

Residuated bilattices

Ramon Jansana · Umberto Rivieccio

the date of receipt and acceptance should be inserted later

Abstract We introduce a new product bilattice construction that generalizes the well-known one for interlaced bilattices and others that were developed more recently, allowing to obtain a bilattice with two residuated pairs as a certain kind of power of an arbitrary residuated lattice. We prove that the class of bilattices thus obtained is a variety, give a finite axiomatization for it and characterize the congruences of its members in terms of those of their lattice factors. Finally, we show how to employ our product construction to define first-order definable classes of bilattices corresponding to any first-order definable subclass of residuated lattices.

Keywords bilattice · product bilattice · twist-structure · residuated lattice

1 Introduction and preliminaries

Bilattices are algebraic structures introduced by Matt Ginsberg [Ginsberg 1988] as a uniform framework for inference in Artificial Intelligence. Since then they have found a variety of applications, sometimes in quite different areas from the original one. The interest in bilattices has thus different sources: among others, computer science and A.I. (see especially the works of Ginsberg, Arieli and Avron), logic programming (Fitting), lattice

theory and algebra [Mobasher et al. 2000] and, more recently, algebraic logic: [Rivieccio 2010], [Bou and Rivieccio 2011], [Bou et al. 201x]. An up-to-date review of the applications of this formalism and also of the motivation behind its study can be found in the the PhD dissertation [Rivieccio 2010].

In the present paper we introduce a new class of structures having a bilattice reduct, that we call *residuated bilattices*, and prove a representation theorem for these algebras analogous to the known ones for interlaced bilattices (and bilattices with additional operators: see for instance [Avron 1996], [Rivieccio 2010], [Bou and Rivieccio 2011]). The motivations behind our work are both logical and algebraic.

On a purely algebraic level, we are interested in generalizing as far as possible the constructions introduced and studied in [Rivieccio 2010] and [Bou et al. 201x] to define different classes of bilattices with implication operators. Constructions of this kind are becoming increasingly common and have proved to be useful in the study of various classes of algebras related to logics, especially residuated lattices (see [Odintsov 2003], [Odintsov 2004], [Odintsov 2009], [Tsinakis and Wille 2009], [Busaniche and Cignoli 2009]).

However, this line of research has so far ignored the parallel work done on bilattices and vice versa. So the connection between bilattices and residuated structures (that seems quite natural in the framework that we are going to describe) has never been pointed out. One of our aims is thus to fill in this gap by providing a formal bridge between residuated lattices and bilattices.

One of the main motivations behind the study of bilattices has been, since their introduction in the 1980s, the attempt to formalize different forms of non-classical reasoning (especially paraconsistent, default and non-monotonic inferences). To this end Arieli and Avron in-

Ramon Jansana
Departament of Logic, History and Philosophy of Science,
Faculty of Philosophy, University of Barcelona, c/ Montalegre, 6
Barcelona 08001, Spain. E-mail: jansana@ub.edu

Umberto Rivieccio
School of Computer Science, University of Birmingham, Edgbaston
B15 2TT, Birmingham, United Kingdom. E-mail: u.rivieccio@cs.bham.ac.uk

roduced bilattice-based logics [Arieli and Avron 1996] in which different kinds of implications (a “strong” and a “weak” one) and consequence relations are explored. However, even the “weak” implication of Arieli and Avron has quite strong algebraic and logical properties: in a certain sense it can be regarded as the bilattice counterpart of classical implication. To state this more precisely, let us recall that in [Rivieccio 2010, Sections 4 and 5] it is proved that the class of algebras that constitute a semantics (the equivalent algebraic semantics, in fact) for the Arieli-Avron logic is categorically equivalent to the class of generalized Boolean algebras (i.e. the 0-free subreducts of Boolean algebras).

On the other hand, researchers in non-classical (especially relevant and paraconsistent) reasoning are usually interested in weaker implication connectives that avoid the so-called paradoxes of classical implication. We believe that, in the context of bilattices, such connectives are to be found within the framework that we sketch in the following sections. In this respect the idea that guided our work was to individuate implication connectives satisfying minimal requirements (corresponding to residuation in the context of lattices) and thus being as general as possible. For these reasons we believe that our algebraic study of residuation within bilattices (that, thanks to the theory of algebraization of logics, has a straightforward logical translation) will suggest new logical systems based on bilattices that may also have some interest for application-oriented research.

The paper is organized as follows. In Section 1.1 we introduce some preliminary notions and results on bilattices that will be needed to develop our study. Section 1.2 introduces the product bilattice construction, a well-known method to build a special kind of bilattices, that we are going to generalize in the subsequent sections. In Section 2 we extend this construction to define bilattices having operations that enjoy residuation properties. In Section 3 we give an axiomatization of the class of algebras corresponding to the bilattices that are representable through the product construction introduced in Section 2. Finally, in Section 4 we define translations from the language of residuated lattices to that of bilattices and vice versa that provide a means to find a “bilattice counterpart” for an arbitrary first-order definable class of residuated lattices.

1.1 Bilattices

In this section we introduce the basic definitions and some well-known results on bilattices.

Definition 1.1 A *bilattice* is an algebra

$$\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \neg \rangle$$

such that the reducts $\langle B, \wedge, \vee \rangle$ and $\langle B, \otimes, \oplus \rangle$ are both lattices and the *negation* \neg is a unary operation satisfying that for every $a, b \in B$,

- (neg 1) if $a \leq_t b$, then $\neg b \leq_t \neg a$
- (neg 2) if $a \leq_k b$, then $\neg a \leq_k \neg b$
- (neg 3) $a = \neg \neg a$.

The order of the lattice $\langle B, \wedge, \vee \rangle$, sometimes called the *truth lattice* or *t-lattice*, is denoted by \leq_t and is called the *truth order*, while the order \leq_k associated with $\langle B, \otimes, \oplus \rangle$, sometimes called the *knowledge lattice* or *k-lattice*, is the *knowledge order*.

Usually in the literature it is required that the two lattices be complete or at least bounded, but here, for the sake of generality, none of these assumptions is made. The minimum and maximum element of the lattice $\langle B, \wedge, \vee \rangle$, in case they exist, will be denoted respectively by f and t . Similarly, \perp and \top will refer to the minimum and maximum of $\langle B, \otimes, \oplus \rangle$, when they exist.

A bilattice is *interlaced* whenever each one of the four operations $\langle \wedge, \vee, \otimes, \oplus \rangle$ is monotonic with respect to both orders \leq_t and \leq_k . That is, when the following quasi-equations hold:

$$\begin{aligned} x \leq_t y &\Rightarrow x \otimes z \leq_t y \otimes z \\ x \leq_t y &\Rightarrow x \oplus z \leq_t y \oplus z \\ x \leq_k y &\Rightarrow x \wedge z \leq_k y \wedge z \\ x \leq_k y &\Rightarrow x \vee z \leq_k y \vee z. \end{aligned}$$

As usual, the inequality $x \leq_t y$ is an abbreviation for the identity $x \wedge y \approx x$ and similarly $x \leq_k y$ stands for $x \otimes y \approx x$. It is easy to see that the interlacing conditions imply the following inequalities:

$$\begin{aligned} x \otimes y \leq_k x \wedge y \leq_k x \oplus y \quad x \wedge y \leq_t x \otimes y \leq_t x \vee y \\ x \otimes y \leq_k x \vee y \leq_k x \oplus y \quad x \wedge y \leq_t x \oplus y \leq_t x \vee y. \end{aligned}$$

Bilattices form a variety, axiomatized by the lattice identities for the two lattices plus the law of double negation

$$x \approx \neg \neg x$$

and the following generalized De Morgan laws:

$$\begin{aligned} \neg(x \wedge y) \approx \neg x \vee \neg y \quad \neg(x \vee y) \approx \neg x \wedge \neg y \\ \neg(x \otimes y) \approx \neg x \oplus \neg y \quad \neg(x \oplus y) \approx \neg x \otimes \neg y. \end{aligned}$$

[Avron 1996] proves that the class of interlaced bilattices is also a variety, axiomatized by the identities for bilattices plus the following ones:

$$\begin{aligned} (x \wedge y) \otimes z \leq_t y \otimes z \quad (x \wedge y) \oplus z \leq_t y \oplus z \\ (x \otimes y) \wedge z \leq_k y \wedge z \quad (x \otimes y) \vee z \leq_k y \vee z. \end{aligned}$$

From an algebraic point of view, the variety of interlaced bilattices is perhaps the most interesting subclass of bilattices. As we are going to see in the next section, its interest comes mainly from the fact that any interlaced bilattice can be represented as a special power of a lattice. This result is well-known for bounded bilattices and it has been more recently generalized to the unbounded case by [Movsisyan et al. 2006] and [Bou and Riviuccio 2011].

The interlacing conditions may be strengthened as in the following definition, due to [Ginsberg 1988]. A bilattice is *distributive* when all possible distributive laws concerning the four lattice operations, i.e. any identity of the following form, hold:

$$x \circ (y \bullet z) \approx (x \circ y) \bullet (x \circ z)$$

for every $\circ, \bullet \in \{\wedge, \vee, \otimes, \oplus\}$. The class of distributive bilattices is a proper subvariety of the interlaced.

