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Necessary Conditions for Nondominated Solutions in Vector Optimization

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Abstract In this paper, we study characterizations and necessary conditions for nondominated points of sets and nondominated solutions of vector-valued functions in vector optimization with variable domination structure. We study not only the case, where the intersection of all the involved domination sets has a nonzero element, but also the case, where it might be the singleton. While the first case has been studied earlier, the second case has not, to the

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best of our knowledge, done yet. Our results extend and improve the existing results in vector optimization with a fixed ordering cone and with a variable ordering structure.

Keywords Vector optimization · Domination structures · Nondominated solutions · Generalized differentiation · Nonlinear scalarization functions

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1 Introduction

This paper addresses problems of set and vector-valued functions in optimization with a variable domination structure. It can be viewed as an extension of vector optimization in which domination sets vary in decision spaces.

The nondomination concept in multiobjetive optimization, with respect to domination structures, was introduced by Yu in [1] and studied in many publications; see, e.g., [2–4] and recent papers [5–9]. This concept is more general than efficiency and it is applicable to decision making, games, image registration in medical engineering, etc.

There are a few necessary conditions for this kind of optimal solutions in the literature obtained for several special classes of ordering structures:

- In [10], Engau formulated necessary conditions for nondominated points to sets with respect to ordering structures, whose domination factor set are ideal-symmetric convex cones. His technique heavily relies on the geometric angles in $\mathbb{R}^2$ and $\mathbb{R}^3$.

- In [9], Eichfelder and Ha established generalized Fermat and Lagrange multiplier rules for nondominated solutions for a special case, where the cones describing this ordering structure are Bishop-Phelps cones.
- In [5], Bao and Mordukhovich obtained necessary conditions for nondominated points of sets and for nondominated solutions of vector optimization problems for the class of convex-cone-valued ordering structures with a nontrivial intersection. In [6], Bao extended them to ordering structures whose images satisfy a local asymptotic closedness property. They used the dual approach based on the extremal principle of variational analysis.
- In [11], Durea et al. derived necessary conditions for nondominated solutions for convex-cone-valued ordering structures in the dual approach; it bases on the characteristic of openness property at nondominated solutions. Besides the nontrivial intersection of domination sets, the structure is assumed to be lower semicontinuous. It is important to emphasize that the authors studied ordering structures acting between the domain and decision spaces as the cost map.

The crucial requirement in the aforementioned papers is that there is a nontrivial vector in the intersection of all ordering cones.

In this paper, we use the Gerstewitz (nonlinear separation) scalarization function introduced in [12,13] (see also Krasnoselskii [14] for assertions in the context of operator theory and compare the scalar optimization problem by

Pascoletti, Serafini [15]) to establish new necessary conditions for nondominated points of sets and nondominated solutions to multiobjective optimization problems with respect to general ordering structures with or without the nonzero intersection of all the domination sets. Note that weak counterparts of nondominated points and nondominated solutions can be studied similarly while we omit such a study in this paper. Note also that the obtained results can be extended to set-valued optimization in the setting of [5,6].

The rest of the paper is organized as follows. Section 2 presents some preliminaries on optimization with ordering structure and tools of nonlinear scalarization. In Section 3, we study characterizations of nondominated points of sets in real linear spaces in two cases: (1) the domination intersection has a nonzero element and (2) it is a singleton; i.e., it is equal to $\{0\}$. Section 4 is devoted to derive necessary conditions for nondominated solutions of vector-valued functions in vector optimization with general domination structures.

2 Preliminaries

Let Y be a real linear space and A be a nonempty set in Y. The set A is said to be pointed if $A \cap (-A) \subseteq \{0\}$, and a cone if $\lambda a \in A$ for all $a \in A$ and $\lambda \geq 0$. A cone A is called nontrivial if $A \notin \{\{0\}, Y\}$. The cone generated by the set A is denoted by $\operatorname{cone} A := \{\lambda a : \lambda \geq 0, a \in A\}$. The notations $\operatorname{core} A$ and $\operatorname{vcl} A$ stand for the algebraic interior and the vector closure of A, respectively, i.e.,

$$\operatorname{core} A := \{y \in A : \forall v \in Y, \exists \lambda > 0 : \; y + [0, \lambda]v \subseteq A\},$$

$$\operatorname{vcl} A := \{y \in Y : \exists v \in Y, \forall \lambda > 0, \exists t \in [0, \lambda] : \; y + tv \in A\}.$$

For a vector $k \in Y$, we denote the vector closure of A in the direction k by

$$\mathrm{vcl}_k A := \{y \in Y : \forall \lambda > 0, \exists t \in [0, \lambda] : \ y + tk \in A\}.$$

It is said that A is k-vectorially closed if $\mathrm{vcl}_k A = A$. When Y is a real linear topological space, notations $\mathrm{int}\, A$ and $\mathrm{cl}\, A$ stand for the topological interior and closure of the set A, respectively. For more results on directionally vector closedness and relationships with vector closedness and topological closedness, see [16,17] and the references therein.

Let C be a nontrivial cone in Y. The set A is said to be free-disposal with respect to (w.r.t.) C if $A + C = A$. This notion was introduced by Debreu in [18] for a convex cone C. The epigraphical set of A w.r.t. C is defined by $\mathcal{E}_C(A) := A + C$. Obviously, $\mathcal{E}_C(A)$ is free-disposal w.r.t. C provided that $C + C \subseteq C$; in particular if C is a convex cone. For the sake of simplicity, we denote $\mathcal{E}_q(A) := A + \mathrm{cone}\, q$ for some $q \in Y$.

2.1 Optimality Concepts

The nondomination concept in vector optimization with a domination structure defined below was introduced by Yu [1] and then further generalized by himself and his colleagues in [2–4]. It is more general than the classical (Pareto) efficiency notion in vector optimization (with a fixed ordering cone).

Let D be a nonempty set in Y. It is called a *domination* set if $0 \in D$ and an *ordering cone* if D is a cone. Denoting the ordering relation on Y by $\leq_D$, we have $y \leq_D v$ if and only if $y \in v - D$. If D is a convex cone, then $\leq_D$ is a preorder on Y.

A domination structure of the linear space Y is a set-valued map $\mathcal{D} : Y \rightrightarrows Y$ assigning to each element y in the decision space Y one domination set $\mathcal{D}(y)$. Given a domination set D, we define the constant domination structure $\mathcal{C}_D : Y \rightrightarrows Y$ by $\mathcal{C}_D(y) = D$ for all $y \in Y$.

Given a domination structure $\mathcal{D}$ in the decision space Y and two elements y and v, we can define two ordering relations. Denoting the *domination* relation by $\leq^n_{\mathcal{D}}$ and the *efficiency* relation by $\leq^e_{\mathcal{D}}$, we define

$$y \leq^n_{\mathcal{D}} v :\Longleftrightarrow v \in y + \mathcal{D}(y), \qquad y \leq^e_{\mathcal{D}} v :\Longleftrightarrow y \in v - \mathcal{D}(v).$$

When $\mathcal{D} = \mathcal{C}_D$, both ordering relations $\leq^n_{\mathcal{D}}$ and $\leq^e_{\mathcal{D}}$ reduce to the common ordering relation $\leq_D$.

For the sake of brevity, we do not study the weak counterparts of the domination and efficiency relations which are defined by replacing a domination set by its interior. We believe that similar results could be obtained by using analogous reasoning or even simpler arguments since the domination sets enjoys an additional nonempty interiority requirement.

