^g w$.

Case 1a. The equality $z = (wx)^g$ is not possible, since $(wx)^g$ is an orientation preserving element.

Case 1b. If $z = (wx)^g w$, since $(yz)^2 = 1$ we have $y(wx)^g wy(wx)^g w = (wx)^{gt(t+1)} ywyw = (yw)^2 = 1$. Then $(xy)^2 = (yz)^2 = (yw)^2$, and $G^* = D_{2g} \times C_2 = \langle x, w\rangle \times \langle y\rangle$. The epimorphism $\theta^* : \Gamma^* \to G^* \cong \Gamma^*/\Gamma$, completely determined up to automorphisms of Γ^* or G^*, is:

$$\theta^*(c_0) = x, \theta^*(c_1) = y, \theta^*(c_2) = (wx)^g w, \theta^*(c_3) = w$$

Note that Γ' is the canonical Fuchsian subgroup of Γ^*. We shall see that the epimorphism θ^* restricted to Γ' is equivalent by automorphisms of Γ' and D_{2g} to the epimorphism constructed in the proof of Theorem 7. A set of generators of a canonical presentation of Γ' expresed in terms of the canonical presentation of Γ^* is:

$$\{x_1' = c_0c_1, x_2' = c_1c_2, x_3' = c_2c_3, x_4' = c_3c_0\}$$

The restriction of θ^* is:

$$\begin{array}{l} x_1' \to xy, x_2' \to y(wx)^g w \\ x_3' \to (wx)^g, x_4' \to wx \end{array}$$

and $\langle xy, wx\rangle \cong D_{2g}$ since $xy(wx)(xy)^{-1} = xw = (wx)^{-1}$. Hence θ^* restricted to Γ' is exactly the epimorphism in the proof of Theorem 7, where $xy = A$ and $D = wx$. Note that $\theta^*(\Gamma') = \langle xy, wx\rangle \cong D_{2g}$, is not the subgroup $\langle z, w\rangle \cong D_{2g}$ used in the construction of G^*.

Case 2. If $z \in yG^*$ then either $z = y(wx)^s$ or $z = y(wx)^s w$. Since $y(wx)^s w$ is orientation preserving, the second case is not possible. Assume $z = y(wx)^s$. From $z^2 = 1$ we have:

$$y(wx)^s y(wx)^s = (wx)^{st+s} = 1$$

so $s(t+1) \equiv 0 \bmod 2g$.

We have $(yz)^2 = 1$ then

$$yy(wx)^s yy(wx)^s = (wx)^{2s} = 1$$

so $s = g$ and $g(t+1) \equiv 0 \bmod 2g$.

Finally we have $(zw)^2 = 1$ then

$$y(wx)^g wy(wx)^g w = 1$$

$$(wx)^{tg}ywy(wx)^g w = 1$$
$$ywy = (wx)^{g(t-1)}w$$

and by $g(t+1) \equiv 0 \bmod 2g$ we have $ywy = w$, then $G^* = D_{2g} \times C_2 = \langle x, w \rangle \times \langle y \rangle$ and

$$\theta^*(c_0) = x, \theta^*(c_1) = y, \theta^*(c_2) = y(wx)^g, \theta^*(c_3) = w$$

The epimorphism θ^* is unique up to automorphism of Γ^* or G^*.

As in the preceding case, we shall see that the epimorphism θ^*, restricted to Γ', is equivalent by automorphisms of Γ' and G^* to the epimorphism constructed in the proof of Theorem 7. As before, a set of generators of a canonical presentation of Γ' expressed in terms of the canonical presentation of Γ^* is:

$$\{x'_1 = c_0c_1, x'_2 = c_1c_2, x'_3 = c_2c_3, x'_4 = c_3c_0\}$$

The restriction of θ^* is:

$$x'_1 \to xy, x'_2 \to (wx)^g$$
$$x'_3 \to (wx)^g w, x'_4 \to wx$$

and $\langle xy, wx \rangle \cong D_{2g}$. Hence θ^* restricted to Γ' is exactly the epimorphism in the proof of Theorem 7.

Case b. Γ^* has signature $(0; +; [2]; (2, 2g))$. Let

$$\left\langle a, c_0, c_1, c_2 : a^2 = c_i^2 = (c_0c_1)^2 = (c_1c_2)^{2g} = xc_0xc_2 = 1 \right\rangle$$

be a canonical presentation of Γ^*. Assume that the epimorphism $\theta^* : \Gamma^* \to G^* \cong \Gamma^*/\Gamma$, is given by

$$\theta^*(a) = x, \theta^*(c_0) = y, \theta^*(c_1) = z, \theta^*(c_2) = w.$$

Then we have that $\{x, y, z, w\}$ is a set of generators of G^* and x, y, z, w, yz have order 2, zw has order $2g$ and $xyxw = 1$.