1.2 Product bilattices

Given an arbitrary lattice $\mathbf{L} = \langle L, \sqcap, \sqcup \rangle$, we can construct the *product bilattice* $\mathbf{L} \odot \mathbf{L} = \langle L \times L, \wedge, \vee, \otimes, \oplus, \neg \rangle$ as follows. For all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in L \times L$,

$$\begin{aligned} \langle a_1, a_2 \rangle \wedge \langle b_1, b_2 \rangle &:= \langle a_1 \sqcap b_1, a_2 \sqcap b_2 \rangle \\ \langle a_1, a_2 \rangle \vee \langle b_1, b_2 \rangle &:= \langle a_1 \sqcup b_1, a_2 \sqcup b_2 \rangle \\ \langle a_1, a_2 \rangle \otimes \langle b_1, b_2 \rangle &:= \langle a_1 \sqcap b_1, a_2 \sqcap b_2 \rangle \\ \langle a_1, a_2 \rangle \oplus \langle b_1, b_2 \rangle &:= \langle a_1 \sqcup b_1, a_2 \sqcup b_2 \rangle \\ \neg \langle a_1, a_2 \rangle &:= \langle a_2, a_1 \rangle. \end{aligned}$$

$\mathbf{L} \odot \mathbf{L}$ is always an interlaced bilattice, and it is distributive if and only if \mathbf{L} is a distributive lattice. From the definition it follows immediately that

$$\begin{aligned} \langle a_1, a_2 \rangle \leq_k \langle b_1, b_2 \rangle &\quad \text{iff} \quad a_1 \leq b_1 \quad \text{and} \quad a_2 \leq b_2 \\ \langle a_1, a_2 \rangle \leq_t \langle b_1, b_2 \rangle &\quad \text{iff} \quad a_1 \leq b_1 \quad \text{and} \quad a_2 \geq b_2 \end{aligned}$$

where \leq is the lattice order of \mathbf{L} .

The following results were proved by [Avron 1996] for bounded bilattices. They were then generalized to the unbounded case by [Movsisyan et al. 2006] and independently by [Bou and Riviuccio 2011].

Theorem 1.2 (Representation of bilattices) *A bilattice \mathbf{B} is interlaced if and only if there is a lattice \mathbf{L} such that $\mathbf{B} \cong \mathbf{L} \odot \mathbf{L}$. Moreover, \mathbf{B} is distributive if and only if \mathbf{L} is a distributive lattice.*

This representation can be used to characterize the congruences of interlaced bilattices in terms of those of their lattice factors: we have that

$$\text{Con}(\mathbf{L} \odot \mathbf{L}) \cong \text{Con}(\mathbf{L}).$$

This result is very important, as it allows to individuate the subdirectly irreducible interlaced bilattices as those that can be obtained as a product bilattice of a subdirectly irreducible lattice. This implies, for instance, that the only subdirectly irreducible distributive bilattice is isomorphic to the bilattice product $\mathbf{2} \odot \mathbf{2}$, where $\mathbf{2}$ denotes the two-element lattice. Therefore *the variety of distributive bilattices is generated by the single four-element bilattice $\mathbf{2} \odot \mathbf{2}$.*

Theorem 1.2 can be proved using different strategies (see [Movsisyan et al. 2006], [Bou and Riviuccio 2011], [Riviuccio 2010]).

Here we give some details of a construction that will be used in the next sections.

Given an interlaced bilattice \mathbf{B} , we consider the set

$$\text{Reg}(\mathbf{B}) = \{a \in B : a = \neg a\}$$

of *regular* elements, i.e. the fixed points of the negation operator. It is easy to see that $\text{Reg}(\mathbf{B})$ is closed under \otimes and \oplus , hence is the universe of a sublattice of the k -lattice of \mathbf{B} . To every $a \in B$ a regular element can be associated according to the following definition:

$$\text{reg}(a) := (a \vee (a \otimes \neg a)) \oplus \neg(a \vee (a \otimes \neg a)).$$

Using the properties of the *reg* operator (see Proposition 3.4), it is possible to prove that there is an isomorphism

$$\iota_{\mathbf{B}} : \mathbf{B} \cong \langle \text{Reg}(\mathbf{B}), \otimes, \oplus \rangle \odot \langle \text{Reg}(\mathbf{B}), \otimes, \oplus \rangle$$

with $\iota_{\mathbf{B}} : B \rightarrow \text{Reg}(\mathbf{B}) \times \text{Reg}(\mathbf{B})$ defined, for all $a \in B$, as

$$\iota_{\mathbf{B}}(a) := \langle \text{reg}(a), \text{reg}(\neg a) \rangle.$$

The inverse map $\iota_{\mathbf{B}}^{-1} : \text{Reg}(\mathbf{B}) \times \text{Reg}(\mathbf{B}) \rightarrow B$ is the map defined, for all $a, b \in \text{Reg}(\mathbf{B})$, by

$$\iota_{\mathbf{B}}^{-1}(\langle a, b \rangle) := (a \otimes (a \vee b)) \oplus (b \otimes (a \wedge b)).$$

Let us also note that, in a product bilattice $\mathbf{L} \odot \mathbf{L}$ (and, as mentioned above, any interlaced bilattice is of this form), the regular elements are those of the form $\langle a, a \rangle$ for some $a \in L$. Given $a_1, a_2 \in L$, we have that $\text{reg}(\langle a_1, a_2 \rangle) = \langle a_1, a_1 \rangle$, so

$$\iota_{\mathbf{B}}(\langle a_1, a_2 \rangle) = \langle \langle a_1, a_1 \rangle, \langle a_2, a_2 \rangle \rangle.$$

Conversely, for any $a_1, a_2 \in L$,

$$\iota_{\mathbf{B}}^{-1}(\langle \langle a_1, a_1 \rangle, \langle a_2, a_2 \rangle \rangle) = \langle a_1, a_2 \rangle.$$

2 Product residuated bilattices

In this section we extend the product bilattice construction to define a bilattice having two residuated pairs as a power of an arbitrary residuated lattice. Our construction is a generalization of the ones described in [Rivieccio 2010, Section 5.1] for “implicative bilattices” and in Bou et al. [Bou et al. 201x, Section 4] for “Brouwerian bilattices”. We were also inspired by results of [Tsinakis and Wille 2006, Theorem 3.3] and of [Busaniche and Cignoli 2009, Theorem 2.3]. In the next section we are going to give an axiomatization for the class of bilattices that are representable as products of this kind.

By a *residuated lattice* we mean an algebra $\mathbf{L} = \langle L, \sqcap, \sqcup, \cdot, \backslash, / \rangle$ such that the reduct $\langle L, \sqcap, \sqcup \rangle$ is a lattice, $\langle L, \cdot \rangle$ is a semigroup (i.e. the operation \cdot is associative) and the following residuation properties are satisfied: for all $a, b, c \in L$,

$$(R) \quad a \cdot b \leq c \quad \text{iff} \quad b \leq a \backslash c \quad \text{iff} \quad a \leq c / b.$$

Notice that this definition is slightly more general than the usual one [Galatos et al. 2007, p. 92] for we do not postulate the existence of a unit element $e \in L$ such that $\langle L, \cdot, e \rangle$ be a monoid.

Definition 2.1 Given a residuated lattice

$$\mathbf{L} = \langle L, \sqcap, \sqcup, \cdot, \backslash, / \rangle$$

we define the *product residuated bilattice*

$$\mathbf{L} \odot \mathbf{L} = \langle L \times L, \wedge, \vee, \otimes, \oplus, \supset, \subset, \neg \rangle$$

as follows:

- (i) the reduct $\langle L \times L, \wedge, \vee, \otimes, \oplus, \neg \rangle$ is the product bilattice $\langle L, \sqcap, \sqcup \rangle \odot \langle L, \sqcap, \sqcup \rangle$ introduced in the previous section;
- (ii) the operations \supset and \subset are defined, for all

$$\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in L_1 \times L_2$$

as

$$\langle a_1, a_2 \rangle \supset \langle b_1, b_2 \rangle := \langle a_1 \backslash b_1, b_2 \cdot a_1 \rangle$$

$$\langle a_1, a_2 \rangle \subset \langle b_1, b_2 \rangle := \langle a_1 / b_1, b_1 \cdot a_2 \rangle.$$

Given a product residuated bilattice $\mathbf{L} \odot \mathbf{L}$, we introduce derived operations defined, for all $a, b \in L \times L$, as

$$a \rightarrow b := (a \supset b) \wedge (\neg a \subset \neg b)$$

$$a \leftarrow b := \neg a \rightarrow \neg b$$

$$a * b := \neg(b \rightarrow \neg a).$$

Applying the definitions, we have that, for all

$$\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in L_1 \times L_2,$$

$$\begin{aligned} \langle a_1, a_2 \rangle \rightarrow \langle b_1, b_2 \rangle &= \langle (a_1 \backslash b_1) \sqcap (a_2 / b_2), b_2 \cdot a_1 \rangle \\ \langle a_1, a_2 \rangle \leftarrow \langle b_1, b_2 \rangle &= \langle (a_1 / b_1) \sqcap (a_2 \backslash b_2), b_1 \cdot a_2 \rangle \\ \langle a_1, a_2 \rangle * \langle b_1, b_2 \rangle &= \langle a_1 \cdot b_1, (b_2 / a_1) \sqcap (b_1 \backslash a_2) \rangle. \end{aligned}$$

As mentioned above, our construction generalizes those of [Rivieccio 2010] and [Bou et al. 201x]. In fact, it is easy to see that the Brouwerian bilattices of Bou et al. [Bou et al. 201x] are just product residuated bilattices having the form $\mathbf{L} \odot \mathbf{L}$ where $\mathbf{L} = \langle L, \sqcap, \sqcup, \cdot, \backslash, / \rangle$ is a Brouwerian lattice, i.e. a residuated lattice satisfying the equation

$$x \sqcap y \approx x \cdot y$$

and consequently also

$$x \backslash y \approx y / x.$$

Similarly, the implicative bilattices of [Rivieccio 2010] are Brouwerian bilattices of the form $\mathbf{L} \odot \mathbf{L}$ where \mathbf{L} satisfies the equation:

$$(x \backslash y) \backslash x \approx x.$$

The following result shows that any product residuated bilattice contains indeed two residuated pairs, thus explaining the choice of our terminology (the same result can also be found in [Tsinakis and Wille 2006, Corollary 3.4]).