Definition 2.1 ($\preceq$-minimality). Given an (ordering) binary relation $\preceq$ in a decision space Y, a nonempty subset A in Y and an element $\bar{a} \in A$, we say

(i) $\bar{a}$ is a minimal point w.r.t. $\preceq$ (for brevity, $\preceq$-minimal) of A if

$$\forall a \in A, a \preceq \bar{a} \Longrightarrow \bar{a} \preceq a.$$

(ii) $\overline{a}$ is a strictly $\preceq$-minimal point of A if there is no point $a \in A \setminus \{\overline{a}\}$ such that $a \preceq \overline{a}$. In other words, if $a \npreceq \overline{a}$, for all $a \in A \setminus \{\overline{a}\}$.

(iii) A (strictly) $\leq^n_{\mathcal{D}}$-minimal (resp. $\leq^e_{\mathcal{D}}$-minimal and $\leq_D$-minimal) point is called a (strictly) nondominated (resp. efficient and minimal) point w.r.t. $\mathcal{D}$ (resp. $\mathcal{D}$ and D) .

Given a domination structure $\mathcal{D} : Y \rightrightarrows Y$ and a nonempty set $A \subseteq Y$, the notations $\mathcal{D}(A)$ and $\mathcal{D}_A$ stand for the image set of $\mathcal{D}$ over A and the intersection of domination sets $\mathcal{D}(y)$ for $y \in A$, respectively. We call them the union and intersection domination set of $\mathcal{D}$ over A. The epigraphical set of A w.r.t. $\mathcal{D}$ is denoted by $\mathcal{E}_{\mathcal{D}}(A) := \bigcup\{a + \mathcal{D}(a) : a \in A\}$.

The pointedness property of cones is often required in vector optimization. It is extended to domination structures in the following definition.

Definition 2.2 (pointed domination structures). Let $\mathcal{D} : Y \rightrightarrows Y$ be a domination structure of a linear space Y, A be a nonempty set in Y, and $\overline{a} \in A$. The domination structure is said to be pointed at $\overline{a}$ over A, if

$$\mathcal{D}(A) \cap (-\mathcal{D}(\overline{a})) = \{0\}.$$

When $\mathcal{D} = \mathcal{C}_D$ for some domination set D, the domination structure $\mathcal{C}_D$ is pointed if and only if D is pointed.

Proposition 2.1 (relations among minimality and strict minimality). *Let $\mathcal{D} : Y \rightrightarrows Y$ be a domination structure of a real linear space Y, A be a nonempty subset in Y, and $\overline{a} \in A$. Assume that $\mathcal{D}$ is pointed at $\overline{a}$ over A. Then,* *if $\overline{a}$ is a $\preceq$-minimal point of A, then it is strictly $\preceq$-minimal to A, where $\preceq$ stands for both $\leq^n_{\mathcal{D}}$ and $\leq^e_{\mathcal{D}}$.*

Proof The proof is straightforward from Definition 2.1 and the pointedness of domination structures and thus it is omitted. □

Remark 2.1 It is obvious that when $\leq^n_D$ and $\leq^e_D$ are anti-symmetric, the conclusions in the above proposition is automatic. In [19, Lemma 1.10(vi)] it is established, for convex-valued and cone-valued structures, that they are anti-symmetric provided that $\mathcal{D}(Y)$ is a convex and pointed cone.

Next, we collect some important relations among these minimal points. Here, we do not assume that domination sets are convex cones.

Proposition 2.2 (relations between nondominated and efficient points of sets). *Let $\mathcal{D} : Y \rightrightarrows Y$ be a domination structure of a real linear space Y, A be a subset in Y, D be a domination set of Y, and $\overline{a} \in A$. The following statements hold:*

(i) If $\overline{a}$ is a strictly nondominated point of A w.r.t. $\mathcal{D}$, then it is a strictly minimal point of A w.r.t. $\mathcal{D}_A$.

(ii) If $\overline{a}$ is a strictly minimal point of A w.r.t. $\mathcal{D}(A)$, then it is a stricty nondominated point of A w.r.t. $\mathcal{D}$.

(iii) If $\overline{a}$ is a minimal (resp., strictly minimal) point of A w.r.t. D, then it is an efficient and a nondominated (resp., strictly efficient and strictly nondominated) point of A w.r.t. the constant domination structure $\mathcal{C}_D$.

(iv) If $\overline{a}$ is a strictly efficient point of A w.r.t. $\mathcal{D}$, then it is a strictly minimal point of A w.r.t. $\mathcal{D}(\overline{a})$.

Proof They are straightforward from Definition 2.1. □

Remark 2.2 [5, Proposition 3.1] presents (i) and (iv) in which the strictly nondominated solutions are called dominated solutions.

(i) does not hold for nondominated points of A w.r.t. $\mathcal{D}$. Indeed, let $Y = \mathbb{R}^2$, $A = \{a_1 = (-1,0), a_2 = (0,0), a_3 = (1,0)\}$ and $\mathcal{D} : Y \rightrightarrows Y$ be given on A by $D(a_1) = D(a_2) = \mathbb{R} \times \{0\}$, $D(a_3) = \mathbb{R}^2_+$. So $\mathcal{D}_A = \mathbb{R}_+ \times \{0\}$; a_2 is a nondominated point of A w.r.t. $\mathcal{D}$ since $a_3 \not\leq^n_{\mathcal{D}} a_2$, $a_1 \leq^n_{\mathcal{D}} a_2$ and $a_2 \leq^n_{\mathcal{D}} a_1$. However, a_2 is not a minimal w.r.t. $\mathcal{D}_A$ since $a_1 \leq_{\mathcal{D}_A} a_2$ but $a_2 \not\leq_{\mathcal{D}_A} a_1$.

(ii) does not hold for minimal points of A w.r.t. $\mathcal{D}(A)$. Indeed, let $Y = \mathbb{R}^2$, $A = \{a_1 = (-1,0), a_2 = (0,0), a_3 = (1,-1)\}$ and $\mathcal{D} : Y \rightrightarrows Y$ be given on A by $D(a_1) = \mathbb{R} \times \mathbb{R}_+,\ D(a_2) = \mathbb{R}^2_+,\ D(a_3) = \mathbb{R}_+ \times \mathbb{R}$. So $\mathcal{D}(A) = \mathbb{R}^2 \setminus \operatorname{int} \mathbb{R}^2_-$; a_2 is a minimal point of A w.r.t. $\mathcal{D}(A)$ since $a_i \leq_{\mathcal{D}(A)} a_j$ for all $i, j \in \{1,2,3\}$, but is not a nondominated point of A w.r.t. $\mathcal{D}$ since $a_1 \leq^n_{\mathcal{D}} a_2$ but $a_2 \not\leq^n_{\mathcal{D}} a_1$.

(iii) does not hold for efficient points of A w.r.t. $\mathcal{D}$. Indeed, let $Y = \mathbb{R}$, $A = \{-1, 0\}$ and $\mathcal{D} : Y \rightrightarrows Y$ be given on A by $\mathcal{D}(-1) = \mathbb{R}$ and $\mathcal{D}(0) = \mathbb{R}_+$. So, $\overline{a} = 0$ is an efficient point of A w.r.t. $\mathcal{D}$ since $-1 \leq^e_{\mathcal{D}} 0$ and $0 \leq^e_{\mathcal{D}} -1$, but is not a minimal point of A w.r.t. $\mathcal{D}(0)$ since $-1 \leq_{\mathcal{D}(0)} 0$ but $0 \not\leq_{\mathcal{D}(0)} -1$.