As $xyxw = 1$ and $G^* \supsetneq G$ we have that $x \notin \langle z, w \rangle \cong D_{2g}$. The group $\langle z, w \rangle$ has index two in G^*, then is a normal subgroup of G^*. Hence:

$$xzx \in \langle z, w \rangle \text{ and } xwx \in \langle z, w \rangle$$

Also we have that xzx and xwx are the images by θ^* of orientation reversing transformations, then:

$$xzx = (zw)^{t_1}z \text{ and } xwx = (zw)^{t_2}z$$

Using that $(yz)^2 = 1$, we have

$$(xwx)z(xwx)z = (zw)^{t_2}zz(zw)^{t_2}zz = (zw)^{2t_2} = 1$$

from where $t_2 = g$. Again by $(yz)^2 = 1$, we have

$$(xwx)z(xwx)z = 1, \text{ then } w(xzx)w(xzx) = 1$$

so

$$w(zw)^{t_1}zw(zw)^{t_1}z = (zw)^{2t_1+2} = 1,$$

then $t_1 = g - 1$.

We have that the group G^* has presentation:

$$\begin{aligned}\langle x, z, w : x^2 = z^2 = w^2 = (zw)^{2g} = 1, xzx = (zw)^{g-1}z, xwx = (zw)^g z\rangle \\ \cong D_{2g} \rtimes_\varphi C_2 = \langle z, w\rangle \rtimes_\varphi \langle x\rangle\end{aligned}$$

where $\varphi : D_{2g} \to D_{2g}$ is $z \to (zw)^{g-1}z$ and $w \to (zw)^g z$.

The epimorphism θ^*, unique up to automorphisms of Γ^* or G^*, is:

$$a \to x; c_0 \to xwx; c_1 \to z; c_2 \to w$$

Note that Γ' is the canonical Fuchsian subgroup of Γ^*. We shall see that the epimorphism θ^* restricted to Γ' is equivalent, by automorphisms of Γ' and D_{2g}, to the epimorphism constructed in the proof of Theorem 7. A set of generators of a canonical presentation of Γ' expressed in terms of the canonical presentation of Γ^* is:

$$\{x'_1 = a, x'_2 = c_0ac_0, x'_3 = c_0c_1, x'_4 = c_1ac_0a = c_1c_2\}$$

The restriction is:

$$\begin{aligned}&x'_1 \to x, x'_2 \to xwxwx = (zw)^g zwx = (zw)^{g+1}x \\ &x'_3 \to xwxz = (zw)^g, x'_4 \to zw\end{aligned}$$

and $\langle x, zw\rangle \cong D_{2g}$ since

$$xzwx = xzxxwx = (zw)^{g-1}z(zw)^g z = (zw)^{-1}$$

Hence θ^* restricted to Γ' is exactly the epimorphism in the proof of Theorem 7, setting $x = A$ and $D = zw$. Note that $\theta^*(\Gamma')$ is $\langle x, zw\rangle \cong D_{2g}$ but it is not the subgroup $\langle z, w\rangle \cong D_{2g}$ used in the construction of G^*.

Since $xzx = (zw)^{g-1}z$, then $(zx)^2 = (zw)^{g-1}$. If g is even then zx has order $4g$ and $D_{2g} \rtimes_\varphi C_2$ is isomorphic to D_{4g}. Finally if g is odd then $(xz)^g$ has of order 2 and it is in the center of $D_{2g} \rtimes_\varphi C_2$, then $D_{2g} \rtimes_\varphi C_2 \cong \langle x, w\rangle \times \langle (xz)^g\rangle \cong D_{2g} \times C_2$. ∎

5 Symmetry type of Riemann surfaces with automorphism group of order $4g$

Theorem 11 *Let X be a Riemann surface of genus g, uniformized by a surface Fuchsian group Γ and with automorphism group G of order $4g$. If Γ' is a Fuchsian group with $\Gamma \leq \Gamma'$ and $X/Aut(X) = \mathbb{D}/\Gamma'$, we assume that the signature of Γ' is $(0; +; [2, 2, 2, 2g])$. Let $\mathrm{Aut}^{\pm}(X) = G^*$ be the extended automorphism group of X and let Γ^* be an NEC group such that $G^* \cong \Gamma^*/\Gamma$. Assume that the signature of Γ^* is $(0; +; [-]; \{(2, 2, 2, 2g)\})$. Then there are four conjugacy classes of anticonformal involutions and the sets of topological types are either $\{+2, 0, -2, -2\}$ if g is odd and $\{+1, 0, -1, -3\}$ if g is even, or $\{-1, -1, -g, -g\}$.*

Proof. By Theorem 10 the automorphism group in this case is isomorphic to

$$D_{2g} \times C_2 = \langle w, x : w^2 = x^2 = (wx)^{2g} = 1 \rangle \times \langle y : y^2 = 1 \rangle .$$