Proposition 2.2 Let $\mathbf{L} \odot \mathbf{L}$ be a product residuated bilattice. Then, for all $a, b, c \in L \times L$,

$$a * b \leq_t c \quad \text{iff} \quad b \leq_t a \rightarrow c \quad \text{iff} \quad a \leq_t c \leftarrow b.$$

Proof Let $a = \langle a_1, a_2 \rangle$, $b = \langle b_1, b_2 \rangle$ and $c = \langle c_1, c_2 \rangle$ where $a_1, a_2, b_1, b_2, c_1, c_2 \in L$. Assume

$$\langle a_1, a_2 \rangle * \langle b_1, b_2 \rangle \leq_t \langle c_1, c_2 \rangle.$$

By the definitions this means that

$$\langle a_1 \cdot b_1, (b_2 / a_1) \sqcap (b_1 \backslash a_2) \rangle \leq_t \langle c_1, c_2 \rangle$$

i.e. that

$$a_1 \cdot b_1 \leq c_1 \quad \text{and} \quad c_2 \leq (b_2 / a_1) \sqcap (b_1 \backslash a_2)$$

where \leq is the lattice order of \mathbf{L} . From the latter inequality we obtain $c_2 \leq b_2 / a_1$ and $c_2 \leq b_1 \backslash a_2$. By residuation (R), we have the following equivalences:

- (i) $a_1 \cdot b_1 \leq c_1$ iff $a_1 \leq c_1 / b_1$ iff $b_1 \leq a_1 \backslash c_1$
- (ii) $c_2 \leq b_2 / a_1$ iff $a_1 \leq c_2 \backslash b_2$ iff $c_2 \cdot a_1 \leq b_2$
- (iii) $c_2 \leq b_1 \backslash a_2$ iff $b_1 \cdot c_2 \leq a_2$ iff $b_1 \leq a_2 / c_2$.

From (i) and (iii) we obtain $b_1 \leq (a_1 \setminus c_1) \sqcap (a_2 / c_2)$, so we have that

$$\langle b_1, b_2 \rangle \leq_t \langle (a_1 \setminus c_1) \sqcap (a_2 / c_2), c_2 \cdot a_1 \rangle$$

i.e. $b \leq_t a \rightarrow c$. From (i) and (ii) we obtain $a_1 \leq (c_1 / b_1) \sqcap (c_2 \setminus b_2)$, so we have that

$$\langle a_1, a_2 \rangle \leq_t \langle (c_1 / b_1) \sqcap (c_2 \setminus b_2), b_1 \cdot c_2 \rangle$$

i.e. $a \leq_t c \leftarrow b$.

Conversely, assume $b \leq_t a \rightarrow c$. By the definitions this means that

$$b_1 \leq (a_1 \setminus c_1) \sqcap (a_2 / c_2) \quad \text{and} \quad c_2 \cdot a_1 \leq b_2.$$

It is then easy, using the equivalences (i), (ii) and (iii) to prove that the assumptions imply $a * b \leq_t c$ and therefore $a \leq_t c \leftarrow b$.

Thus, for any product residuated bilattice

$$\mathbf{L} \odot \mathbf{L} = \langle L \times L, \wedge, \vee, \otimes, \oplus, \supset, \subset, \neg \rangle$$

the structure $\langle L \times L, \wedge, \vee, *, \rightarrow, \leftarrow, \neg \rangle$ is a residuated lattice endowed with an involutive negation. The following properties are also easily checked:

- if the semigroup operation \cdot is commutative in \mathbf{L} , then \mathbf{L} satisfies

$$x \setminus y \approx y / x$$

and consequently $\mathbf{L} \odot \mathbf{L}$ satisfies

$$x \supset y \approx y \subset x$$

$$x \rightarrow y \approx y \leftarrow x$$

$$x * y \approx y * x$$

- if there is an element $1 \in L$ such that $\langle L, \cdot, 1 \rangle$ is a monoid and 1 is the top element of the lattice order of \mathbf{L} , then $\langle L \times L, *, \langle 1, 1 \rangle \rangle$ is also a monoid, $\langle 1, 1 \rangle$ is the maximum element of the \leq_k order of $\mathbf{L} \odot \mathbf{L}$ (i.e. the element we have denoted above by \top) and $\langle L \times L, \wedge, \vee, *, \rightarrow, \leftarrow, \neg, \langle 1, 1 \rangle \rangle$ is an *involutive residuated lattice* according to the terminology used in [Galatos and Raftery 2004].
- if there is an element $e \in L$ such that $\langle L, \cdot, e \rangle$ is a monoid, then $\langle L \times L, *, \langle e, e \rangle \rangle$ is also a monoid and $\langle L \times L, \wedge, \vee, *, \rightarrow, \leftarrow, \neg \rangle$ is an *involutive residuated lattice* according to the terminology used in [Galatos and Raftery 2004];
- if \mathbf{L} is integral (i.e. $\langle L, \cdot, e \rangle$ is a monoid and e is the top element of the lattice order of \mathbf{L}), then $\langle e, e \rangle$ is the maximum element of the \leq_k order of $\mathbf{L} \odot \mathbf{L}$ (i.e. the element we have denoted above by \top).

A remarkable feature of the construction introduced in Definition 2.1 is that, like the product described in the previous section, it preserves the congruences, in the sense that the lattice of congruences of any residuated lattice \mathbf{L} is isomorphic to the lattice of congruences of the corresponding product residuated bilattice $\mathbf{L} \odot \mathbf{L}$. The remaining part of this section is devoted to proving this result.

As we anticipated in the previous section, it has been proved by [Bou and Riviello 2011, Proposition 3.13] that the congruences of any interlaced bilattice $\mathbf{L} \odot \mathbf{L}$ are isomorphic to those of \mathbf{L} via the map

$$H : \text{Con}(\mathbf{L}) \cong \text{Con}(\mathbf{L} \odot \mathbf{L})$$

defined, for all $\theta \in \text{Con}(\mathbf{L})$ and all $a_1, a_2, b_1, b_2 \in L$, as follows:

$$\langle \langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \rangle \in H(\theta)$$

iff $(a_1, b_1), (a_2, b_2) \in \theta$.

$$\langle \langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \rangle \in H(\theta) \quad \text{iff} \quad (a_1, b_1), (a_2, b_2) \in \theta$$

The inverse map H^{-1} can be defined, for all $\theta \in \text{Con}(\mathbf{L} \odot \mathbf{L})$ and all $a, b \in L$, as follows: $(a, b) \in H^{-1}(\theta)$ iff there are $c, d \in L$ such that $\langle \langle a, c \rangle, \langle b, d \rangle \rangle \in \theta$. We are going to see that the same maps give an isomorphism in the expanded algebraic signature.

Theorem 2.3 *Let $\mathbf{L} = \langle L, \sqcap, \sqcup, \cdot, \setminus, / \rangle$ be a residuated lattice whose associated product residuated bilattice is $\mathbf{L} \odot \mathbf{L} = \langle L \times L, \wedge, \vee, \otimes, \oplus, \supset, \subset, \neg \rangle$. Then $\text{Con}(\mathbf{L}) \cong \text{Con}(\mathbf{L} \odot \mathbf{L})$.*

Proof We already know that the congruences of the reduct $\langle L, \sqcap, \sqcup \rangle$ are isomorphic to those of

$$\langle L \times L, \wedge, \vee, \otimes, \oplus, \neg \rangle.$$

So we just need to check that a relation

$$\theta \in \text{Con}(\langle L, \sqcap, \sqcup \rangle)$$

is compatible with the operations $\langle \cdot, \setminus, / \rangle$ if and only if $H(\theta)$ is compatible with $\langle \supset, \subset \rangle$. Assume then that $\theta \in \text{Con}(\mathbf{L})$ and $\langle \langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \rangle, \langle \langle c_1, c_2 \rangle, \langle d_1, d_2 \rangle \rangle \in H(\theta)$, which by the definition of H means that

$$(a_1, b_1), (a_2, b_2), (c_1, d_1), (c_2, d_2) \in \theta.$$

We have to prove that

$$\langle \langle a_1, a_2 \rangle \supset \langle c_1, c_2 \rangle, \langle b_1, b_2 \rangle \supset \langle d_1, d_2 \rangle \rangle \in H(\theta).$$

Applying the definitions, this amounts to proving that

$$\langle \langle a_1 \setminus c_1, c_2 \cdot a_1 \rangle, \langle b_1 \setminus d_1, d_2 \cdot b_1 \rangle \rangle \in H(\theta)$$

i.e. that $(a_1 \setminus c_1, b_1 \setminus d_1), (c_2 \cdot a_1, d_2 \cdot b_1) \in \theta$. Then the result easily follows using the compatibility of θ with the operations \cdot and \setminus . The same reasoning shows that $H(\theta)$ is compatible with the operation \subset .