2.2 Nonlinear Scalarization Functions

Let us recall an important nonlinear scalarization tool from [12,13] by Gerstewitz (Tammer) and Weidner (cf. [20]).

Definition 2.3 Let $k \in Y \setminus \{0\}$ and $D \subseteq Y$ be a nonempty set being free-disposal w.r.t. cone(k). The Gerstewitz nonlinear scalarization function $\varphi_{k,D} : Y \to \mathbb{R} \cup \{\pm\infty\}$ associated with k and D is defined by

$$\varphi_{k,D}(y) := \inf\{t \in \mathbb{R} : y \in tk - D\} \text{ for all } y \in Y. \tag{1}$$

Remark 2.3 In vector optimization dealing with minimization, one needs to ensure that values of a function are not equal to $-\infty$. By (1), $\varphi_{k,D}(y) = -\infty$ if and only if $-y + \mathbb{R}k \subseteq D$.

The Gerstewitz scalarization function was originally defined in real topological spaces for (topologically) closed sets D satisfying $D + [0, +\infty[k \subseteq D$. In [17,21], the scalarization function was extended to real linear spaces for any nonzero direction q and any nonempty set H. In this paper, we will call $\varphi_{q,H}$ a generalized Gerstewitz scalarization function.

The following lemma was proved in [21].

Lemma 2.1 ([21, Lemma 2.2]). *Let $q \in Y \setminus \{0\}$, and $\emptyset \neq H \subseteq Y$. The generalized Gerstewitz scalarization function $\varphi_{q,H}$ defined as in* (1) *has the following properties:*

(a) $\varphi_{q,H}(y) < +\infty$ *if and only if* $y \in \mathbb{R}q - \mathrm{vcl}_q H$.

(b) $[\varphi_{q,H}(y) < 0] =]-\infty, 0[q - \mathrm{vcl}_q H$.

(c) $[\varphi_{q,H}(y) \leq 0] =]-\infty, 0]q - \text{vcl}_q H$.

The next proposition shows that each generalized Gersterwitz scalarization function can be equivalently formulated as a Gersterwitz scalarizing function.

Proposition 2.3 *Let $q \in Y \setminus \{0\}$, $\emptyset \neq H \subseteq Y$, and $\mathcal{E}_q(H)$ be the epigraphical set of H w.r.t. the cone* $\text{cone}\, q$. *Then:*

(a) $\mathcal{E}_q(H)$ *is free-disposal w.r.t.* $\text{cone}\, q$.

(b) $\forall y \in Y, \varphi_{q,H}(y) = \varphi_{q,\mathcal{E}_q(H)}(y)$.

Proof Part (a) is obvious. To prove (b), let us observe that $H \subseteq \mathcal{E}_q(H)$ and so

$$\varphi_{q,\mathcal{E}_q(H)}(y) \leq \varphi_{q,H}(y), \quad \forall y \in Y. \tag{2}$$

Now, we consider three cases:

Case 1: If $\varphi_{q,\mathcal{E}_q(H)}(y) = +\infty$, then in view of (2), part (b) is obvious.

Case 2: If $\varphi_{q,\mathcal{E}_q(H)}(y) = -\infty$, then we could find a sequence $\{\gamma_n\}$ with $\gamma_n \to -\infty$ such that $y \in \gamma_n q - \mathcal{E}_q(H) = \gamma_n q - H - \text{cone}\, q$. Now, the last equality, for an arbitrary fixed integer n, ensures the existence of a number $\theta_n \geq 0$ such that $y \in (\gamma_n - \theta_n)q - H$ and thus $\varphi_{q,H}(y) = \inf\{t \in \mathbb{R} : y \in tq - H\} \leq \gamma_n - \theta_n \leq \gamma_n$. Since it holds for all n, then when $n \to +\infty$ we have $\varphi_{q,H}(y) = -\infty$.

Case 3: If $\varphi_{q,\mathcal{E}_q(H)}(y) = \gamma \in \mathbb{R}$, then in view of (2) we need to show that $\varphi_{q,H}(y) \leq \gamma$. Since $\varphi_{q,\mathcal{E}_q(H)}(y) = \inf\{t \in \mathbb{R} : y \in tq - \mathcal{E}_q(H)\} = \gamma$, for every $\varepsilon > 0$, there is $t \leq \gamma + \varepsilon$ such that $y \in tq - \mathcal{E}_q(H) = tq - H - \text{cone}\, q$. Then, we could find $\theta \geq 0$ such that $y \in (t-\theta)q - H$. By the definition of $\varphi_{q,H}$, we have

$$\varphi_{q,H}(y) = \inf\{t \in \mathbb{R} : y \in tq - H\} \leq t - \theta \leq t \leq \gamma + \varepsilon.$$

Since ε was arbitrarily positive, we have $\varphi_{q,H}(y) \leq \gamma$ and the proof is complete.□

Remark 2.4 By Proposition 2.3(a)-(b) and Remark 2.3, it follows that $\varphi_{q,H}(y) > -\infty$ if and only if $-y + \mathbb{R}q \nsubseteq H + \operatorname{cone} q$.

3 Characterizations of Nondominated Points via Nonlinear Scalarization

First, we establish characterizations of nondominated points of sets w.r.t. domination structures. We consider two cases: (1) the intersection domination set $\mathcal{D}_A$ has a nonzero element and (2) it might be $\{0\}$.

The standing assumptions for sets. Y is a linear space, $\emptyset \neq A \subseteq Y$, $\overline{a} \in A$, $k \in Y \setminus \{0\}$, $\mathcal{D}$ is a domination structure of Y pointed at $\overline{a}$ over A.

Proposition 3.1 (characterizations for nondominated points of sets, case 1). *Let the standing assumptions hold. Assume that* $k \in \mathcal{D}_A$.

(i) If

$$y \in A \setminus \{\overline{a}\}, \varphi_{-k,-\mathcal{D}(y)-y}(\overline{a}) > 0, \tag{3}$$

then $\overline{a}$ *is a strictly nondominated point of* A *w.r.t.* $\mathcal{D}$.

(ii) If $\overline{a}$ *is a nondominated point of* A *w.r.t.* $\mathcal{D}$*, then for every* $y \in A \setminus \{\overline{a}\}$ *such that* $\mathcal{D}(y)$ *is free-disposal w.r.t.* $\operatorname{cone} k$ *and* k*-vectorially closed, we have* $\varphi_{-k,-\mathcal{D}(y)-y}(\overline{a}) > 0$.

(iii) If $\overline{a}$ *is a nondominated point of* A *w.r.t.* $\mathcal{D}$*, then for every* $y \in A \setminus \{\overline{a}\}$ *such that* $\mathcal{D}(y)$ *is free-disposal w.r.t.* $\operatorname{cone} k$*, we have* $\varphi_{-k,-\mathcal{D}(y)-y}(\overline{a}) \geq 0$.