There are two possible epimorphisms $\theta_i^* : \Gamma^* \to G^*$, $i = 1, 2$:

$$\theta_1^*(c_0) = x, \theta_1^*(c_1) = y, \theta_1^*(c_2) = (wx)^g w, \theta_1^*(c_3) = w$$

and

$$\theta_2^*(c_0) = x, \theta_2^*(c_1) = y, \theta_2^*(c_2) = y(wx)^g, \theta_2^*(c_3) = w.$$

There are four conjugacy classes of involutions in $D_{2g} \times C_2$ not in $\theta_i^*(\Gamma')$, a set of representatives of each class is $\{x, y, y(wx)^g, w\}$.

For each involution ι in $\mathrm{Aut}^{\pm}(X) = G^*$ the number of fixed ovals of ι is given by the following formula of G. Gromadzki (cf. [12]):

$$\textstyle\sum_{c_i,\ \text{s.t. } \theta_i^*(c_i)^h = \iota} \left[C(G^*, \theta_i^*(c_i)) : \theta_i^*(C(\Gamma^*, c_i))\right].$$

For the epimorphism θ_1^* we have the following centralizers:

$C(G^*, \theta_1^*(c_0)) = C(G^*, x) = \langle x, (wx)^g \rangle \times \langle y \rangle$ and $\theta_1^*(C(\Gamma^*, c_0)) = \langle x, (wx)^g \rangle \times \langle y \rangle$

$C(G^*, \theta_1^*(c_1)) = C(G^*, y) = G^*$ and $\theta_1^*(C(\Gamma^*, c_1)) = \langle x, (wx)^g w \rangle \times \langle y \rangle$

$C(G^*, \theta_1^*(c_2)) = C(G^*, (wx)^g w) = \langle w, (wx)^g \rangle \times \langle y \rangle$ and $\theta_1^*(C(\Gamma^*, c_3)) = \langle w, (wx)^g \rangle \times \langle y \rangle$

$C(G^*, \theta_1^*(c_3)) = C(G^*, w) = \langle w, (wx)^g \rangle \times \langle y \rangle$ and $\theta_1^*(C(\Gamma^*, c_3)) = \langle w, (wx)^g \rangle$.

For the class of involutions $[x]$ we have either:

$[C(G^*, \theta_1^*(c_0)) : \theta_i^*(C(\Gamma^*, c_0))] = 1$ oval, if g is even or

$[C(G^*, \theta_1^*(c_0)) : \theta_i^*(C(\Gamma^*, c_0))] + [C(G^*, \theta_1^*(c_2)) : \theta_i^*(C(\Gamma^*, c_2))] = 2$ ovals, if g is odd.

Note that $\langle x, (wx)^g w\rangle$ is isomorphic to D_{2g} if g is even and it is isomorphic to D_g if g is odd. Hence the class of involutions $[y]$ has 2 ovals if g is odd and 1 oval if g is even.

There is no reflection c_i such that $\theta_1^*(c_i)$ is in the conjugacy class represented by $y(wx)^g$, then the anticonformal involutions in the class $[y(wx)^g]$ have no ovals.

Finally for the class of involutions $[w]$ we have either:

$[C(G^*, \theta_1^*(c_2)) : \theta_i^*(C(\Gamma^*, c_2))] + [C(G^*, \theta_1^*(c_3)) : \theta_i^*(C(\Gamma^*, c_3))] = 3$ ovals, if g is even or

$[C(G^*, \theta_1^*(c_3)) : \theta_i^*(C(\Gamma^*, c_3))] = 2$ ovals, if g is odd.

The set of topological types is $\{\pm 2, \pm 2, \pm 2, 0\}$ if g is odd and $\{\pm 3, \pm 1, \pm 1, 0\}$ if g is even. Now applying Theorem 3.4.4 of [5], the topological types of the anticonformal involutions are $\{+2, 0, -2, -2\}$ if g is odd and $\{+1, 0, -1, -3\}$ if g is even.

For the epimorphism θ_2^* we have:

$C(G^*, \theta_2^*(c_0)) = C(G^*, x) = \langle x, (wx)^g\rangle \times \langle y\rangle$ and $\theta_1^*(C(\Gamma^*, c_0)) = \langle x, (wx)^g\rangle \times \langle y\rangle$, then $[x]$ has one oval.

$C(G^*, \theta_2^*(c_1)) = C(G^*, y) = G^*$ and $\theta_1^*(C(\Gamma^*, c_1)) = \langle x, (wx)^g\rangle \times \langle y\rangle$, thus $[y]$ has g ovals.

$C(G^*, \theta_2^*(c_2)) = C(G^*, y(wx)^g) = G^*$ and $\theta_1^*(C(\Gamma^*, c_2)) = \langle w, (wx)^g\rangle \times \langle y\rangle$, then $[y(wx)^g]$ has g ovals.