Conversely, assume $\theta \in \text{Con}(\mathbf{L} \odot \mathbf{L})$ and

$$(a_1, a_2), (b_1, b_2) \in H^{-1}(\theta).$$

This means that there are $c_1, c_2, d_1, d_2 \in L$ such that

$$(\langle a_1, c_1 \rangle, \langle a_2, c_2 \rangle), (\langle b_1, d_1 \rangle, \langle b_2, d_2 \rangle) \in \theta$$

and we have to prove that $(a_1 \star b_1, a_2 \star b_2) \in H^{-1}(\theta)$ for any operation $\star \in \{\cdot, \setminus, /\}$. From the assumption that $\theta \in \text{Con}(\mathbf{L} \odot \mathbf{L})$ it follows that $\langle a_1, c_1 \rangle \supset \langle b_1, d_1 \rangle = \langle a_1 \setminus b_1, d_1 \cdot a_1 \rangle \theta \langle a_2 \setminus b_2, d_2 \cdot a_2 \rangle = \langle a_2, c_2 \rangle \supset \langle b_2, d_2 \rangle$ which implies that $(a_1 \setminus b_1, a_2 \setminus b_2) \in H^{-1}(\theta)$. The same reasoning, using the compatibility of θ with the operation \subset , shows that $(a_1 / b_1, a_2 / b_2) \in H^{-1}(\theta)$. To prove that

$$(a_1 \cdot b_1, a_2 \cdot b_2) \in H^{-1}(\theta),$$

notice that $(\langle a_1, c_1 \rangle, \langle a_2, c_2 \rangle) \in \theta$ implies

$$(\langle c_1, a_1 \rangle, \langle c_2, a_2 \rangle) \in \theta$$

because θ is compatible with the bilattice negation. Then we have that $\langle b_1, d_1 \rangle \supset \langle c_1, a_1 \rangle = \langle b_1 \setminus c_1, a_1 \cdot b_1 \rangle \theta \langle b_2 \setminus c_2, a_2 \cdot b_2 \rangle = \langle b_2, d_2 \rangle \supset \langle c_2, a_2 \rangle$ which, using again compatibility with negation, implies that

$$(\langle a_1 \cdot b_1, b_1 \setminus c_1 \rangle, \langle a_2 \cdot b_2, b_2 \setminus c_2 \rangle) \in \theta.$$

Now we just apply the definitions to conclude that $(a_1 \cdot b_1, a_2 \cdot b_2) \in H^{-1}(\theta)$.

As we have seen in the previous section for bilattices, the above result implies that any property that depends on the lattice of congruences (congruence-distributivity, congruence-permutability, etc.) transfers from residuated lattices to product residuated bilattices; also, the subdirectly irreducible algebras of the two classes are in one-to-one correspondence.

3 Residuated bilattices

We are now going to introduce a class of bilattices that will be proved to coincide up to isomorphism with those that can be obtained through the construction described in the previous section.

Definition 3.1 A *residuated bilattice* is an algebra

$$\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \supset, \subset, \neg \rangle$$

such that $\langle B, \wedge, \vee, \otimes, \oplus, \neg \rangle$ is an interlaced bilattice and the following equations and quasi-equations are satisfied:

$$x \supset (y \wedge z) \approx (x \supset y) \wedge (x \supset z) \quad (\text{RB1})$$

$$(x \vee y) \supset z \approx (x \supset z) \wedge (y \supset z) \approx (x \oplus y) \supset z \quad (\text{RB2})$$

$$(x \wedge y) \subset z \approx (x \subset z) \wedge (y \subset z) \quad (\text{RB3})$$

$$x \subset (y \vee z) \approx (x \subset y) \wedge (x \subset z) \approx x \subset (y \oplus z) \quad (\text{RB4})$$

$$x \supset (y \otimes z) \leq_k (x \supset y) \otimes (x \supset z) \quad (\text{RB5})$$

$$(x \otimes y) \subset z \leq_k (x \subset z) \otimes (y \subset z) \quad (\text{RB6})$$

$$(x \supset y) \otimes (\neg x \subset \neg y) \approx (x \supset y) \wedge (\neg x \subset \neg y) \quad (\text{RB7})$$

$$x \rightarrow (y \rightarrow z) \approx \neg(x \rightarrow \neg y) \rightarrow z \quad (\text{RB8})$$

$$\begin{aligned} x \leq_t x \otimes (y \supset z) &\Leftrightarrow y \leq_t y \otimes (z \subset x) \\ &\Leftrightarrow (x \supset \neg y) \otimes \neg z \leq_t x \supset \neg y \end{aligned} \quad (\text{RB9})$$

where $x \rightarrow y$ is an abbreviation for $(x \supset y) \wedge (\neg x \subset \neg y)$. We shall also use the following abbreviations:

$$x \leftarrow y := \neg x \rightarrow \neg y$$

$$x * y := \neg(y \rightarrow \neg x).$$

We denote by **ReBiLat** the class of residuated bilattices, which is by definition a quasi-variety (in the next section we are going to see that it is in fact a variety).

Our next aim is to show that any residuated bilattice can be represented via the construction introduced in Definition 2.1. Let $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \supset, \subset, \neg \rangle$ be a residuated bilattice. We define the relation \sim as follows:

$$\sim := \{ \langle a, b \rangle \in B \times B : a \otimes b = a \vee b \}.$$

An alternative but equivalent definition is the following (see [Bou and Rivieccio 2011, Proposition 3.6]):

$$\sim := \{ \langle a, b \rangle \in B \times B : a \oplus b = a \wedge b \}.$$

This relation was studied in [Bou and Rivieccio 2011], and [Rivieccio 2010] in order to prove the representation theorem for interlaced bilattices. It was shown in [Bou and Rivieccio 2011, Proposition 3.8] that \sim is a congruence of the reduct $\langle B, \wedge, \vee, \otimes, \oplus \rangle$ of any interlaced bilattice \mathbf{B} . We are now going to see that \sim is also compatible with the operations $\langle *, \supset, \subset \rangle$. We shall need the following properties.

Proposition 3.2 *Let $\mathbf{B} \in \text{ReBiLat}$ and $a, b, c \in B$. Then:*

- (i) if $a \leq_t b$, then $c \supset a \leq_t c \supset b$ and $b \supset c \leq_t a \supset c$
- (ii) if $a \leq_t b$, then $a \subset c \leq_t b \subset c$ and $c \subset b \leq_t c \subset a$
- (iii) if $a \leq_k b$, then $c \supset a \leq_k c \supset b$ and $a \subset c \leq_k b \subset c$

- (iv) if $a \wedge b = a \oplus b$, then $a \supset c = b \supset c$ and $c \subset a = c \subset b$
 (v) $a * b \sim \neg(b \supset \neg a)$
 (vi) $a * (b * c) = (a * b) * c$.

Proof (i). Assume $a \leq_t b$. Then, by (RB1), $c \supset a = c \supset (a \wedge b) = (c \supset a) \wedge (c \supset b) \leq_t c \supset b$. By (RB2) we have $b \supset c = (a \vee b) \supset c = (a \supset c) \wedge (b \supset c) \leq_t a \supset c$.

(ii). Assume $a \leq_t b$. Then, by (RB3), $a \subset c = (a \wedge b) \subset c = (a \subset c) \wedge (b \subset c) \leq_t b \subset c$. By (RB4) we have $c \subset b = c \subset (a \vee b) = (c \subset a) \wedge (c \subset b) \leq_t c \subset a$.

(iii). Assume $a \leq_k b$. Then by (RB5) $c \supset a = c \supset (a \otimes b) \leq_k (c \supset a) \otimes (c \supset b)$. The same reasoning, using (RB6) instead of (RB5), shows that $a \subset c \leq_k b \subset c$.

(iv). Assume $a \wedge b = a \oplus b$. Then, by the absorption laws, $a \vee (a \oplus b) = a \vee (a \wedge b) = a$. Applying (RB2), we have

$$\begin{aligned} a \supset c &= (a \vee (a \oplus b)) \supset c \\ &= (a \supset c) \wedge ((a \oplus b) \supset c) \\ &= (a \supset c) \wedge (a \supset c) \wedge (b \supset c) \\ &= (a \supset c) \wedge (b \supset c). \end{aligned}$$

Hence $a \supset c \leq_t b \supset c$ and, by symmetry, we obtain $a \supset c = b \supset c$. To prove that $c \subset a = c \subset b$, we reason in the same way, using (RB4) instead of (RB2).

(v). By (RB7) we have that, for all $a, b \in B$,

$$(b \supset \neg a) \otimes (\neg b \subset a) = (b \supset \neg a) \wedge (\neg b \subset a).$$

Negating both sides of this equality and applying De Morgan laws, we obtain

$$\neg(b \supset \neg a) \otimes \neg(\neg b \subset a) = \neg(b \supset \neg a) \vee \neg(\neg b \subset a)$$

which means that $\neg(b \supset \neg a) \sim \neg(\neg b \subset a)$. Since \sim is a congruence of the bilattice reduct of \mathbf{B} , this implies that

$$\begin{aligned} \neg(b \supset \neg a) &= \neg(b \supset \neg a) \vee \neg(b \supset \neg a) \\ &\sim \neg(b \supset \neg a) \vee \neg(\neg b \subset a). \end{aligned}$$

Using De Morgan laws it is easy to check that $a * b = \neg(b \supset \neg a) \vee \neg(\neg b \subset a)$ and this concludes our proof.