(iv) Assume that $\mathcal{D}_A$ *is free-disposal w.r.t.* $\operatorname{cone} k$. *If* $\overline{a}$ *is a nondominated point of* A *w.r.t.* $\mathcal{D}$, *then* $\overline{a}$ *is a minimal solution of the function* $\varphi_{k,\mathcal{D}_A-\overline{a}}$ *over* A. *Assume in addition that* $\mathcal{D}_A$ *is* k*-vectorially closed. Then,* $\overline{a}$ *is a strictly minimal solution of* $\varphi_{k,\mathcal{D}_A-\overline{a}}$ *over* A.

Proof (i) Arguing by contradiction, we assume that $\overline{a}$ is not a strictly nondominated point of A w.r.t. $\mathcal{D}$. Then, we could find $y \in A \setminus \{\overline{a}\}$ such that $\overline{a} \in y + \mathcal{D}(y) = 0 \cdot k - (-y - \mathcal{D}(y))$. By the definition of $\varphi_{-k,-y-\mathcal{D}(y)}$, we have $\varphi_{-k,-y-\mathcal{D}(y)}(\overline{a}) \leq 0$ contradicting (3).

(ii) Fix an arbitrary point $y \in A \setminus \{\overline{a}\}$ such that $\operatorname{vcl}_k \mathcal{D}(y) = \mathcal{D}(y)$ and $\mathcal{D}(y) + \operatorname{cone} k = \mathcal{D}(y)$. Since $\mathcal{D}$ is pointed at $\overline{a}$ over A and $\overline{a}$ is a nondominated point of A w.r.t. $\mathcal{D}$, Proposition 2.1 yields

$$\begin{aligned}\overline{a} \notin y + \mathcal{D}(y) &= y + \mathcal{D}(y) + \operatorname{cone} k \\ &= y + \operatorname{vcl}_k \mathcal{D}(y) + \operatorname{cone} k = -\operatorname{cone}(-k) - \operatorname{vcl}_{-k}(-\mathcal{D}(y) - y).\end{aligned}$$

Then, by Lemma 2.1 (d), we have $\varphi_{-k,-\mathcal{D}(y)-y}(\overline{a}) > 0$. As y is arbitrary, (ii) holds.

(iii) By proceeding similarly as in the proof of (ii), we have

$$\overline{a} \notin y + \mathcal{D}(y) = y + \mathcal{D}(y) + \operatorname{cone} k = -\operatorname{cone}(-k) - (-\mathcal{D}(y) - y). \tag{4}$$

• If $\overline{a} \notin -\operatorname{cone}(-k) - \operatorname{vcl}_{-k}(-\mathcal{D}(y) - y)$, then Lemma 2.1 (d) ensures that $\varphi_{-k,-\mathcal{D}(y)-y}(\overline{a}) > 0$.

• If $\overline{a} \in -\operatorname{cone}(-k) - \operatorname{vcl}_{-k}(-\mathcal{D}(y) - y)$, then Lemma 2.1 (d) ensures that $\varphi_{-k,-\mathcal{D}(y)-y}(\overline{a}) \leq 0$. If $\varphi_{-k,-\mathcal{D}(y)-y}(\overline{a}) < 0$, by Lemma 2.1 (c) we have

$$\overline{a} \in]-\infty, 0[(-k) - \operatorname{vcl}_{-k}(-\mathcal{D}(y) - y) \subseteq -(-\mathcal{D}(y) - y) = y + \mathcal{D}(y),$$

where the last inclusion follows by [21, Lemma 2.3 (c), for $D = \operatorname{cone} k$]. This contradicts (4) and thus verifies $\varphi_{-k,-\mathcal{D}(y)-y}(\overline{a}) = 0$. Hence, (iii) holds.

(iv) Assume that $\overline{a}$ is a nondominated point of A w.r.t. $\mathcal{D}$. Since $\mathcal{D}$ is pointed at $\overline{a}$ over A, by Proposition 2.1 and Proposition 2.2 (i), $\overline{a}$ is a strictly minimal point of A w.r.t. $\leq_{\mathcal{D}_A}$; i.e.,

$$\forall y \in A \setminus \{\overline{a}\}, y \notin \overline{a} - \mathcal{D}_A.$$

By an analogous reasoning as in the proof of part (iii), we have $\varphi_{k,\mathcal{D}_A-\overline{a}}(y) \geq 0$ for all $y \in A \setminus \{\overline{a}\}$, with strict inequality under the additional closedness assumption, clearly verifying that $\overline{a}$ is a minimal solution and a strictly minimal solution of $\varphi_{k,\mathcal{D}_A-\overline{a}}$, respectively. □

Remark 3.1 Observe that

$$\begin{aligned} \varphi_{-k,-\mathcal{D}(y)-y}(\overline{a}) &= \inf\{t \in \mathbb{R} : \overline{a} \in -tk + \mathcal{D}(y) + y\} \\ &= \inf\{t \in \mathbb{R} : y \in tk - (\mathcal{D}(y) - \overline{a})\} = \varphi_{k,\mathcal{D}(y)-\overline{a}}(y). \end{aligned} \tag{5}$$

Thus, in parts (i)-(iii) of the proposition above we can replace $\varphi_{-k,-\mathcal{D}(y)-y}(\overline{a})$ by $\varphi_{k,\mathcal{D}(y)-\overline{a}}(y)$.

When $\mathcal{D} = \mathcal{C}_D$ over A for some domination set D, we derive from this remark and Proposition 3.1 refined characterizations for minimal points of sets w.r.t. D; in particular, the nonconvex separation theorem in [20, Theorem 2.3.6] in the nonsolid case; i.e., $\operatorname{int}(D) = \emptyset$.

Remark 3.2 Proposition 3.1 is an extension of Eichfelder's results in [8], where the domination structure takes values on closed, pointed and convex cones.

Note that if K is a convex cone, K is free-disposal w.r.t. $\operatorname{cone} k$ for every $k \in K \setminus \{0\}$. Eichfelder used the scalarization $\varphi^{\overline{a}}_{k,\mathcal{D}} : Y \to \mathbb{R} \cup \{\pm\infty\}$ given by

$$\varphi^{\overline{a}}_{k,\mathcal{D}}(y) := \inf\{t \in \mathbb{R} \mid y - \overline{a} \in tk - \mathcal{D}(y)\},$$

where $\inf \emptyset = +\infty$. We have the equality

$$\begin{aligned}\varphi_{-k,-\mathcal{D}(y)-y}(\overline{a}) &= \inf\{t \in \mathbb{R} : \overline{a} \in -tk + \mathcal{D}(y) + y\} \\ &= \inf\{t \in \mathbb{R} : y - \overline{a} \in tk - \mathcal{D}(y)\} = \varphi^{\overline{a}}_{k,\mathcal{D}}(y)\end{aligned}$$

for all $y \in Y$. Therefore, Proposition 3.1 (i) could be written in the following form: if $\overline{a}$ is a strictly minimal solution of the function $\varphi^{\overline{a}}_{k,\mathcal{D}}$ over A, then $\overline{a}$ is a strictly nondominated point of A w.r.t. $\mathcal{D}$.

Next, we consider case 2, where the intersection domination set $\mathcal{D}_A$ might be $\{0\}$. We use the technique in [24] to construct a solid cone K such that the nondominated point of A under consideration is a minimal point of the epigraphical set $\mathcal{E}_{\mathcal{D}}(A)$ w.r.t. to K. Then, the known scalarization technique could be applied.