$C(G^*, \theta_2^*(c_3)) = C(G^*, w) = \langle w, (wx)^g\rangle \times \langle y\rangle$ and $\theta_1^*(C(\Gamma^*, c_3)) = \langle w, (wx)^g\rangle \times \langle y\rangle$, thus $[w]$ has one oval.

The set of topological types is $\{\pm 1, \pm 1, -g, -g\}$.

By Theorem 3.4.4 of [5] the topological types of the anticonformal involutions are $\{-1, -1, -g, -g\}$. ■

Theorem 12 *Let X be a Riemann surface of genus g, uniformized by a surface Fuchsian group Γ and with automorphism group G of order $4g$. If Γ' is a Fuchsian group with $\Gamma \le \Gamma'$ and $X/Aut(X) = \mathbb{D}/\Gamma'$, we assume that the signature of Γ' is $(0; +; [2, 2, 2, 2g])$. Let $Aut^{\pm}(X) = G^*$ be the extended automorphism group of X and Γ^* be an NEC group such that $G^* \cong \Gamma^*/\Gamma$. Assume that the signature of Γ^* is $(0; +; [2]; \{(2, 2g)\})$. The set of topological types of the anticonformal involutions of X is $\{0, 0, -2, -2\}$ if the genus g is odd and $\{-2\}$ if the genus g is even.*

Proof. By Theorem 10 the automorphism group in this case is isomorphic to

$$\left\langle x, z, w : x^2 = z^2 = w^2 = (zw)^{2g} = 1, xzx = (zw)^{g-1}z, xwx = (zw)^g z\right\rangle$$
$$= D_{2g} \rtimes_{\varphi} C_2$$

If g is odd, there are four conjugacy classes of orientation reversing order two elements in $Aut^{\pm}(X) \cong D_{2g} \times C_2 = \langle x, w\rangle \times \langle (xz)^g\rangle$, a set of representatives of each class is $\{z, w, (xz)^g, (xw)^g\}$. If g is even the group $D_{2g} \rtimes_{\varphi} C_2$ is isomorphic to D_{4g} and there is only a conjugacy class of orientation reversing involutions represented by z.

The epimorphism $\theta^* : \Gamma^* \to G^*$ is:

$$a \to x; c_0 \to xwx; c_1 \to z; c_2 \to w$$

Assume that g is odd. To use the formula of Gromadzky ([12]) we need to compute the centralizers:

$$C(G^*, \theta^*(c_1)) = C(G^*, z) = \langle z, (zw)^g, (xz)^g\rangle \cong C_2 \times C_2 \times C_2$$
$$C(G^*, \theta^*(c_2)) = C(G^*, w) = \langle w, (zw)^g, (xw)^g\rangle \cong C_2 \times C_2 \times C_2.$$

Now we have:

$$[C(G^*, \theta^*(c_2)) : \theta^*(C(\Gamma^*, c_2))] = [C(G^*, \theta^*(c_1)) : \theta^*(C(\Gamma^*, c_1))] = 2$$

The number of ovals of the involutions in the conjugacy classes $[z]$ and $[w]$ is 2.

The conjugacy classes $[(xz)^g]$ and $[(xw)^g]$ correspond to involutions without ovals.

If g is even we have:

$$C(G^*, \theta^*(c_1)) = C(G^*, z) = \left\langle z, (xz)^{2g}\right\rangle \cong C_2 \times C_2$$
$$C(G^*, \theta^*(c_2)) = C(G^*, w) = \left\langle w, (xz)^{2g}\right\rangle \cong C_2 \times C_2.$$

Therefore the number of ovals of the involutions in $[z]$ is:

$$[C(G^*, \theta^*(c_0)) : \theta^*(C(\Gamma^*, c_0))] + [C(G^*, \theta^*(c_1)) : \theta^*(C(\Gamma^*, c_1))] = 2.$$

Applying Theorem 3.3.2 of [5], the topological types are: $\{0, 0, -2, -2\}$ if g is odd and $\{-2\}$ if g is even. ■

6 On the set of points with automorphism group of order $4g$ in the moduli space of Riemann surfaces

In this section we study family $\mathcal{F}_g$ of surfaces with $4g$ automorphims as subspace of the moduli space $\mathcal{M}_g$.