(vi). Applying the definition of $*$, we have

$$\begin{aligned} a * (b * c) &= \neg(\neg(c \rightarrow \neg b) \rightarrow \neg a) \\ &= \neg(c \rightarrow (b \rightarrow \neg a)) \quad \text{by (RB8)} \\ &= \neg(c \rightarrow \neg\neg(b \rightarrow \neg a)) \\ &= (a * b) * c. \end{aligned}$$

Proposition 3.3 For any residuated bilattice

$$\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \supset, \subset, \neg \rangle$$

the relation \sim is a congruence of the reduct

$$\langle B, \wedge, \vee, \otimes, \oplus, \supset, \subset \rangle$$

and is compatible with the operation $*$.

Proof We have to check compatibility with the operations $\langle \supset, \subset, * \rangle$. Given $a, b, c, d \in B$, assume $a \sim b$ and $c \sim d$. Then, as observed above, we have that $a \wedge b = a \oplus b$ and $c \wedge d = c \oplus d$. Note also that, by Proposition 3.2 (iv), $b \supset d = a \supset d$. We have then

$$\begin{aligned} (a \supset c) \wedge (b \supset d) &= (a \supset c) \wedge (a \supset d) && \text{by hyp.} \\ &= a \supset (c \wedge d) && \text{by (RB1)} \\ &= a \supset (c \oplus d) && \text{by hyp.} \\ &\geq_k (a \supset c) \oplus (a \supset d) && \text{Prop. 3.2 (iii)} \\ &= (a \supset c) \oplus (b \supset d) && \text{by hyp.} \end{aligned}$$

Hence $a \supset c \sim b \supset d$. A similar argument allows to conclude that $a \subset c \sim b \subset d$. To prove that $a * c \sim b * d$, note that it is sufficient to show that $\neg(c \supset \neg a) \sim \neg(d \supset \neg b)$. This is so because by Proposition 3.2 (v) we have that $a * b \sim \neg(b \supset \neg a)$ for all $a, b \in B$. By the assumptions and Proposition 3.2 (iv), we have that

$$\begin{aligned} \neg(c \supset \neg a) \vee \neg(d \supset \neg b) &= \neg(c \supset \neg a) \vee \neg(c \supset \neg b) \\ &= \text{by De Morgan laws} \\ &= \neg((c \supset \neg a) \wedge (c \supset \neg b)) \\ &= \text{by (RB1)} \\ &= \neg(c \supset (\neg a \wedge \neg b)) \\ &= \text{by De Morgan laws} \\ &= \neg(c \supset \neg(a \vee b)) \\ &= \text{by the assumptions} \\ &= \neg(c \supset \neg(a \otimes b)) \\ &= \text{by De Morgan laws} \\ &= \neg(c \supset (\neg a \otimes \neg b)) \\ &= \text{by (RB5)} \\ &\leq_k \neg((c \supset \neg a) \otimes (c \supset \neg b)) \\ &= \text{by De Morgan laws} \\ &= \neg(c \supset \neg a) \otimes \neg(c \supset \neg b). \end{aligned}$$

Hence, the result easily follows using the interlacing conditions.

Using the previous results, let us prove that some more useful properties hold.

Proposition 3.4 Let $\mathbf{B} \in \text{ReBiLat}$ and $a, b \in B$. Then:

- (i) $a \sim \text{reg}(a)$
- (ii) $a \sim b$ iff $\text{reg}(a) = \text{reg}(b)$
- (iii) $\text{reg}(a \wedge b) = \text{reg}(a \otimes b) = \text{reg}(a) \otimes \text{reg}(b)$
- (iv) $\text{reg}(a \vee b) = \text{reg}(a \oplus b) = \text{reg}(a) \oplus \text{reg}(b)$
- (v) $\text{reg}(a \supset b) = \text{reg}(\text{reg}(a) \supset \text{reg}(b))$
- (vi) $\text{reg}(a \subset b) = \text{reg}(\text{reg}(a) \subset \text{reg}(b))$
- (vii) $\text{reg}(a * b) = \text{reg}(\text{reg}(a) * \text{reg}(b)) = \text{reg}(\text{reg}(a) * b) = \text{reg}(a * \text{reg}(b)) = \text{reg}(\neg(b \supset \neg a)) = \text{reg}(\neg(\neg b \subset a))$
- (viii) $\text{reg}(a) \leq_k \text{reg}(b)$ iff $a \leq_t a \otimes b$.

Proof For a proof of items (i)-(iv), we refer the reader to [Bou and Rivieccio 2011] or [Rivieccio 2010, Proposition 2.2.5]. Items (v), (vi) and the first three equalities of (vii) follow easily from (i), (ii) and the fact that \sim is compatible with the operations $\langle *, \supset, \subset \rangle$. The last two equalities of (vii) follow easily from Proposition 3.2 (v). As to (viii), note that $\text{reg}(a) \leq_k \text{reg}(b)$ if and only if

$$\text{reg}(a) = \text{reg}(a) \otimes \text{reg}(b) = \text{reg}(a \otimes b).$$

By (ii), this is equivalent to $a \wedge (a \otimes b) = a \oplus (a \otimes b) = a$, i.e. $a \leq_t a \otimes b$.

Now, given a residuated bilattice

$$\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \supset, \subset, *, \neg \rangle$$

we consider the algebra

$$\mathbf{B}^* = \langle \text{Reg}(\mathbf{B}), \otimes, \oplus, \cdot, \setminus, / \rangle$$

where $\langle \text{Reg}(\mathbf{B}), \otimes, \oplus \rangle$ is the sublattice of the k -lattice of \mathbf{B} and the operations $\{\cdot, \setminus, /\}$ are defined, for all $a, b \in \text{Reg}(\mathbf{B})$, as follows:

$$\begin{aligned} a \cdot b &:= \text{reg}(a * b) \\ a \setminus b &:= \text{reg}(a \supset b) \\ a / b &:= \text{reg}(a \subset b). \end{aligned}$$

We have the following:

Proposition 3.5 *Let $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \supset, \subset, \neg \rangle$ be a residuated bilattice. Then:*

- (i) $a \cdot b = \text{reg}(\neg(b \supset a)) \oplus \text{reg}(\neg(b \subset a))$ for all $a, b \in \text{Reg}(\mathbf{B})$
- (ii) \mathbf{B}^* is a residuated lattice.

Proof (i). Let $a, b \in \text{Reg}(\mathbf{B})$. Then:

$$\begin{aligned} a \cdot b &= \text{reg}(\neg((b \supset a) \wedge (\neg b \subset a))) \\ &= \text{because } a, b \in \text{Reg}(\mathbf{B}) \\ &= \text{reg}(\neg((b \supset a) \wedge (b \subset a))) \\ &= \text{by De Morgan laws} \\ &= \text{reg}(\neg(b \supset a) \vee \neg(b \subset a)) \\ &= \text{by Prop. 3.4 (iv)} \\ &= \text{reg}(\neg(b \supset a) \oplus \neg(b \subset a)) \\ &= \text{reg}(\neg(b \supset a)) \oplus \text{reg}(\neg(b \subset a)). \end{aligned}$$

(ii). We have seen in Proposition 3.2 (vi) that the operation $*$ is associative. This implies that the operation \cdot is also associative, because one has

$$\begin{aligned} a \cdot (b \cdot c) &= \text{reg}(a * \text{reg}(b * c)) \\ &= \text{reg}(a * (b * c)) && \text{by Prop. 3.4 (vii)} \\ &= \text{reg}((a * b) * c) && \text{by associativity of } * \\ &= \text{reg}(\text{reg}(a * b) * c) && \text{by Prop. 3.4 (vii)} \\ &= (a \cdot b) \cdot c. \end{aligned}$$

To prove that the operations $\{\setminus, /\}$ are the residua of \cdot , assume that $a \cdot b \leq_k c$ for some $a, b, c \in \text{Reg}(\mathbf{B})$. By (i) we have that

$$a \cdot b = \text{reg}(\neg(b \supset a)) \oplus \text{reg}(\neg(b \subset a)).$$

Then from the assumptions it follows that $\text{reg}(\neg(b \supset a)) \leq_k c = \text{reg}(c)$ and $\text{reg}(\neg(b \subset a)) \leq_k c = \text{reg}(c)$. By Proposition 3.4 (viii), this means that $\neg(b \supset a) \leq_t \neg(b \supset a) \otimes c$ and $\neg(b \subset a) \leq_t \neg(b \subset a) \otimes c$. From the first we obtain $(b \supset a) \otimes \neg c \leq_t b \supset a$ and now, applying (RB9), we have $b \leq_t b \otimes (\neg a \supset c) = b \otimes (a \supset c)$. Hence, by Proposition 3.4 (viii),

$$\text{reg}(b) = b \leq_k \text{reg}(a \supset c) = a \setminus c.$$

Similarly, from $\neg(b \subset a) \leq_t \neg(b \subset a) \otimes c$, using (RB9), we obtain $\neg a \leq_t \neg a \otimes (c \subset b)$. Hence $a \leq_k \text{reg}(c \subset b) = c / b$.