Proposition 3.2 (characterizations for nondominated points of sets, case 2). *Let the standing assumptions for sets hold, $\overline{a}$ be a nondominated point of A w.r.t. $\mathcal{D}$, and assume that $\operatorname{cone}(\mathcal{E}_{\mathcal{D}}(A) - \overline{a})$ is not the whole space Y. Then, there is a nontrivial cone K satisfying*

(K1) there is $k \in K \setminus \{0\}$ such that K is free-disposal w.r.t. $\operatorname{cone} k$,

(K2) $\operatorname{vcl}_k K \cap -\operatorname{cone}(\mathcal{E}_{\mathcal{D}}(A) - \overline{a}) = \{0\}$,

such that $\overline{a}$ is a strictly minimal solution of the scalarization function $\varphi_{k,K-\overline{a}}(y)$ over A; i.e., $\forall y \in A \setminus \{\overline{a}\}, \varphi_{k,K-\overline{a}}(y) > 0$.

Proof Since $\overline{a}$ is a nondominated point of A w.r.t. $\mathcal{D}$ and $\mathcal{D}$ is pointed at $\overline{a}$ over A, by Proposition 2.1 we have

$$\forall y \in A \setminus \{\overline{a}\}, \overline{a} \notin y + \mathcal{D}(y) \text{ (and thus } \quad 0 \notin y + \mathcal{D}(y) - \overline{a}).$$

Since $\operatorname{cone}(\mathcal{E}_{\mathcal{D}}(A) - \overline{a})$ is not the whole space Y, there is a nonzero element $k \notin -\operatorname{cone}(\mathcal{E}_{\mathcal{D}}(A) - \overline{a})$ and thus $\operatorname{cone} k \cap (-\operatorname{cone}(\mathcal{E}_{\mathcal{D}}(A) - \overline{a})) = \{0\}$. We could take $K = \operatorname{cone} k$. Note that (K2) is equivalent to $(\overline{a} - \operatorname{vcl}_k K) \cap \mathcal{E}_{\mathcal{D}}(A) = \{\overline{a}\}$, i.e., $\overline{a}$ is a minimal point of $\mathcal{E}_{\mathcal{D}}(A)$ w.r.t. the ordering cone $\operatorname{vcl}_k K$. Thus, by Proposition 3.1 (ii) and relation (5), we deduce that $\varphi_{k,K-\overline{a}}(y) > 0$ for all $y \in A \setminus \{\overline{a}\}$. □

In the Banach space setting, we could construct a closed, convex and solid cone satisfying conditions (K1) and (K2).

Proposition 3.3 *Let Y be a Banach space. If $\operatorname{cl}\operatorname{cone}(\mathcal{E}_{\mathcal{D}}(A) - \overline{a}) \neq Y$, then there is a closed, convex and solid cone K satisfying conditions (K1) and*

(K2') $K \cap -\operatorname{cl}\operatorname{cone}(\mathcal{E}_{\mathcal{D}}(A) - \overline{a}) = \{0\}$.

Proof Since $\operatorname{cl}\operatorname{cone}(\mathcal{E}_{\mathcal{D}}(A) - \overline{a})$ is not the whole space, there is

$$k \notin -\operatorname{cl}\operatorname{cone}(\mathcal{E}_{\mathcal{D}}(A) - \overline{a})$$

with $\|k\| = 1$. Choose η such that

$$0 < 2\eta < \min\{1, \inf\{\|k + y\| : y \in \operatorname{cone}(\mathcal{E}_{\mathcal{D}}(A) - \overline{a})\}\}.$$

We construct the following cone

$$K := \operatorname{cone} \mathbb{B}_\eta(k) \ \text{ with } \ \mathbb{B}_\eta(k) = \{y \in Y : \|y - k\| \leq \eta\}. \tag{6}$$

By [20, Lemma 3.2.51] with $C = \operatorname{cone} k$ and $B = \{k\}$, the cone K is closed, convex and solid and clearly satisfies condition (K1). It remains to show the fulfillment of condition (K2'), so fix an arbitrary nonzero vector $y \in -K \setminus \{0\}$. There is $t > 0$ such that $-ty \in \mathbb{B}_\eta(k)$. Therefore, we have $\|-ty-k\| = \|k+ty\| \le \eta$. By the choice of η, $ty \notin \operatorname{cl}\operatorname{cone}(\mathcal{E}_\mathcal{D}(A) - \overline{a})$ and thus $y \notin \operatorname{cl}\operatorname{cone}(\mathcal{E}_\mathcal{D}(A) - \overline{a})$. Since y was arbitrary in $-K \setminus \{0\}$, we have $(-K) \cap \operatorname{cl}\operatorname{cone}(\mathcal{E}_\mathcal{D}(A) - \overline{a}) = \{0\}$. The proof is complete. □

As a consequence of Proposition 3.2, we extend and improve the characterizations of minimal points in vector optimization.

Corollary 3.1 *Let $K \subseteq Y$ be a nontrivial convex and pointed cone and $k \in K \setminus \{0\}$. Suppose that K is k-vectorially closed. If $\overline{a}$ is a minimal point of A w.r.t. K, then there is a convex cone $C \supseteq K$ satisfying*

$$(-\operatorname{vcl}_k C) \cap \operatorname{cone}(\mathcal{E}_K(A) - \overline{a}) = \{0\}$$

such that $\overline{a}$ is a strictly minimal solution of $\varphi_{k,C-\overline{a}}$ over $\mathcal{E}_K(A)$.

Proof Since K is a convex, pointed and k-vectorially closed cone, we get from the minimality of $\overline{a}$ to A w.r.t. K that $(-\operatorname{vcl}_k K) \cap \operatorname{cone}(A + K - \overline{a}) = \{0\}$. Therefore, we could take at least $C = K$ and thus the proof is complete by Proposition 3.1(ii). □

Example 3.1 Let $Y = \mathbb{R}^2$, $A = \{(y_1, y_2) \in \mathbb{R}^2 : y_2^2 = -y_1\}$, $\overline{a} = (0,0)$, and the domination structure $\mathcal{D} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ with domination sets

$$\mathcal{D}(y_1, y_2) = \{0\} \times \mathbb{R}_+, \text{ if } y_2 > 0;\ \mathcal{D}(y_1, y_2) = \mathbb{R}^2_+, \text{ if } y_2 = 0;$$

$$\mathcal{D}(y_1, y_2) = \mathbb{R}_+ \times \{0\}, \text{ if } y_2 < 0.$$

Since $\mathcal{D}_A = \{0\}$, Proposition 3.1 is not applicable. However, Proposition 3.2 can be applied since cone $(\mathcal{E}_{\mathcal{D}}(A) - \overline{a}) = \mathbb{R}^2 \backslash] - \infty, 0[\times \{0\}$.

4 Necessary Conditions for Nondominated Solutions of Vector-Valued Functions.

This section is devoted to establish necessary conditions for nondominated solutions of vector-valued functions w.r.t. domination structures in terms of Mordukhovich limiting differentiation which enjoys a full calculus in Asplund spaces. We assume that all the spaces under consideration are Asplund unless otherwise stated. Recall that a Banach space is *Asplund* if every convex continuous function $\varphi : U \to \mathbb{R}$ defined on an open convex subset U of X is Fréchet differentiable on a dense subset of U.

First, we present several definitions and properties of the basic generalized differential constructions in the book [22].