Theorem 13 *The set of points $\mathcal{F}_g \subset \mathcal{M}_g$ corresponding to Riemann surfaces of genus $g \geq 2$ given in Theorem 7 is the Riemann sphere with three punctures.*

Proof. Let $\mathbf{T}_g$ be the Teichmüller space of classes of surface Fuchsian groups of genus g and let $\pi : \mathbf{T}_g \to \mathcal{M}_g$ be the canonical projection. The points in $\pi^{-1}(\mathcal{F}_g)$ are classes of surface Fuchsian groups contained in Fuchsian groups with signature $(0; +; [2, 2, 2, 2g])$. By Theorem 7, up to automorphisms of Fuchsian groups and dihedral groups, there is only one possible normal inclusion of surface groups of genus g in groups with signature $(0; +; [2, 2, 2, 2g])$, this inclusion produces $i_* : \mathbf{T}_{(0;+;[2,2,2,2g])} \to \mathbf{T}_g$ and $\pi \circ i_*(\mathbf{T}_{(0;+;[2,2,2,2g])}) \supset \mathcal{F}_g$. The set $i_*(\mathbf{T}_{(0;+;[2,2,2,2g])})$ is an open disc and the map $\pi \circ i_* |_{(\pi \circ i_*)^{-1}(\mathcal{F}_g)}$ is the projection given by the action of a properly discontinuous group, then $\mathcal{F}_g$ is a real non-compact Riemann surface.

Since the epimorphism θ in the proof of Theorem 7 is unique (up to automorphisms), the family $\mathcal{F}_g$ admits a covering that is the space $O_{2,2,2,2g}$ of orbifolds with three conic points of order 2, and one of order $2g$ minus a point p, corresponding to the surface of the family with more than $4g$ automorphisms (the Wiman surface X_{8g}). The point p is the orbifold constructed with four hyperbolic triangles with angles $\pi/2, \pi/4$ and $\pi/4g$. The three conic points of order two may be pairwise as near as we want giving in this way three punctures in the space of orbifolds $O_{2,2,2,2g}$, hence topologically is the sphere without three points. Now there is an automorphism of order two in $O_{2,2,2,2g} \setminus \{p\}$ compatible with θ, so $\mathcal{F}_g$ is isomorphic to the Riemann sphere with three punctures.

The Riemann surface $\mathcal{F}_g$ admits an anticonformal involution whose fixed point set is formed by the real Riemann surfaces in $\mathcal{F}_g$. ■

Theorem 14 *The real Riemann surface $\mathcal{F}_g$ has an anticonformal involution whose fixed point set consists of three arcs a_1, a_2, b, corresponding to the real Riemann surfaces in the family. The topological closure of $\mathcal{F}_g$ in $\widehat{\mathcal{M}_g}$ has an anticonformal involution whose fixed point set $\overline{a_1 \cup a_2 \cup b}$ (closure of $a_1 \cup a_2 \cup b$ in $\widehat{\mathcal{M}_g}$) is a closed Jordan curve. The set $\overline{a_1 \cup a_2 \cup b} \setminus (a_1 \cup a_2 \cup b)$ consists of three points: two nodal surfaces and the Wiman surface of type II.*

Proof. In $\mathcal{F}_g$, the surfaces have exactly $4g$ automorphisms, therefore to complete $\mathcal{F}_g$ to $\overline{\mathcal{F}}_g$ (the topological closure of $\mathcal{F}_g$ in $\widehat{\mathcal{M}_g}$), it is necessary to add surfaces with more than $4g$ automorphisms and nodal surfaces in $\widehat{\mathcal{M}_g} \setminus \mathcal{M}_g$.

The surfaces in $\mathcal{F}_g$ having anticonformal automorphisms correspond to the two inclusions i_1 and i_2 of Fuchsian groups with signature $(0;+;[2,2,2,2g])$ in NEC groups with signature $(0;+;[-];\{(2,2,2,2g)\})$ and the inclusion j in NEC groups with signature $(0;+;[2];\{(2,2g)\})$, see Theorem 10. The set of points in $\mathcal{F}_g$ having anticonformal involutions are the following subsets of $\mathcal{M}_g$:

$$\pi \circ i_* \circ i_{1*}(\mathbf{T}_{(0;+;[-];\{(2,2,2,2g)\})}) = a_1$$
$$\pi \circ i_* \circ i_{2*}(\mathbf{T}_{(0;+;[-];\{(2,2,2,2g)\})}) = a_2$$
$$\pi \circ i_* \circ j_*(\mathbf{T}_{(0;+;[2];\{(2,2g)\})}) = b.$$

Since $\mathbf{T}_{(0;+;[-];\{(2,2,2,2g)\})}$ and $\mathbf{T}_{(0;+;[2];\{(2,2g)\})}$ are of real dimension 1, by Proposition 1 the sets a_1, a_2 and b are connected 1-manifolds. Let $\overline{a}_1, \overline{a}_2$ and $\overline{b}$ be the closures of a_1, a_2 and b in $\widehat{\mathcal{M}}_g$. Now we shall describe the surfaces in $\overline{a_1 \cup a_2 \cup b} \setminus (a_1 \cup a_2 \cup b)$.