Conversely, assume for instance that $b \leq_k a \setminus c = \text{reg}(a \supset c)$ for some $a, b, c \in \text{Reg}(\mathbf{B})$. By Proposition 3.4 (viii) we have $b \leq_t b \otimes (a \supset c)$ and this, by (RB9), implies $(b \supset \neg a) \otimes \neg c \leq_t b \supset \neg a$. From this, by the properties of the bilattice negation, we obtain $\neg(b \supset \neg a) \leq_t \neg(b \supset \neg a) \otimes c$. Now we can apply Proposition 3.4 (viii) again to conclude that $\text{reg}(\neg(b \supset a)) \leq_k c$. By a similar reasoning we obtain $\text{reg}(\neg(b \subset a)) \leq_k c$ and then, applying the interlacing conditions, we have

$$\text{reg}(\neg(b \supset a)) \oplus \text{reg}(\neg(b \subset a)) = a \cdot b \leq_k c.$$

Finally, if $a \leq_k c / b = \text{reg}(c \subset b)$, then by Proposition 3.4 (viii) we obtain $a \leq_t a \otimes (c \subset b)$. By (RB9) this is equivalent to $b \leq_t b \otimes (a \supset c)$. Now we apply Proposition 3.4 (viii) again to obtain $b = \text{reg}(b) \leq_k \text{reg}(a \supset c) = a \setminus c$ and this completes the proof.

We are now able to state our main result:

Theorem 3.6 *Any residuated bilattice*

$$\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \supset, \subset, \neg \rangle$$

is isomorphic to the product residuated bilattice

$$\mathbf{B}^* \odot \mathbf{B}^*$$

where $\mathbf{B}^ = \langle \text{Reg}(\mathbf{B}), \otimes, \oplus, \cdot, \setminus, / \rangle$ is a residuated lattice, through the map*

$$\iota : B \rightarrow \text{Reg}(\mathbf{B}) \times \text{Reg}(\mathbf{B})$$

defined, for all $a \in B$, as

$$\iota(a) := \langle \text{reg}(a), \text{reg}(\neg a) \rangle.$$

Proof We know from [Bou and Riviuccio 2011, Proposition 3.13] that there is an isomorphism

$$\iota: B \rightarrow \text{Reg}(\mathbf{B}) \times \text{Reg}(\mathbf{B})$$

between

$$\langle B, \wedge, \vee, \otimes, \oplus, \neg \rangle$$

and

$$\langle \text{Reg}(\mathbf{B}), \otimes, \oplus \rangle \odot \langle \text{Reg}(\mathbf{B}), \otimes, \oplus \rangle.$$

So we only need to check that this map respects the operations \supset and \subset . Let us denote by \supset^* and \subset^* these operations in $\mathbf{B}^* \odot \mathbf{B}^*$. As to the first, we have

$$\begin{aligned} \iota(a \supset b) &= \\ &= \langle \text{reg}(a \supset b), \text{reg}(\neg(a \supset b)) \rangle \\ &= \text{by Proposition 2.4 (vi)} \\ &= \langle \text{reg}(a \supset b), \text{reg}(\neg b * a) \rangle \\ &= \text{by Proposition 2.4 (vi)} \\ &= \langle \text{reg}(\text{reg}(a) \supset \text{reg}(b)), \text{reg}(\text{reg}(\neg b) * \text{reg}(a)) \rangle \\ &= \text{by definition} \\ &= \langle \text{reg}(a) \setminus \text{reg}(b), \text{reg}(\neg b) \cdot \text{reg}(a) \rangle \\ &= \text{by definition} \\ &= \langle \text{reg}(a), \text{reg}(\neg a) \rangle \supset^* \langle \text{reg}(b), \text{reg}(\neg b) \rangle \\ &= \iota(a) \supset^* \iota(b). \end{aligned}$$

As to the second, we have

$$\begin{aligned} \iota(a \subset b) &= \\ &= \langle \text{reg}(a \subset b), \text{reg}(\neg(a \subset b)) \rangle \\ &= \text{by Proposition 2.4 (vi)} \\ &= \langle \text{reg}(a \subset b), \text{reg}(b * \neg a) \rangle \\ &= \text{by Proposition 2.4 (vi)} \\ &= \langle \text{reg}(\text{reg}(a) \subset \text{reg}(b)), \text{reg}(\text{reg}(b) * \text{reg}(\neg a)) \rangle \\ &= \text{by definition} \\ &= \langle \text{reg}(a) / \text{reg}(b), \text{reg}(b) \cdot \text{reg}(\neg a) \rangle \\ &= \text{by definition} \\ &= \langle \text{reg}(a), \text{reg}(\neg a) \rangle \subset^* \langle \text{reg}(b), \text{reg}(\neg b) \rangle \\ &= \iota(a) \subset^* \iota(b). \end{aligned}$$

We immediately obtain the following:

Corollary 3.7 *For any residuated bilattice \mathbf{B} ,*

$$\text{Con}(\mathbf{B}) \cong \text{Con}(\mathbf{B}^*).$$

Proof Follows immediately from Theorem 2.3 and Theorem 3.6.

The isomorphism $H: \text{Con}(\mathbf{B}) \cong \text{Con}(\mathbf{B}^*)$ can be defined, for all $\theta \in \text{Con}(\mathbf{B})$, as follows (cf. Bou et al. [Bou et al. 201x, Theorem 4.13]):

$$H(\theta) := \theta \cap (\text{Reg}(\mathbf{B}) \times \text{Reg}(\mathbf{B}))$$

and conversely, for all $\theta \in \text{Con}(\mathbf{B}^*)$ and all $a, b \in B$, $(a, b) \in H^{-1}(\theta)$ iff

$$(\text{reg}(a), \text{reg}(b)), (\text{reg}(\neg a), \text{reg}(\neg b)) \in \theta.$$

4 Translations

The results of the previous section (Theorem 3.6 and Corollary 3.7) allow to establish a relationship not only between residuated bilattices and residuated lattices, but also between subclasses of them, in particular between the sub-quasi-varieties of these two classes. Indeed, there is an easily definable isomorphism between classes of residuated lattices and classes of residuated bilattices, each closed under isomorphic images. The isomorphism is established by the map that sends a class \mathbf{K} of residuated lattices closed under isomorphisms to the class of residuated bilattices

$$\mathbf{K}^{\mathbf{B}} := \{\mathbf{B} : \mathbf{B} \cong \mathbf{L} \odot \mathbf{L} \text{ for some } \mathbf{L} \in \mathbf{K}\}$$

The inverse of this map sends a class \mathbf{K} of residuated bilattices closed under isomorphisms to the class of residuated lattices

$$\mathbf{K}^{\mathbf{L}} := \{\mathbf{L} : \mathbf{B} \cong \mathbf{L} \odot \mathbf{L} \text{ for some } \mathbf{B} \in \mathbf{K}\}.$$

Both maps are obviously order perserving with respect to the inclusion order, and since they apply to classes closed under isomorphisms, it is easy to see that $\mathbf{K} = (\mathbf{K}^{\mathbf{B}})^{\mathbf{L}}$ and $\mathbf{K} = (\mathbf{K}^{\mathbf{L}})^{\mathbf{B}}$. The maps $(\cdot)^{\mathbf{B}}$ and $(\cdot)^{\mathbf{L}}$ send quasi-varieties to quasi-varieties and varieties to varieties. In particular, $(\cdot)^{\mathbf{B}}$ maps the variety of residuated lattices to the class of residuated bilattices, thus implying that the latter class is a variety because the class of residuated lattices is [Galatos et al. 2007, Theorem 2.7].

We will prove the above-mentioned facts by introducing a procedure to obtain an axiomatization of any class of residuated bilattices definable by first-order sentences from an axiomatization of the class of residuated lattices associated with it by the isomorphism, as well as a procedure to obtain an axiomatization of any class of residuated lattices definable by first-order sentences from an axiomatization of the class of residuated bilattices associated with it by the isomorphism.

Let \mathcal{T} be the set of terms in the language of residuated bilattices:

$$\langle \wedge, \vee, \otimes, \oplus, \supset, \subset, *, \neg \rangle.$$

For any $\varphi \in \mathcal{T}$, we define the term

$$\text{reg}(\varphi) := (\varphi \vee (\varphi \otimes \neg\varphi)) \oplus \neg(\varphi \vee (\varphi \otimes \neg\varphi)).$$

Let \mathcal{S} be the set of terms in the language of residuated lattices:

$$\langle \sqcap, \sqcup, \setminus, /, \cdot \rangle.$$

For every term $\varphi \in \mathcal{S}$ we can define the term $\varphi^\bullet \in \mathcal{T}$ obtained from φ by replacing all occurrences of the symbols $\sqcap, \sqcup, \setminus, /, \cdot$ respectively by occurrences of $\otimes, \oplus, \supset, \subset, *$. This is done inductively as follows: for any variable $x \in \mathcal{S}$ and all terms $\varphi, \psi \in \mathcal{S}$,

$$\begin{aligned} x^\bullet &:= x & (\varphi \setminus \psi)^\bullet &:= \varphi^\bullet \supset \psi^\bullet \\ (\varphi \sqcap \psi)^\bullet &:= \varphi^\bullet \otimes \psi^\bullet & (\varphi / \psi)^\bullet &:= \varphi^\bullet \subset \psi^\bullet \\ (\varphi \sqcup \psi)^\bullet &:= \varphi^\bullet \oplus \psi^\bullet & (\varphi \cdot \psi)^\bullet &:= \varphi^\bullet * \psi^\bullet. \end{aligned}$$

The translation $\tau : \mathcal{S} \longrightarrow \mathcal{T}$ is then given by

$$\tau(\varphi) := \text{reg}(\varphi^\bullet).$$

We have not included constants in either of the two languages (0 and 1 for lattices, f, t, \perp and \top for bilattices), but our translations can be easily extended to these enriched languages by defining $0^\bullet := \perp$ and $1^\bullet := \top$, so we have that $\tau(0) = \perp$ and $\tau(1) = \top$.