Let X be an Asplund space and $\Omega \subset X$ be a subset of X. The *Fréchet normal cone* to Ω at $x \in \Omega$ is defined by

$$\widehat{N}(x; \Omega) := \Big\{ x^* \in X^* : \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \leq 0 \Big\}, \tag{7}$$

where $u \xrightarrow{\Omega} x$ means $u \to x$ with $u \in \Omega$. Let $\bar{x} \in \Omega$. Assume that Ω is locally closed around $\bar{x} \in \Omega$, i.e., there is a neighborhood U of $\bar{x}$ such that $\Omega \cap \operatorname{cl} U$ is a closed set. The (basic, limiting, Mordukhovich) *normal cone* to Ω at $\bar{x}$ is defined by

$$N(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{x\to\bar{x}} \widehat{N}(x;\Omega)$$
$$= \Big\{x^* \in X^* : \exists\, x_k \to \bar{x},\ x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in \widehat{N}(x_k;\Omega)\Big\}, \tag{8}$$

where Lim sup stands for the sequential Painlevé-Kuratowski outer limit. Note that, in contrast to (7), the basic normal cone (8) is often *nonconvex* enjoying nevertheless *full calculus*, and that both the cones (7) and (8) reduce to the normal cone of convex analysis when Ω is convex.

Given a set-valued map $F\colon X \rightrightarrows Y$, its graph and domain are defined by

$$\operatorname{gph} F := \{(x,y) \in X \times Y : y \in F(x)\} \ \text{ and } \ \operatorname{dom} F := \{x \in X : F(x) \neq \emptyset\}.$$

The *limiting/Mordukhovich coderivative* $D_L^*F(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$ of F at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ is defined by

$$D_L^*F(\bar{x},\bar{y})(y^*) := \{x^* \in X^* : (x^*,-y^*) \in N((\bar{x},\bar{y});\operatorname{gph} F)\}, \tag{9}$$

which is a positively homogeneous function of $y^* \in Y^*$. Remind that any single-valued function $f : X \to Y$ can be considered as a set-valued map in the usual way. We omit $\bar{y} = f(\bar{x})$ in (9) if $F = f : X \to Y$ is single-valued. If f is *strictly differentiable* at $\bar{x}$ (which is automatic when it is C^1 around this point), then $D_L^*f(\bar{x})(y^*) = \big\{\nabla f(\bar{x})^*y^*\big\}$ for all $y^* \in Y^*$.

Note that coderivatives of set-valued maps enjoys the full calculus in the Asplund setting including the sum rule [22, Theorem 3.10] and the chain rule [22, Theorem 3.13 (ii)]. For the sake of briefness, we do not recall them, but the reader needs to aware of the mixed qualification condition and the partially sequential normal compactness (PSNC) conditions in each calculus rule; the reader is referred to [22, Definition 1.67]. In this paper, we formulate our results for functions and/or mappings enjoying the Lipschitzian property so that both mentioned conditions automatically hold. Recall that F is *Lipschitz-like* around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ and a constant $\xi \geq 0$ such that

$$F(x) \cap V \subset F(u) + \xi\|x - u\|\mathbb{B} \text{ for all } x, u \in U.$$

This clearly reduces to the classical local Lipschitz continuity for single-valued functions.

Let $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$. The epigraphical multifunction $\mathcal{E}_\varphi : X \rightrightarrows \mathbb{R}$ is defined by $\mathcal{E}_\varphi(x) := [\varphi(x), +\infty[$, if $\varphi(x) \in \mathbb{R}$; $\mathcal{E}_\varphi(x) := \emptyset$, otherwise.

Assume that φ is lower semicontinuous around $\bar{x} \in \operatorname{dom} \varphi$, where $\operatorname{dom} \varphi := \{x \in X : |\varphi(x)| < \infty\}$. The basic subdifferential of φ at $\bar{x}$ is defined by

$$\partial_L \varphi(\bar{x}) := \{x^* \in X^* \ : \ (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})); \operatorname{epi} \varphi)\}.$$

It follows that $\partial_L \varphi(\bar{x}) := D_L^* \mathcal{E}_\varphi(\bar{x})(1)$ for lower semicontinuous functions. If in addition φ is convex, then $\partial_L \varphi(\bar{x}) = \partial \varphi(\bar{x})$, where ∂ stands for the classical subdifferential of convex analysis.

Next, we recall subdifferentials of scalarization functions associated with solid cones.

Lemma 4.1 ([23, Lemma 2.1]). *Let Y be a real topological linear space, K be a nontrivial, closed, solid and convex cone in Y, and $k \in \operatorname{int} K \setminus \{0\}$. The* *scalarization function* $\varphi_{k,K}$ *defined by* (1) *is continuous and sublinear and the subdifferential of* $\varphi_{k,K}$ *at* $y = 0$ *is given by*

$$\partial\varphi_{k,K}(0) = K^+ \cap H_1(k),$$

where $K^+ := \{y^* \in Y^* : \forall y \in K, y^*(y) \geq 0\}$ *is the positive polar cone of* K*, and* $H_1(k) := \{y^* \in Y^* : y^*(k) = 1\}$.

Next, we state the lower subdifferential condition for local minimal solutions under a geometric constraint.

Lemma 4.2 (cf. [22, Proposition 5.3]). *Let* X *be an Asplund space,* $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ *be an extended-real-valued function,* Ω *be a nonempty set in* X*, and* $\bar{x}$ *be a local minimal solution of* φ *over* Ω*. Assume that* Ω *is locally closed and* φ *is locally Lipschitz continuous at* $\bar{x}$*. Then, one has*

$$0 \in \partial_L\varphi(\bar{x}) + N(\bar{x}; \Omega).$$

The standing assumptions for functions. X and Y are Asplund spaces, $k \in Y \setminus \{0\}$, $f : X \to Y$ is a vector-valued function, $\bar{x} \in \operatorname{dom} f$, $\bar{y} = f(\bar{x})$, and $\mathcal{D} : Y \rightrightarrows Y$ is a domination structure of Y being pointed at $\bar{y}$ over a neighborhood of $\bar{y}$.

Definition 4.1 (local nondominated and efficient solutions). It is said that $\bar{x}$ is a local nondominated (resp. efficient) solution of f w.r.t. $\mathcal{D}$ if there are a neighborhood U of $\bar{x}$ and a neighborhood V of $\bar{y}$ such that $\bar{x}$ is a nondominated solution w.r.t. $\mathcal{D}$ of the restricted function of f denoted by $f_V : U \to V$ with $\operatorname{gph} f_V = \operatorname{gph} f \cap (U \times V)$; i.e.,

$$\forall x \in U \cap \operatorname{dom} f \cap f^{-1}(V), f_V(x) = f(x),$$

where $f^{-1} : Y \rightrightarrows X$ is a set-valued map with $f^{-1}(y) := \{x \in X : y = f(x)\}$. Equivalently, this means that $f(\bar{x})$ is a nondominated point of $f_V(U)$.

Let us denote $\mathcal{D}_V := \cap\{\mathcal{D}(f(x)) : x \in \operatorname{dom} f_V\}$.

The next proposition plays an important role in establishing necessary conditions for nondominated solutions of functions w.r.t. domination structures.