The arc a_1 contains the surfaces with anticonformal involutions of topological types $\{+2,0,-2,-2\}$ or $\{+1,0,-1,-3\}$, a_2 the surfaces with anticonformal involutions of topological types $\{-1,-1,-g,-g\}$, and b the surfaces with anticonformal involutions of topological types $\{0,0,-2,-2\}$ or $\{-2\}$.

Let $\mathcal{G}$ be the graph of a nodal surface in the clousure of $\mathcal{F}_g$ in $\widehat{\mathcal{M}}_g$. By the main theorem of [9] the graph $\mathcal{G}$ has $[D_{2g} : H(\theta \circ \delta)]$ vertices, where $\theta : \Gamma' \to D_{2g}$ is the epimorphism given in Theorem 7, δ is an automorphism of the group Γ' with signature $(0;+;[2,2,2,2g])$ and $H(\theta\circ\delta) = \langle \theta\circ\delta(x_1x_2), \theta\circ\delta(x_3), \theta\circ\delta(x_4)\rangle$.

First of all, we shall consider the nodal surfaces in the clousure of the arc a_1. Let $X = \mathbb{D}/\Gamma$ be a Riemann surface in the arc a_1. Then there is an NEC group Γ^* of signature $(0;+;[-];\{(2,2,2,2g)\})$ such that $Aut^{\pm}(X) \cong \Gamma^*/\Gamma$. The group $Aut^{\pm}(X)$ is isomorphic to:

$$D_{2g} \times C_2 = \langle w, x : w^2 = x^2 = (wx)^{2g} = 1\rangle \times \langle y : y^2 = 1\rangle$$

and the epimorphism $\theta_1^* : \Gamma^* \to \Gamma^*/\Gamma \cong D_{2g} \times C_2$ is defined in a canonical presentation of Γ^* by:

$$\theta_1^*(c_0) = x, \theta_1^*(c_1) = y, \theta_1^*(c_2) = (wx)^g w, \theta_1^*(c_3) = w.$$

The restriction θ_1^{*+} of θ_1^* to $(\Gamma^*)^+$ is

$$\theta_1^*(c_0c_1) = \theta_1^{*+}(x_1) = xy, \theta_1^*(c_1c_2) = \theta_1^{*+}(x_2) = y(wx)^g w,$$
$$\theta_1^*(c_2c_3) = \theta_1^{*+}(x_3) = (wx)^g, \theta_1^*(c_3c_0) = \theta_1^{*+}(x_3) = wx.$$

The nodal surfaces that are limits of the real surfaces in the arc a_1 are given by automorphisms δ of the group $(\Gamma^*)^+$ such that $\theta\circ\delta$ is $\theta_1^{*+}\circ\gamma$, where γ is an automorphism of the group Γ^*. This fact reduces the possible graphs of the nodal surfaces in $\overline{a_1}\cap(\widehat{\mathcal{M}}_g\setminus\mathcal{M}_g)$ to two: $\mathcal{G}^1(\theta)$ and $\mathcal{G}^2(\theta)$. If $H^1(\theta_1^*)$ is the subgroup of D_{2g} generated by $\theta_1^{*+}(x_1x_2)$, $\theta_1^{*+}(x_3)$, $\theta_1^{*+}(x_4)$, the number of vertices of $\mathcal{G}^1(\theta)$ is given by $[D_{2g}:H^1(\theta_1^*)]$, and for $\mathcal{G}^2(\theta)$ the number of vertices is given by the index of the subgroup $H^2(\theta_1^*)$ of D_{2g} generated by $\theta_1^{*+}(x_1)$, $\theta_1^{*+}(x_2x_3)$, $\theta_1^{*+}(x_4)$. Hence the number of components (vertices of the corresponding graphs) of such nodal surfaces are, respectively:

$$[D_{2g}:\langle\theta_1^*(c_0c_2),\theta_1^*(c_2c_3),\theta_1^*(c_3c_0)\rangle] =$$
$$[D_{2g}:\left\langle(wx)^{g-1},(wx)^g,wx\right\rangle] = [D_{2g}:\langle wx\rangle] = 2$$

and

$$[D_{2g}:\langle\theta_1^*(c_0c_1),\theta_1^*(c_1c_3),\theta_1^*(c_3c_0)\rangle] =$$
$$[D_{2g}:\langle xy,yw,wx\rangle] = 1.$$

By the main theorem of [9] the degree of the vertices of the graph $\mathcal{G}^1(\theta_1^*)$ is $[H^1(\theta_1^*):\langle\theta_1^*(x_1x_2)\rangle]$. Since $[H^1(\theta_1^*):\langle\theta_1^*(x_1x_2)\rangle]=1$ if g is even and 2 if g is odd, the vertices of the graph $\mathcal{G}^1(\theta_1^*)$ have degree 1 or 2, and the graph has two vertices and one or two edges joining them, the graph $\mathcal{G}^1(\theta_1^*)$ is a 1- or 2-dipole.