Let v be an assignment on a residuated lattice \mathbf{L} . We define the assignment v^+ on $\mathbf{L} \odot \mathbf{L}$ as follows:

$$v^+(x) := \langle v(x), v(x) \rangle.$$

It is easily seen by induction that, for every $\varphi \in \mathcal{S}$,

$$v^+(\tau(\varphi)) = \langle v(\varphi), v(\varphi) \rangle.$$

If v is an assignment on an algebra \mathbf{A} and $a \in A$, then v_x^a is the assignment that maps x to a and every variable $y \neq x$ to $v(y)$. Note that if v is an assignment on a residuated lattice \mathbf{L} and $a \in L$, then the assignment $(v_x^a)^+$ on $\mathbf{L} \odot \mathbf{L}$ satisfies that $(v_x^a)^+ \langle a, a \rangle = (v_x^a)^+$.

Let now w be an assignment on $\mathbf{L} \odot \mathbf{L}$. For every $\varphi \in \mathcal{T}$, denote by $w(\varphi)_1$ and $w(\varphi)_2$ respectively the first and second component of the pair $w(\varphi)$. It is easy to see that, for every $\varphi \in \mathcal{T}$,

$$w(\text{reg}(\varphi)) = \langle w(\varphi)_1, w(\varphi)_1 \rangle.$$

It is also easy to check that if w' is another assignment on $\mathbf{L} \odot \mathbf{L}$ such that $w(x)_1 = w'(x)_1$ for every variable x , then $w(\text{reg}(\varphi)) = w'(\text{reg}(\varphi))$ for every $\varphi \in \mathcal{T}$. Let w^- be the assignment on \mathbf{L} defined by $w^-(x) := w(x)_1$ for every variable x . By induction, we have that, for every $\varphi \in \mathcal{S}$,

$$w^-(\varphi) = w(\varphi^\bullet)_1.$$

Then, for every $\varphi \in \mathcal{S}$,

$$w(\tau(\varphi)) = \langle w^-(\varphi), w^-(\varphi) \rangle.$$

Note that if x is a variable and $\langle a, b \rangle \in L \times L$, then $(w_x^{\langle a, b \rangle})^- = (w^-)_x^a$.

We extend the translation τ to first-order formulas as follows: for all terms $\varphi, \psi \in \mathcal{S}$ and all every first-order formulas σ, σ' of \mathcal{S} ,

$$\begin{aligned} \tau(\varphi \approx \psi) &:= \tau(\varphi) \approx \tau(\psi) \\ \tau(\sim \sigma) &:= \sim \tau(\sigma) \\ \tau(\sigma \Rightarrow \sigma') &:= \tau(\sigma) \Rightarrow \tau(\sigma') \\ \tau(\exists x \sigma) &:= \exists x \tau(\sigma) \end{aligned}$$

where the symbols \sim, \Rightarrow and \exists denote respectively classical negation, implication and existential quantifier. We have then the following:

Proposition 4.1 *For every first-order formula σ of \mathcal{S} ,*

- (i) $\mathbf{L} \models \sigma[w^-]$ iff $\mathbf{L} \odot \mathbf{L} \models \tau(\sigma)[w]$, for every assignment w on $\mathbf{L} \odot \mathbf{L}$
- (ii) $\mathbf{L} \models \sigma[v]$ iff $\mathbf{L} \odot \mathbf{L} \models \tau(\sigma)[v^+]$, for every assignment on \mathbf{L} .

Proof (i). By induction. Let w be an assignment on $\mathbf{L} \odot \mathbf{L}$. It is clear from the observations above that $w^-(\varphi) = w^-(\psi)$ if and only if $w(\tau(\varphi)) = w(\tau(\psi))$. Thus (i) holds for equations. For negations and implications it is immediate using the inductive hypothesis. Assume it holds for σ . We show that it holds for $\exists x \sigma$. Suppose that $\mathbf{L} \models \exists x \sigma[w^-]$ and let $a \in L$ be such that $\mathbf{L} \models \sigma[(w^-)_x^a]$. Then, note that $(w^-)_x^a = (w_x^{\langle a, b \rangle})^-$ for every $b \in L$. So, using the inductive hypothesis, we obtain $\mathbf{L} \odot \mathbf{L} \models \tau(\sigma)[w_x^{\langle a, b \rangle}]$. Therefore, $\mathbf{L} \odot \mathbf{L} \models \exists x \tau(\sigma)[w]$. Suppose now that $\mathbf{L} \odot \mathbf{L} \models \exists x \tau(\sigma)[w]$. Let $a, b \in L$ be such that $\mathbf{L} \odot \mathbf{L} \models \tau(\sigma)[w_x^{\langle a, b \rangle}]$. Then, by inductive hypothesis, $\mathbf{L} \models \sigma[(w_x^{\langle a, b \rangle})^-]$. Now, since $(w^-)_x^a = (w_x^{\langle a, b \rangle})^-$, we have that $\mathbf{L} \models \sigma[(w^-)_x^a]$. It follows that $\mathbf{L} \models \exists x \sigma[w^-]$. (ii). Follows from (i) because $v = (v^+)^-$ for every assignment v on \mathbf{L} .

As a consequence of the previous proposition, we have the following:

Proposition 4.2 *If \mathbf{K} is a class of residuated lattices axiomatizable by a set of first-order sentences Π , then $\mathbf{K}^{\mathbf{B}}$ is the class of residuated bilattices axiomatizable by the set of sentences $\tau[\Pi]$ obtained by applying the translation τ to the sentences in Π .*

Proof If $\mathbf{B} \cong \mathbf{L} \odot \mathbf{L}$ and $\mathbf{L} \in \mathbf{K}$, then since every sentence in Π is valid in \mathbf{L} , every sentence in $\tau[\Pi]$ is valid in \mathbf{B} . Conversely, if every sentence in $\tau[\Pi]$ is valid in a residuated bilattice \mathbf{B} , then since $\mathbf{B} \cong \mathbf{L} \odot \mathbf{L}$ for some residuated lattice \mathbf{L} , we have that the sentences in Π are valid on \mathbf{L} . Therefore, $\mathbf{L} \in \mathbf{K}$.

Note that if Π is a set of equations, then $\tau[\Pi]$ is also a set of equations, and if Π is a set of quasi-equations, then so is $\tau[\Pi]$. Therefore the map $(\cdot)^B$ sends varieties to varieties and quasi-varieties to quasi-varieties.

Conversely, drawing inspiration from the representation given in Theorem 3.6, we may define inductively a translation $\rho : \mathcal{T} \rightarrow \mathcal{S} \times \mathcal{S}$ as follows. First of all we make a partition of the set of variables into two infinite sets and consider its cartesian product. Then we consider a bijection h between the set of variables and this cartesian product. If x is variable, let $h(x) := \langle x_1, x_2 \rangle$. Then, for any variable $x \in \mathcal{T}$ and terms $\varphi, \psi \in \mathcal{T}$, we define

$$\begin{aligned} \rho(x) &:= h(x) \\ \rho(\neg\varphi) &:= \langle \rho(\varphi)_2, \rho(\varphi)_1 \rangle \\ \rho(\varphi \wedge \psi) &:= \langle \rho(\varphi)_1 \sqcap \rho(\psi)_1, \rho(\varphi)_2 \sqcup \rho(\psi)_2 \rangle \\ \rho(\varphi \vee \psi) &:= \langle \rho(\varphi)_1 \sqcup \rho(\psi)_1, \rho(\varphi)_2 \sqcap \rho(\psi)_2 \rangle \\ \rho(\varphi \otimes \psi) &:= \langle \rho(\varphi)_1 \sqcap \rho(\psi)_1, \rho(\varphi)_2 \sqcap \rho(\psi)_2 \rangle \\ \rho(\varphi \oplus \psi) &:= \langle \rho(\varphi)_1 \sqcup \rho(\psi)_1, \rho(\varphi)_2 \sqcup \rho(\psi)_2 \rangle \\ \rho(\varphi \supset \psi) &:= \langle \rho(\varphi)_1 \setminus \rho(\psi)_1, \rho(\psi)_2 \cdot \rho(\varphi)_1 \rangle \\ \rho(\varphi \subset \psi) &:= \langle \rho(\varphi)_1 / \rho(\psi)_1, \rho(\psi)_1 \cdot \rho(\varphi)_2 \rangle \\ \rho(\varphi * \psi) &:= \langle \rho(\varphi)_1 \cdot \rho(\psi)_1, \\ &\quad \rho(\psi)_2 / \rho(\varphi)_1 \sqcap (\rho(\psi)_1 \setminus \rho(\varphi)_2) \rangle \end{aligned}$$

where for any formula φ we use the convention to refer by $\rho(\varphi)_1$ to the first member of the pair $\rho(\varphi)$ and by $\rho(\varphi)_2$ to the second member, so that

$$\rho(\varphi) = \langle \rho(\varphi)_1, \rho(\varphi)_2 \rangle.$$

If we wanted to include constants in the language, then we would let

$$\begin{aligned} \rho(\mathbf{f}) &:= \langle 0, 1 \rangle & \rho(\mathbf{t}) &:= \langle 1, 0 \rangle \\ \rho(\perp) &:= \langle 0, 0 \rangle & \rho(\top) &:= \langle 1, 1 \rangle. \end{aligned}$$