Proposition 4.1 *Let the standing assumptions for functions hold. Assume that $\bar{x}$ is a local nondominated solution of f w.r.t. $\mathcal{D}$, and let $U, V, \mathcal{D}_V$ be the sets given in Definition 4.1. Suppose that $\mathcal{D}_V$ is nonzero and convex, and that $k \in \operatorname{int} \mathcal{D}_V$, ensuring the existence of a closed, convex and solid cone*

$$K := \operatorname{cone}(k + \mathbb{B}_\eta(0)) \quad \textit{with} \quad k + \mathbb{B}_\eta(0) \subseteq \operatorname{int} \mathcal{D}_V, \tag{10}$$

where $\mathbb{B}_\eta(0) = \{y \in Y : \|y\| \leq \eta\}$ and $\eta < \|k\|/2$. Assume also that $\mathcal{D}(y) + \mathcal{D}_V \subseteq \mathcal{D}(y)$, for all $y \in V$. Then, $(\bar{x}, \bar{y})$ is a local minimal solution of the problem

$$\textit{minimize} \quad s(x, y) \quad \textit{subject to } (x, y) \in \operatorname{gph}(f + \mathcal{D} \circ f), \tag{11}$$

where $s : X \times Y \to \mathbb{R}$ is defined by $s(x, y) := \varphi_{k,K-\bar{y}}(y)$.

Proof According to Definition 4.1, $\bar{x}$ is a nondominated solution of f_V w.r.t. $\mathcal{D}$, i.e., $\bar{y}$ is a nondominated point of $f_V(U)$ w.r.t. $\mathcal{D}$. We assume without loss of generality that $V \subseteq \bar{y} + \mathbb{B}_\eta(0)$, and that $\mathcal{D}$ is pointed at $\bar{y}$ over V.

We affirm that $(\bar{x}, \bar{y})$ is a minimal solution of the problem

$$\text{minimize} \quad s(x, y) \quad \text{subject to } (x, y) \in \operatorname{gph}(f_V + \mathcal{D} \circ f) \cap (U \times V). \tag{12}$$

Indeed, suppose by reasoning to the contrary that $(\bar{x}, \bar{y})$ is not a solution of (12). This means that there exists $(x, w) \in \operatorname{gph}(f_V + \mathcal{D} \circ f) \cap (U \times V)$ such that $\varphi_{k,K-\bar{y}}(w) < 0$. Thus, there exists $t < 0$ such that $w \in tk - K + \bar{y}$. In addition, we have that $x \in U$, $w \in V$ and $w = f(x) + d$, for some $d \in \mathcal{D}(f(x))$. Hence, $w = f(x) + d \in tk - K + \bar{y} \subseteq -K + \bar{y}$. Also, as $w \in V \subseteq \bar{y} + \mathbb{B}_\eta(0)$, we have $\bar{y} - w \in K \cap \mathbb{B}_\eta(0)$.

It follows that $K \cap \mathbb{B}_\eta(0) \subseteq \mathcal{D}_V$. Indeed, fix an arbitrary element $b \in K \cap \mathbb{B}_\eta(0)$. By the structure of K, there is $\bar{t} > 0$ and $b' \in \mathbb{B}_\eta(0)$ such that $b = \bar{t}(k + b') \in \mathbb{B}_\eta(0)$. We have

$$\bar{t}\|k\| = \|\bar{t}k\| = \|\bar{t}k + \bar{t}b' - \bar{t}b'\| \leq \|\bar{t}k + \bar{t}b'\| + \|\bar{t}b'\| \leq \eta + \bar{t}\eta$$

and thus $\bar{t} \leq \eta/(\|k\| - \eta) \leq 1$. Since $k + b' \in \mathcal{D}_V$, $0 \in \mathcal{D}_V$ and $\mathcal{D}_V$ is convex, we have $b = \bar{t}(k + b') = (1 - \bar{t})0 + \bar{t}(k + b') \in \mathcal{D}_V$. Then, $\bar{y} \in w + \mathcal{D}_V \subseteq f(x) + \mathcal{D}(f(x)) + \mathcal{D}_V \subseteq f(x) + \mathcal{D}(f(x)) = (f + \mathcal{D} \circ f)(x)$, clearly contradicting the nondominatedness of $\bar{y}$ over $f_V(U)$. This contradiction verifies the minimality of $(\bar{x}, \bar{y})$ to problem (12), which implies that $(\bar{x}, \bar{y})$ is a local minimal solution of problem (11), and the proof is complete. □

Now, we are ready to formulate and prove the first necessary condition for nondominated solution w.r.t. domination structures.

Theorem 4.1 (necessary conditions for nondominated solutions, case 1). *Let the standing assumptions for functions hold. Suppose that $\bar{x}$ is a local nondominated solution of f w.r.t. $\mathcal{D}$, and let $U, V, \mathcal{D}_V$ be the sets given in Definition 4.1. If the following conditions hold*

(H1) $k \in \operatorname{int} \mathcal{D}_V$, $\mathcal{D}_V$ *is nonzero and convex, and* $\mathcal{D}(y) + \mathcal{D}_V \subseteq \mathcal{D}(y)$ *for all* $y \in V$,

(H2) f *is locally Lipschitz at* $\bar{x}$,

(H3) Either f^{-1} *or* $\mathcal{D}$ *is Lipschitz-like at* $(f(\bar{x}), \bar{x})$ *or* $(\bar{y}, 0)$*, respectively.*

(H4) $\operatorname{gph} \mathcal{D}$ *is locally closed at* $(\bar{y}, 0)$,

then there is $y^* \in K^+$ *with* $y^*(k) = 1$ *such that*

$$0 \in D_L^* f(\bar{x})(y^*) + D_L^* f(\bar{x}) \circ D_L^* \mathcal{D}(\bar{y}, 0)(y^*), \tag{13}$$

where K *is presented in* (10).

Proof By Proposition 4.1, $(\bar{x}, \bar{y})$ is a local minimal solution of the problem

$$\text{minimize} \quad s(x, y) \;\; \text{subject to} \;\; (x, y) \in \operatorname{gph} (f + \mathcal{D} \circ f),$$

where $s(x, y) = \varphi_{k, K - \bar{y}}(y)$, i.e., $(\bar{x}, \bar{y})$ is a minimal solution of problem (12). Let us observe that $\varphi_{k, K-\bar{y}}(y) = \varphi_{k,K}(y - \bar{y})$, and so $\partial \varphi_{k,K-\bar{y}}(\bar{y}) = \partial \varphi_{k,K}(0)$. By Lemma 4.1, s is locally Lipschitz and the subdifferential of s at $(\bar{x}, \bar{y})$ is given by $\partial_L s(\bar{x}, \bar{y}) = \{0\} \times (K^+ \cap H_1(k))$. Applying Lemma 4.2 to problem (11), there is $(0, y^*) \in \partial_L s(\bar{x}, \bar{y})$ such that

$$(0, -y^*) \in N((\bar{x}, \bar{y}); \operatorname{gph} (f_V + \mathcal{D} \circ f)),$$

By the definition of coderivative, we have

$$0 \in D_L^* \left(f_V + \mathcal{D} \circ f\right)(\bar{x}, \bar{y})(y^*) \subseteq D_L^* f(\bar{x})(y^*) + D_L^* (\mathcal{D} \circ f)(\bar{x}, 0)(y^*),$$

where the upper estimation holds due to the sum rule in [22, Theorem 3.10]. It can be applied since both the mixed qualification condition and the partially sequential normal compactness condition are satisfied by (H2) and the inner semi-continuity assumption is automatically fulfilled by the single-valued function f.