By [9] and since $[H^2(\theta_1^*):\langle\theta_1^*(x_2x_3)\rangle]=g$, the graph $\mathcal{G}^2(\theta_1^*)$ has one vertex and g loops. We call X_D the nodal surface corresponding to $\mathcal{G}^1(\theta_1^*)$ and X_R the nodal surface corresponding to $\mathcal{G}^2(\theta_1^*)$.

Each vertex of $\mathcal{G}^i(\theta_1^*)$ corresponds to one component of the nodal surface. The uniformization groups of the components of X_D and X_R are $\ker\omega_1$ and $\ker\omega_2$ respectively, where the homorphisms $\omega_i:\widehat{\Gamma}\to D_{2g}$, $i=1,2$ are defined by:

$$\omega_1:\gamma_1\to\theta_1^*(c_0c_2)=\theta_1^*(x_1x_2)=x(wx)^g w,$$
$$\gamma_2\to\theta_1^*(c_2c_3)=\theta_1^*(x_3)=(wx)^g,$$
$$\gamma_3\to\theta_1^*(c_3c_0)=\theta_1^*(x_4)=wx.$$

from a Fuchsian group $\widehat{\Gamma}$ with signature $(0;+;[\infty,2,2g])$ (one parabolic class of transformations) and presentation $\left\langle\gamma_i:\gamma_1\gamma_2\gamma_3=\gamma_2^2=\gamma_2^{2g}\right\rangle$. As a consequence each component of X_D has genus $\frac{g}{2}$ if g is even and $\frac{g-1}{2}$ if g is odd.

Now

$$\omega_2 : \gamma_1 \to \theta_1^*(c_0c_1) = \theta_1^*(x_1) = xy,$$
$$\gamma_2 \to \theta_1^*(c_1c_3) = \theta_1^*(x_2x_3) = yw,$$
$$\gamma_3 \to \theta_1^*(c_3c_0) = \theta_1^*(x_4) = wx$$

where $\widehat{\Gamma}$ has presentation $\left\langle \gamma_i : \gamma_1\gamma_2\gamma_3 = \gamma_1^2 = \gamma_2^{2g} \right\rangle$ and signature $(0;+;[\infty,2,2g])$. The component of X_R has genus 0.

The set $\overline{a}_1$ intersects $\widehat{\mathcal{M}_g} \setminus \mathcal{M}_g$ in two points to : X_D and X_R, thus a_1 is an arc.

Let X_{8g} be the Wiman surface of type II with automorphism group of order $8g$ (for $g = 2$, $\mathrm{Aut}(X_{16}) = GL(2,3)$) and such that the signature of the Fuchsian group Δ uniformizing $X_{8g}/\mathrm{Aut}^{\pm}(X_{8g})$ is $(0;+;[-];\{(2,4,4g)\})$ (signature $(0;+;[-];\{(2,3,8)\})$ for $g = 2$). The surface X_{8g} belongs to the clousure of the arc a_2 since a group of signature $(0;+;[-];\{(2,2,2,2g)\})$ is contained in Δ and the epimorphism θ_2^* may be extended to Δ.

There is also one point in $\overline{a_2} \cap (\widehat{\mathcal{M}_g} \setminus \mathcal{M}_g)$. The graph of $\overline{a_2} \cap (\widehat{\mathcal{M}_g} \setminus \mathcal{M}_g)$ has only one vertex since:

$$\theta_2^*(c_0) = x, \theta_2^*(c_1) = y, \theta_2^*(c_2) = y(wx)^g, \theta_2^*(c_3) = w$$

$$\theta_2^*(c_0c_1) = xy, \theta_2^*(c_1c_2) = (wx)^g, \theta_2^*(c_2c_3) = y(wx)^g w, \theta_2^*(c_3c_0) = wx$$

$$\theta_2^*(c_0c_2) = xy(wx)^g, \theta_2^*(c_2c_3) = y(wx)^g w, \theta_2^*(c_3c_0) = wx$$

$$\theta_2^*(c_0c_1) = xy, \theta_2^*(c_1c_3) = yw, \theta_2^*(c_3c_0) = wx$$

and $\langle yx(wx)^g, y(wx)^g w, wx \rangle = \langle yx, yw, wx \rangle \cong D_{2g}$. Hence $X_R \in \overline{a_2} \cap (\widehat{\mathcal{M}_g} \setminus \mathcal{M}_g)$. Therefore $\overline{a}_2 \setminus a_2$ has two points: X_R and X_{8g}, thus a_2 is an arc.