We extend the translation ρ to first-order formulas in the language of residuated bilattices as follows: for all terms $\varphi, \psi \in \mathcal{T}$ and all every first-order formulas σ, σ' of \mathcal{T} ,

$$\begin{aligned} \rho(\varphi \approx \psi) &:= \rho(\varphi)_1 \approx \rho(\psi)_1 \ \& \ \rho(\varphi)_2 \approx \rho(\psi)_2 \\ \rho(\sim \sigma) &:= \sim \rho(\sigma) \\ \rho(\sigma \Rightarrow \sigma') &:= \rho(\sigma) \Rightarrow \rho(\sigma') \\ \rho(\exists x \sigma) &:= \exists x_1 x_2 \rho(\sigma), \text{ where } h(x) = \langle x_1, x_2 \rangle. \end{aligned}$$

Let \mathbf{L} be a residuated lattice and v an assignment on $\mathbf{L} \odot \mathbf{L}$. We define the assignment v^* on \mathbf{L} as follows. Let $x \in \mathcal{T}$ be a variable and $h(x) = \langle x_1, x_2 \rangle$. If $v(x) = \langle a_1, a_2 \rangle$, we set

$$v^*(x_1) := a_1 \quad \text{and} \quad v^*(x_2) := a_2.$$

Since every variable is the first or second component of $h(y)$ for some variable $y \in \mathcal{T}$, we have indeed defined an assignment. By induction it follows that, for every $\varphi \in \mathcal{T}$,

$$v(\varphi) = \langle v^*(\rho(\varphi)_1), v^*(\rho(\varphi)_2) \rangle. \quad (4.1)$$

Let us now assume that w is an assignment on \mathbf{L} . We define the assignment w_* on $\mathbf{L} \odot \mathbf{L}$ as follows:

$$w_*(x) := \langle w(x_1), w(x_2) \rangle$$

for every variable $x \in \mathcal{T}$, where $h(x) = \langle x_1, x_2 \rangle$. Then clearly $(w_*)^* = w$. Thus, have the following:

Proposition 4.3 *For every first-order formula σ of \mathcal{T} ,*

- (i) $\mathbf{L} \odot \mathbf{L} \models \sigma[v]$ iff $\mathbf{L} \models \rho(\sigma)[v^*]$ for every assignment v on $\mathbf{L} \odot \mathbf{L}$
- (ii) $\mathbf{L} \odot \mathbf{L} \models \sigma[w_*]$ iff $\mathbf{L} \models \rho(\sigma)[w]$ for every assignment w on \mathbf{L} .

Proof (ii) follows from (i) taking into account that if w is an assignment on \mathbf{L} , then $(w_*)^* = w$. We prove (i) by induction. For equations it easily follows using (4.1). Applying the inductive hypothesis it is straightforward to see that it holds for negations and implications. Assume it holds for σ . We prove that it holds for $\exists x \sigma$. Suppose that $\mathbf{L} \odot \mathbf{L} \models \exists x \sigma[v]$. Then $\mathbf{L} \odot \mathbf{L} \models \sigma[v_x^{(a,b)}]$ for some $a, b \in L$. So, by inductive hypothesis, $\mathbf{L} \models \rho(\sigma)[(v_x^{(a,b)})^*]$. Now note that $(v_x^{(a,b)})^* = (v^*)_{x_1, x_2}^{a, b}$. So, $\mathbf{L} \models \rho(\sigma)[(v^*)_{x_1, x_2}^{a, b}]$ and therefore $\mathbf{L} \models \exists x_1 x_2 \rho(\sigma)[(v^*)]$, i.e. $\mathbf{L} \models \rho(\exists x \sigma)[v^*]$. Conversely, if $\mathbf{L} \models \rho(\exists x \sigma)[v^*]$, then $\mathbf{L} \models \rho(\sigma)[(v^*)_{x_1, x_2}^{a, b}]$ for some $a, b \in L$, and so $\mathbf{L} \odot \mathbf{L} \models \sigma[v_x^{(a,b)}]$. Therefore, by inductive hypothesis, $\mathbf{L} \odot \mathbf{L} \models \sigma[v_x^{(a,b)}]$, which implies $\mathbf{L} \odot \mathbf{L} \models \exists x \sigma[v]$.

As a consequence of the previous proposition, we have the following:

Proposition 4.4 *If \mathbf{K} is a class of residuated bilattices axiomatizable by a set of first-order sentences Π , then the class \mathbf{K}^L is the class of residuated lattices axiomatizable by the set of first-order sentences $\rho[\Pi]$ obtained from Π by applying the translation ρ to the sentences in Π .*

Note that if Π is a set of equations, then $\rho[\Pi]$ can be taken as a set of equations too, and if Π is a set of quasi-equations, then $\rho[\Pi]$ can be taken a set of quasi-equations. Therefore, the map $(\cdot)^L$ sends varieties to varieties and quasi-varieties to quasi-varieties.

In Bou et al. [Bou et al. 201x, Section 5] the correspondence between a number of quasi-varieties of lattices and bilattices is extended to a categorical equivalence between categories associated with lattices and bilattices (see also [Rivieccio 2010, Section 5.6]). The

definition of the categories is in all cases the natural one, i.e. the objects of the category are algebras of the corresponding class of lattices or bilattices and the morphisms are algebraic homomorphisms. This result can be straightforwardly generalized, using the same definitions for categories and functors, to the context of residuated lattices and bilattices, thus establishing categorical equivalences between all the sub-quasi-varieties of residuated lattices and all the sub-quasi-varieties of residuated bilattices. We will not pursue this here, but it is not difficult to see that all the relevant proofs of [Bou et al. 201x] can be straightforwardly adapted to the present setting.

5 Acknowledgments

The research of the first author was partially supported by grant 2009SGR-1433 of the AGAUR of the Generalitat de Catalunya and by grant MTM2008-01139 of the Spanish Ministerio de Ciencia e Innovación, which includes EU FEDER funds.

References

- [Arieli and Avron 1996] Arieli O, Avron A (1996) Reasoning with logical bilattices. *Journal of Logic, Language and Information* 5(1):25–63
- [Avron 1996] Avron A (1996) The structure of interlaced bilattices. *Mathematical Structures in Computer Science* 6(3):287–299
- [Bou et al. 201x] Bou F, Jansana R, Rivieccio U (201x) Varieties of interlaced bilattices. *Algebra Universalis*, to appear.
- [Bou and Rivieccio 2011] Bou F, Rivieccio U (2011) The logic of distributive bilattices. *Logic Journal of the IGPL* 19(1):183–216.
- [Busaniche and Cignoli 2009] Busaniche M, Cignoli R (2009) Residuated lattices as an algebraic semantics for paraconsistent Nelson’s logic. *Journal of Logic and Computation* 19(6):1019–1029.
- [Galatos and Raftery 2004] Galatos N, Raftery JG (2004) Adding Involution to Residuated Structures. *Studia Logica* 77(2):181–207.
- [Galatos et al. 2007] Galatos N, Jipsen P, Kowalski T, Ono H (2007) Residuated Lattices: an algebraic glimpse at substructural logics, *Studies in Logic and the Foundations of Mathematics*, vol 151. Elsevier, Amsterdam.
- [Ginsberg 1988] Ginsberg ML (1988) Multivalued logics: A uniform approach to inference in artificial intelligence. *Computational Intelligence* 4:265–316, DOI 10.1111/j.1467-8640.1988.tb00280.x
- [Mobasher et al. 2000] Mobasher B, Pigozzi D, Slutzki G, Voutsadakis G (2000) A duality theory for bilattices. *Algebra Universalis* 43(2-3):109–125.
- [Movsisyan et al. 2006] Movsisyan YM, Romanowska AB, Smith JDH (2006) Superproducts, hyperidentities, and algebraic structures of logic programming. *J Combin Math Combin Comput* 58:101–111.
- [Odintsov 2003] Odintsov SP (2003) Algebraic semantics for paraconsistent Nelson’s logic. *Journal of Logic and Computation* 13(4):453–468, DOI 10.1093/logcom/13.4.453
- [Odintsov 2004] Odintsov SP (2004) On the representation of N4-lattices. *Studia Logica* 76(3):385–405, DOI 10.1023/B:STUD.0000032104.14199.08
- [Odintsov 2009] Odintsov SP (2009) On axiomatizing shramko-wansing’s logic. *Studia Logica* 91(3):407–428, DOI 10.1007/s11225-009-9181-6
- [Rivieccio 2010] Rivieccio U (2010) An algebraic study of bilattice-based logics. Ph. D. Dissertation, University of Barcelona
- [Tsinakis and Wille 2006] Tsinakis C, Wille AM (2006) Minimal varieties of involutive residuated lattices. *Studia Logica* 83(1-3):407–423, DOI 10.1007/s11225-006-8311-7