To complete the proof, we apply the chain rule in [22, Theorem 3.13 (ii)] to the composition $\mathcal{D} \circ f$. Since f is single-valued and Lipschitz continuous around $\bar{x}$, the inner semi-compactness assumption is satisfied. Both the mixed qualification and the partially sequential normal compactness conditions are satisfied by (H3). The chain rule gives

$$D_L^*(\mathcal{D} \circ f)(\bar{x}, 0)(y^*) \subseteq D_L^* f(\bar{x}) \circ D_L^* \mathcal{D}(\bar{y}, 0)(y^*).$$

Substituting this into the previous inclusion, we obtain the necessary condition (13) and complete the proof. □

Remark 4.1 **(comparisons with existing results).** (1) When we use domination structures $\mathcal{P} : X \rightrightarrows Y$ acting from the domain space to the image space, we do not need to use the chain rule to find the upper estimate of the codetivative of $\mathcal{P}$ in terms of those of $\mathcal{D}$ and f. As a consequence, the necessary condition (13) becomes $0 \in D_L^* f(\bar{x})(y^*) + D_L^* \mathcal{P}(\bar{x}, 0)(y^*)$, and we do not need to assume that f^{-1} is locally Lipschitz at $(f(\bar{x}), \bar{x})$. It is important to emphasize that working with domination structures $\mathcal{P}$ is easier since we could avoid to deal with chain rules for generalized differentiation. See [11] for earlier developments.

(2) Our approach is based on the nonlinear scalarization function $\varphi_{k,K-f(\bar{x})}$ instead of the extremal principle in the variational approach in [5, Theorem 4.2]. We restrict ourselves to cost functions with Lipschitz behaviors so

that we do not need to assume the so-called sequentially normal compactness and qualification conditions so that we could emphasize on the idea and the way to use calculus rules of the proof. The obtained results could be easy to extend to the broader class of functions whose graphs are locally closed around the point under consideration; see [6] for tools and technique. In addition, we do not assume that $\mathcal{D}(y)$ is a nonempty convex cone for all $y \in Y$ and that $\mathcal{D}(\bar{y})$ is not a subspace of Y and that the limiting coderivative $D^*_L\mathcal{D}(\bar{y},0)$ satisfies $-y^* \in D^*_L\mathcal{D}(\bar{y},0)(y^*) \quad\Longrightarrow\quad y^*=0$ as in the aforementioned result.

(3) Note that in the finite-dimensional space setting, condition (H3) can be replaced by

$$\text{(H3}'\text{)}\quad D^*_L\mathcal{D}(\bar{y},0)(0)\cap\big(-D^*_Lf^{-1}(\bar{y},\bar{x})(0)\big)=\{0\}.$$

The rest of this section presents new necessary conditions for nondominated solutions of vector-valued functions for the case, where the intersection domination set $\mathcal{D}_V$ might be just $\{0\}$.

We denote $\mathcal{E}_f(X) := \bigcup_{x\in X}\{f(x)+\mathcal{D}(f(x))\}$.

Theorem 4.2 (necessary conditions for nondominated solutions, case 2). *Let the standing assumptions for functions hold. Assume that $\bar{x}$ is a local nondominated solution of f w.r.t. $\mathcal{D}$, and let $U, V, \mathcal{D}_V$ be the sets introduced in Definition 4.1. Let (H1') be the following condition:*

(H1') $\operatorname{cl}\operatorname{cone}(\mathcal{E}_f(X)-\bar{y})$ *is not the whole image space Y.*

If (H1'), (H2), (H3) and (H4) hold, then there is $y^*\in K^+$ with $y^*(k)=1$ such that (13) *holds, where K is presented in* (6).

Proof By hypothesis, we know that $\bar{x}$ is a nondominated solution of f_V. Therefore, $\bar{y} = f(\bar{x})$ is a nondominated point of the image set $f_V(U)$. By Propositions 3.2 and 3.3, there is a closed, convex and solid cone K defined by (6) such that $\bar{y}$ is a minimal solution of

$$\text{minimize} \quad \varphi_{k,K-\bar{y}}(y) \quad \text{subject to} \quad y \in f_V(U).$$

We claim that $(\bar{x}, \bar{y})$ is a minimal solution of problem (12). Arguing by contradiction, assume that it does not hold; i.e., there is $(x, y) \in \operatorname{gph}(f_V + \mathcal{D} \circ f) \cap (U \times V)$ such that $s(x, y) = \varphi_{k,K-\bar{y}}(y) < 0$. Thus, we have $y \in \bar{y} - K$ and $y \neq \bar{y}$. By (K2), we have $y \notin (f_V + \mathcal{D} \circ f)(X)$, a contradiction. Using the same arguments as in the proof of Theorem 4.1, we can find $y^* \in K^+$ with $y^*(k) = 1$ such that (13) holds. The proof is complete. □

Remark 4.2 When $\mathcal{D} = \mathcal{C}_D$, then Theorem 4.2 extends [24, Theorem 3.10] and improves [25, Theorem 4.5] from an ordering cone to a general domination set.

To close this section, we illustrate the necessary condition obtained in Theorem 4.2.

Example 4.1 Let $f : [-1, 1] \to \mathbb{R}^2$ be a vector-valued function defined by $f(x) := (x, -x)$ and $\mathcal{D} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be a cone-valued domination structure in $\mathbb{R}^2$ with domination sets

$$\mathcal{D}(y) := \begin{cases} C_1, & \text{if } y_1 < 0, \\ \mathbb{R}^2_+, & \text{if } y_1 = 0, \\ C_2, & \text{if } y_1 > 0, \end{cases}$$

where $C_1 := \text{cone}\,([0,1]\times\{2\})$, $C_2 := \text{cone}\,(\{2\}\times[0,1])$, and $y := (y_1, y_2) \in \mathbb{R}^2$. It is easy to check that $\bar{x} = 0$ is a nondominated solution of f w.r.t. $\mathcal{D}$ and that (H1′), (H2) and (H4) are satisfied. Condition (H3′) is fulfilled since

$$N(((y_1,y_2),0;\text{gph}\,\mathcal{D}) = \begin{cases} \{0\}\times N(0;C_1), & \text{if } y_1 < 0,\\ \{0\}\times \big(N(0;C_1)\cup N(0;C_2)\big), & \text{if } y_1 = 0,\\ \{0\}\times N(0;C_2), & \text{if } y_1 > 0. \end{cases}$$

Therefore, the necessary condition in Theorem 4.2 holds. Indeed, we can choose $K = \mathbb{R}^2_+$ and $y^* = (1,1) \in K^+$ satisfying

$$0 \in D^*_L f(0)(y^*) + D^*_L f(0) \circ D^*_L \mathcal{D}(0,0)(y^*) = \nabla f(0)^*(y^*) + D^*_L f(0)(0).$$

5 Conclusions

In this paper, we use the nonlinear scalarization function to characterize nondominated points of a set and nondominated solutions of a vector optimization problem with a domination structure. In contrast to existing results, we do not assume that domination sets are convex cones and that the intersection of all domination sets is different from zero.

By using scalarization approach, necessary conditions for nondominated solutions can be derived from the Fermat rule for scalarized optimization problem and the calculus rules for generalized differentiation. The results obtained in this paper could be extended to set-valued cost maps, local nondominated solutions, and other types of weak and proper nondominated solutions.

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