Finally, in a similar way one sees that b joins X_D and X_{8g}, so b is an arc and $\overline{a_1 \cup a_2 \cup b}$ is a closed Jordan curve, the fixed point set of an anticonformal involution of $\overline{\mathcal{F}}_g$. ■

Remark 15 *The surfaces in the arc a_2 are the surfaces having maximal number of ovals among the Riemann surfaces of genus g with four non-conjugate anticonformal involutions, two of which do not commute (see Theorem 1 in [13]).*

Remark 16 *As a consequence of the above theorem we have that $\overline{\mathbb{R}\mathcal{F}_g} \cap \mathcal{M}_g$ has two connected components, then we cannot always continuously deform a real algebraic curve with $4g$ automorphisms to another real algebraic curve with the same characteristics mantaining the real character and the number of automorphisms along the path.*

References

[1] Accola, R. D. M. On the number of automorphisms of a closed Riemann surface. *Trans. Amer. Math. Soc.* **131** (1968) 398–408.

[2] Bers L. On spaces of Riemann surfaces with nodes. *Bull. Amer. Math. Soc.* **80** (1974), no. 6, 1219-1222.

[3] Bujalance, E. On compact Klein surfaces with a special automorphism group. *Ann. Acad. Sci. Fenn. Math.* **22** (1997), no. 1, 15–20.

[4] Bujalance, E.; Cirre, F. J. A family of Riemann surfaces with orientation reversing automorphisms. In the tradition of Ahlfors-Bers. V, 25–33, Contemp. Math., 510, Amer. Math. Soc., Providence, RI, 2010.

[5] Bujalance, E.; Cirre, F. J.; Gamboa, J. M.; Gromadzki, G. Symmetry types of hyperelliptic Riemann surfaces. *Mém. Soc. Math. Fr. (N.S.)* No. **86** (2001), vi+122 pp.

[6] Bujalance, E.; Etayo, J. J.; Gamboa, J. M.; Gromadzki, G. Automorphism groups of compact bordered Klein surfaces. A combinatorial approach. Lecture Notes in Mathematics, 1439. Springer-Verlag, Berlin, 1990. xiv+201 pp.

[7] Bujalance, E.; Turbek, P. On Klein surfaces with $2p$ or $2p+2$ automorphisms. *J. Algebra* **235** (2001), no. 2, 559–588.

[8] Bujalance, E.; Gromadzki, G.; Turbek, P. On non-orientable Riemann surfaces with $2p$ or $2p+2$ automorphisms. *Pacific J. Math.* **201** (2001), no. 2, 267–288.

[9] Costa A. F.; González-Aguilera V. Limits of equisymmetric 1-complex dimensional families of Riemann surfaces, Preprint 2015.

[10] Costa, A. F.; Parlier, H. On Harnack's theorem and extensions: a geometric proof and applications. *Conform. Geom. Dyn.* **12** (2008), 174–186.

[11] Deligne, P. and Mumford, D. The irreducibility of the space of curves. *Publications Mathématiques de L'I.H.E.S.* **36** (1965) 75-109, .

[12] Gromadzki, G.: On a Harnack–Natanzon theorem for the family of real forms of Riemann surfaces, *J. Pure Appl. Algebra* **121** (1997), 253–269.

[13] Gromadzki, G.; Izquierdo, M. On ovals of Riemann surfaces of even genera. *Geom. Dedicata* **78** (1999), no. 1, 81–88.

[14] Hirose, S. On periodic maps over surfaces with large periods. *Tohoku Math. J.* (2) **62** (2010), no. 1, 45–53.

[15] Kulkarni, R. S. A note on Wiman and Accola-Maclachlan surfaces. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **16** (1991), no. 1, 83–94.

[16] Kulkarni, R. S. Riemann surfaces admitting large automorphism groups. Extremal Riemann surfaces (San Francisco, CA, 1995), 63–79, *Contemp. Math.*, 201, Amer. Math. Soc., Providence, RI, 1997.

[17] Macbeath, A. M.; Singerman, D. Spaces of subgroups and Teichmüller space. *Proc. London Math. Soc.* **31** (1975), no. 2, 211–256.

[18] Maclachlan, C. A bound for the number of automorphisms of a compact Riemann surface. *J. London Math. Soc.* **44** (1969) 265–272.

[19] Nakagawa, K. On the orders of automorphisms of a closed Riemann surface. *Pacific J. Math.* **115** (1984), no. 2, 435–443.

[20] Natanzon, S. M. Moduli of Riemann surfaces, real algebraic curves, and their superanalogs. *Translations of Mathematical Monographs*, 225. American Mathematical Society, Providence, RI, 2004. viii+160 pp.

[21] Singerman, D. Finitely maximal Fuchsian groups. *J. London Math. Soc.* (2) **6** (1972), 29–38.

[22] Wiman, A. Über die hyperelliptischen Curven und diejenigen von Geschlechte p - Jwelche eindeutige Tiansformationen in sich zulassen. *Bihang till K. Svenska Vet.-Akad. Handlingar, Stockholm* **21** (1895-6) 1-28.